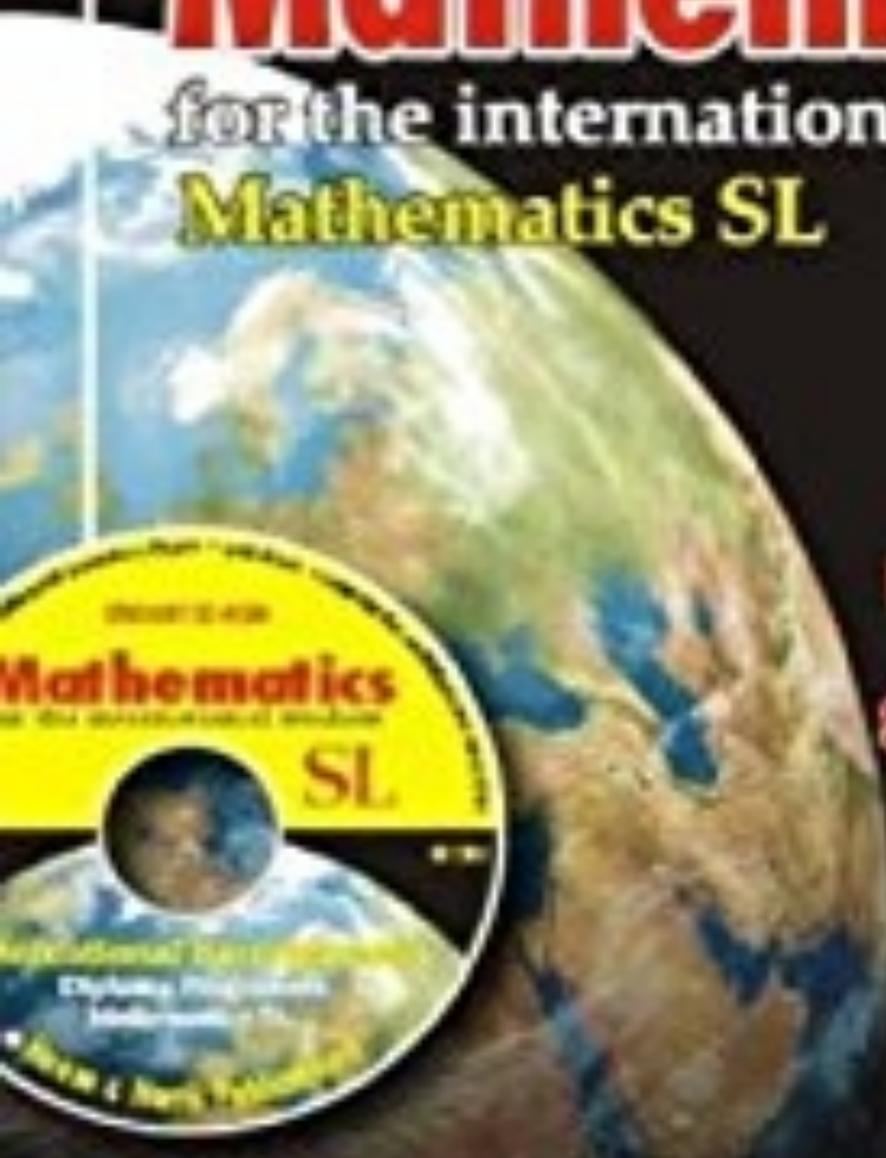
HAESE & HARRIS PUBLICATIONS

Mathematics

for the international student

Mathematics SL



John Owen

Robert Haese

Sandra Haese

Mark Bruce

International Baccalaureate
Diploma Programme

HAESE & HARRIS PUBLICATIONS
Specialists in mathematics publishing

Mathematics

for the international student

Mathematics SL

John Owen
Robert Haese
Sandra Haese
Mark Bruce

International Baccalaureate Diploma Programme

MATHEMATICS FOR THE INTERNATIONAL STUDENT
International Baccalaureate Mathematics SL Course

John Owen	B.Sc., Dip.T.
Robert Haese	B.Sc.
Sandra Haese	B.Sc.
Mark Bruce	B.Ed.

Haese & Harris Publications
3 Frank Collopy Court, Adelaide Airport, SA 5950, AUSTRALIA
Telephone: +61 8 8355 9444, Fax: + 61 8 8355 9471
Email: info@haeseandharris.com.au
Web: www.haeseandharris.com.au

National Library of Australia Card Number & ISBN 1 876543 03 5

Published by Raksar Nominees Pty Ltd
3 Frank Collopy Court, Adelaide Airport, SA 5950, AUSTRALIA

First Edition 2004
Reprinted 2005 three times *(with minor corrections)*, 2006, 2007

Cartoon artwork by John Martin. Artwork by Piotr Poturaj and David Purton
Cover design by Piotr Poturaj
Computer software by David Purton

Typeset in Australia by Susan Haese (Raksar Nominees). Typeset in Times Roman $10\frac{1}{2}/11\frac{1}{2}$

Acknowledgements: While every attempt has been made to trace and acknowledge copyright, the authors and publishers apologise for any accidental infringement where copyright has proved untraceable. They would be pleased to come to a suitable agreement with the rightful owner.

FOREWORD

Mathematics for the International Student: Mathematics SL has been written to embrace the syllabus for the new two-year Mathematics SL Course, which is one of the courses of study in the International Baccalaureate Diploma Programme. It is not our intention to define the course. Teachers are encouraged to use other resources. We have developed the book independently of the International Baccalaureate Organization (IBO) in consultation with many experienced teachers of IB Mathematics. The text is not endorsed by the IBO.

This package is language rich and technology rich. The combination of textbook and interactive Student CD will foster the mathematical development of students in a stimulating way. Frequent use of the interactive features on the CD is certain to nurture a much deeper understanding and appreciation of mathematical concepts.

The book contains many problems from the basic to the advanced, to cater for a wide range of student abilities and interests. While some of the exercises are simply designed to build skills, every effort has been made to contextualise problems, so that students can see everyday uses and practical applications of the mathematics they are studying, and appreciate the universality of mathematics.

Emphasis is placed on the gradual development of concepts with appropriate worked examples, but we have also provided extension material for those who wish to go beyond the scope of the syllabus. Some proofs have been included for completeness and interest although they will not be examined.

For students who may not have a good understanding of the necessary background knowledge for this course, we have provided printable pages of information, examples, exercises and answers on the Student CD. To access these pages, simply click on the 'Background knowledge' icon when running the CD.

It is not our intention that each chapter be worked through in full. Time constraints will not allow for this. Teachers must select exercises carefully, according to the abilities and prior knowledge of their students, to make the most efficient use of time and give as thorough coverage of work as possible. Investigations throughout the book will add to the discovery aspect of the course and enhance student understanding and learning. Many Investigations are suitable for portfolio assignments and have been highlighted in the table of contents. Review sets appear at the end of each chapter and a suggested order for teaching the two-year course is given at the end of this Foreword.

The extensive use of graphics calculators and computer packages throughout the book enables students to realise the importance, application and appropriate use of technology. No single aspect of technology has been favoured. It is as important that students work with a pen and paper as it is that they use their calculator or graphics calculator, or use a spreadsheet or graphing package on computer.

The interactive features of the CD allow immediate access to our own specially designed geometry packages, graphing packages and more. Teachers are provided with a quick and easy way to demonstrate concepts, and students can discover for themselves and re-visit when necessary.

Instructions appropriate to each graphic calculator problem are on the CD and can be printed for students. These instructions are written for Texas Instruments and Casio calculators.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

Email: info@haeseandharris.com.au
Web: www.haeseandharris.com.au

JTO RCH

SHH MFB

Thank you

The authors and publishers would like to thank all those teachers who offered advice and encouragement. Many of them read the page proofs and offered constructive comments and suggestions. These teachers include: Marjut Mäenpää, Cameron Hall, Paul Urban, Fran O'Connor, Glenn Smith, Anne Walker, Malcolm Coad, Ian Hilditch, Phil Moore, Julie Wilson, David Martin, Kerrie Clements, Margie Karbassioun, Brian Johnson, Carolyn Farr, Rupert de Smidt, Terry Swain, Marie-Therese Filippi, Nigel Wheeler, Sarah Locke, Rema George.

TEACHING THE TWO-YEAR COURSE – A SUGGESTED ORDER

For the first year, it is suggested that students work progressively from Chapter 1 through to Chapter 16, although some teachers may prefer to leave Chapter 16 'Vectors in 3 dimensions' until the second year.

'Descriptive statistics' and 'Probability' (Chapters 18 and 19) could possibly be taught the first year. Alternatively, calculus could be introduced (Chapters 20-22), but most teachers will probably prefer to leave calculus until the second year and have students work progressively from Chapter 20 though to Chapter 29.

We invite teachers who have their preferred order, to email their suggestions to us. We can put these suggestions on our website to be shared with other teachers.

USING THE INTERACTIVE STUDENT CD

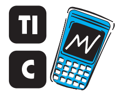
The CD is ideal for independent study. Frequent use will nurture a deeper understanding of Mathematics. Students can revisit concepts taught in class and undertake their own revision and practice. The CD also has the text of the book, allowing students to leave the textbook at school and keep the CD at home.

The icon denotes an active link on the CD. Simply 'click' the icon to access a range of interactive features:

- spreadsheets and worksheets
- video clips
- graphing and geometry software
- graphics calculator instructions
- computer demonstrations and simulations
- background knowledge

For those who want to make sure they have the prerequisite levels of understanding for this new course, printable pages of background information, examples, exercises and answers are provided on the CD. Click the 'Background knowledge' icon.

Graphics calculators: Instructions for using graphics calculators are also given on the CD and can be printed. Instructions are given for Texas Instruments and Casio calculators. Click on the relevant symbol (TI or C) to access printable instructions.



NOTE ON ACCURACY

Students are reminded that in assessment tasks, including examination papers, unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

ERRATA

If you find an error in this book please notify us by emailing errata@haeseandharris.com.au.

As a help to other teachers and students, we will include the correction on our website and correct the book at the first reprint opportunity.

TABLE OF CONTENTS

* denotes ideas for possible **Portfolio Assignments**

Abbreviations used in this book 10

BACKGROUND KNOWLEDGE – to access, 'click' active icon on CD 11

A Operations with surds (radicals) CD
B Standard form (scientific notation) CD
C Number systems and set notation CD
D Algebraic simplification CD
E Linear equations and inequalities CD
F Absolute value (modulus) CD
G Product expansion CD
H Factorisation CD
Investigation: Another factorisation technique CD
I Formula rearrangement CD
J Adding and subtracting algebraic fractions CD
K Congruence and similarity CD
ANSWERS CD
Summary of circle properties 11
Summary of measurement facts 12

1 FUNCTIONS 15

A Relations and functions 16
B Interval notation, domain and range 19
C Function notation 22
Investigation: Fluid filling functions 24
D Composite functions, $f \circ g$ 26
E The reciprocal function $x \mapsto \frac{1}{x}$ 27
F Inverse functions 28
G The identity function 31
Review set 1A 32
Review set 1B 33

2 SEQUENCES AND SERIES 35

A Number patterns 36
B Sequences of numbers 38
C Arithmetic sequences 41
D Geometric sequences 44
E Series 51
F Sigma notation 57
* ***Investigation***: Von Koch's Snowflake curve 58
Review set 2A 59
Review set 2B 59
Review set 2C 60

3 EXPONENTS 61

A Index notation 62
B Negative bases 63
C Index laws 65
D Rational indices 71
E Algebraic expansion 73
F Exponential equations 74
G Graphs of exponential functions 75
* ***Investigation***: Exponential graphs 76
H Growth 79
I Decay 81
Review set 3A 84
Review set 3B 84
Review set 3C 85
Review set 3D 86

4 LOGARITHMS 87

A Introduction 88
B Logarithms in base 10 90
Investigation: Discovering the laws of logarithms 92
C Laws of logarithms 93
D Exponential equations (using logarithms) 96
E Growth and decay revisited 97
F Compound interest revisited 99
G The change of base rule 100
Review set 4A 101
Review set 4B 102

5 NATURAL LOGARITHMS 103

A Introduction 104
Investigation 1: e occurs naturally 104
Investigation 2: Continuous compound interest 105
B Natural logarithms 108
Investigation 3: The laws of natural logarithms 109
C Laws of natural logarithms 110
D Exponential equations involving e 112
E Growth and decay revisited 112
F Inverse functions revisited 114
Review set 5A 116
Review set 5B 117

6 GRAPHING AND TRANSFORMING FUNCTIONS 119

A Families of functions 120
* ***Investigation***: Function families 120
B Key features of functions 122
C Transformations of graphs 123
D Functional transformations 126
Review set 6 127

7 COORDINATE GEOMETRY 129

A Assumed knowledge 131
* ***Investigation***: Finding where lines meet using technology 135
B Equations of lines 138
C Distance between two points 141
D Midpoints and perpendicular bisectors 144
Review set 7A 146

Review set 7B 147
Review set 7C 148

8 QUADRATIC EQUATIONS AND FUNCTIONS 149

A Function notation $f: \mapsto ax^2 + bx + c$ 153
B Graphs of quadratic functions 154
* ***Investigation 1***: Graphing $y = a(x-\alpha)(x-\beta)$ 155
* ***Investigation 2***: Graphing $y = a(x-h)^2 + k$ 155
C Completing the square 160
D Quadratic equations 162
E The quadratic formula 168
F Solving quadratic equations with technology 170
G Problem solving with quadratics 171
H Quadratic graphs (review) 174
I The discriminant, Δ 178
J Determining the quadratic from a graph 182
K Where functions meet 185
L Quadratic modelling 186
Review set 8A 190
Review set 8B 191
Review set 8C 191
Review set 8D 192
Review set 8E 193

9 THE BINOMIAL THEOREM 195

A Binomial expansions 196
Investigation 1: The binomial expansions of $(a+b)^n, n \geqslant 4$ 196
Investigation 2: ${}_nC_r$ or C_r^n values 199
B The general binomial expansion 199
Review set 9 202

10 PRACTICAL TRIGONOMETRY WITH RIGHT ANGLED TRIANGLES 203

A Pythagoras' rule (review) 205
B Pythagoras' rule in 3-D problems 207
Investigation: Shortest distance 208
C Right angled triangle trigonometry 209
D Finding sides and angles 211
E Problem solving using trigonometry 217
F The slope of a straight line 221
Review set 10A 222
Review set 10B 223
Review set 10C 223

11 THE UNIT CIRCLE 225

A The unit quarter circle 226
B Obtuse angles 228
C The unit circle 231
Investigation: Parametric equations 232
Review set 11 233

12 NON RIGHT ANGLED TRIANGLE TRIGONOMETRY 235

A Areas of triangles 236
B Sectors and segments 239
C The cosine rule 241
D The sine rule 244
Investigation: The ambiguous case 245
E Using the sine and cosine rules 248
Review set 12A 251
Review set 12B 252

13 PERIODIC PHENOMENA 255

A Observing periodic behaviour 257
B Radian measure and periodic properties of circles 260
C The unit circle (revisited) 265
D The sine function 270
* ***Investigation 1***: The family $y = A\sin x$ 271
* ***Investigation 2***: The family $y = \sin Bx,\ B > 0$ 271
* ***Investigation 3***: The families $y = \sin(x - C)$ and $y = \sin x + D$ 273
E Modelling using sine functions 275
F Equations involving sine 278
* ***Investigation 4***: The area under an arch of $y = \sin\theta$ 284
G The cosine function 285
H Solving cosine equations 287
I Trigonometric relationships 289
J Double angle formulae 292
Investigation 5: Double angle formulae 292
K The tangent function 295
L Tangent equations 298
M Other equations involving $\tan x$ 301
Review set 13A 302
Review set 13B 302
Review set 13C 303
Review set 13D 304
Review set 13E 305

14 MATRICES 307

A Introduction 308
B Addition and subtraction of matrices 311
C Multiples of matrices 314
D Matrix algebra for addition 316
E Matrix multiplication 317
F Using technology 321
G Some properties of matrix multiplication 325
H The inverse of a 2×2 matrix 328
I Solving a pair of linear equations 331
J The 3×3 determinant 334
K The inverse of a 3×3 matrix 337
L 3×3 systems with unique solutions 337

*	***Investigation***: Using matrices in cryptography	341
	Review set 14A	342
	Review set 14B	343
	Review set 14C	344
	Review set 14D	345
	Review set 14E	345
15	**VECTORS IN 2-DIMENSIONS**	**347**
A	Vectors	348
B	Operations with vectors	352
C	Vectors in component form	360
D	Vector equations	365
E	Vectors in coordinate geometry	366
F	Parallelism	368
G	Unit vectors	369
H	Angles and scalar product	371
	Review set 15A	376
	Review set 15B	376
	Review set 15C	377
	Review set 15D	378
16	**VECTORS IN 3-DIMENSIONS**	**379**
A	3-dimensional coordinates	380
B	3-dimensional vectors	383
C	Algebraic operations with 3-D vectors	386
D	Parallelism	389
E	Unit vectors	390
F	Collinear points and ratio of division – extension	391
G	The scalar product of 3-D vectors	392
	Review set 16A	395
	Review set 16B	396
17	**LINES IN THE PLANE AND IN SPACE**	**397**
A	Vector and parametric form of a line in 2-dimensional geometry	399
B	The velocity vector of a moving object	401
C	Constant velocity problems	403
	Investigation: The two yachts problem	405
D	The closest distance	406
E	Geometric applications of $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	409
F	Lines in space	411
G	Line classification	414
	Review set 17A	416
	Review set 17B	417
18	**DESCRIPTIVE STATISTICS**	**419**
	BACKGROUND KNOWLEDGE IN STATISTICS – to access, 'click' active icon on CD	**420**
A	Statistical enquiries	CD
	Investigation: Statistics from the internet	CD
B	Populations and samples	CD
C	Presenting and interpreting data	CD
	ANSWERS	CD
A	Continuous numerical data and histograms	421
B	Measuring the centre of data	425
	Investigation: Merits of the mean and median	427
C	Cumulative data	442
D	Measuring the spread of data	445
E	Statistics using technology	453
F	Variance and standard deviation	456
G	The significance of standard deviation	461
	Review set 18A	463
	Review set 18B	465
19	**PROBABILITY**	**467**
A	Experimental probability	470
	Investigation 1: Tossing drawing pins	470
	Investigation 2: Coin tossing experiments	471
	Investigation 3: Dice rolling experiments	472
B	Sample space	474
C	Theoretical probability	475
D	Using grids to find probabilities	479
E	Compound events	480
	Investigation 4: Probabilities of compound events	481
	Investigation 5: Revisiting drawing pins	481
F	Using tree diagrams	485
G	Sampling with and without replacement	487
*	***Investigation 6***: Sampling simulation	490
	Investigation 7: How many should I plant?	491
H	Pascal's triangle revisited	492
I	Sets and Venn diagrams	494
J	Laws of probability	499
K	Independent events revisited	503
	Review set 19A	505
	Review set 19B	505
20	**INTRODUCTION TO CALCULUS**	**507**
	Investigation 1: The speed of falling objects	508
A	Rate of change	509
B	Instantaneous rates of change	513
	Investigation 2: Instantaneous speed	513
	Review set 20	519
21	**DIFFERENTIAL CALCULUS**	**521**
A	The idea of a limit	522
	Investigation 1: The slope of a tangent	522
B	Derivatives at a given x-value	525
C	The derivative function	530
	Investigation 2: Finding slopes of functions with technology	531
D	Simple rules of differentiation	533
	Investigation 3: Simple rules of differentiation	534
E	The chain rule	537

Investigation 4: Differentiating composites 538
F Product and quotient rules 541
G Tangents and normals 545
H The second derivative 550
Review set 21A 552
Review set 21B 553
Review set 21C 554

22 APPLICATIONS OF DIFFERENTIAL CALCULUS 555
A Functions of time 556
B Time rate of change 558
C General rates of change 560
D Motion in a straight line 564
Investigation: Displacement, velocity and acceleration graphs 568
E Curve properties 571
F Rational functions 579
G Inflections and shape type 583
H Optimisation 587
I Economic models 597
Review set 22A 600
Review set 22B 602

23 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS 605
A Derivatives of exponential functions 606
Investigation 1: The derivative of $y = a^x$ 606
Investigation 2: Finding a when $y = a^x$ and $\dfrac{dy}{dx} = a^x$ 607
B Using natural logarithms 611
C Derivatives of logarithmic functions 615
Investigation 3: The derivative of $\ln x$ 615
D Applications 618
Review set 23A 622
Review set 23B 623

24 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS 625
A The derivative of $\sin x$, $\cos x$, $\tan x$ 627
B Maxima/minima with trigonometry 632
Review set 24 634

25 AREAS WITHIN CURVED BOUNDARIES 637
A Areas where boundaries are curved 638
Investigation 1: Finding areas using rectangles 640
B Definite integrals 642
Investigation 2: $\int_0^1 (1 - x^2)\, dx$, $\int_0^1 (x^2 - 1)\, dx$ 646
Review set 25 648

26 INTEGRATION 649
A Reviewing the definite integral 650
B The area function 655
Investigation 1: The area function 656
C Antidifferentiation 656
D The fundamental theorem of calculus 658
E Integration 662
F Integrating e^{ax+b} and $(ax+b)^n$ 668
G Integrating $f(u)\,u'(x)$ by substitution 670
H Distance from velocity 673
I Definite integrals 675
Investigation 2: $\int_a^b f(x)\, dx$ and areas 677
J Finding areas 677
K Problem solving by integration 682
Review set 26A 685
Review set 26B 686
Review set 26C 687

27 TRIGONOMETRIC INTEGRATION 689
A Basic trigonometric integrals 690
B Integrals of trigonometric functions of the form $f(ax+b)$ 692
C Definite integrals 695
D Area determination 698
Review set 27A 700
Review set 27B 700

28 VOLUMES OF REVOLUTION 701
A Solids of revolution 702
B Volumes for two defining functions 705
Review set 28 708

29 STATISTICAL DISTRIBUTIONS 709
A Discrete random variables 710
B Discrete probability distributions 712
C Expectation 714
Investigation 1: Concealed number tickets 717
D The mean and standard deviation of a discrete random variable 717
E The binomial distribution 721
F Mean and standard deviation of a binomial random variable 724
Investigation 2: The mean and standard deviation of a binomial random variable 726
G Normal distributions 727
Investigation 3: Standard deviation significance 730
Investigation 4: Mean and standard deviation of $z = \dfrac{x - \overline{x}}{s}$ 733
H The standard normal distribution (z-distribution) 734
I Applications of the normal distribution 739
Review set 29A 741
Review set 29B 742
Review set 29C 743

ANSWERS AND INDEX 745

SYMBOLS AND NOTATION USED IN THIS BOOK

This notation is based on that indicated by the International Organisation of Standardisation.

$\mathbf{N}$	the set of all natural numbers $\{0, 1, 2, 3, 4, 5, \ldots\ldots\}$
$\mathbf{Z}$	the set of all integers $\{0, \pm1, \pm2, \pm3, \pm4, \pm5, \ldots\ldots\}$
$\mathbf{Z}^+$	the set of all positive integers $\{1, 2, 3, 4, 5, \ldots\ldots\}$
$\mathbf{Q}$	the set of all rational numbers
$\mathbf{Q}^+$	the set of all positive rational numbers
$\mathcal{R}$	the set of all real numbers
$\mathcal{R}^+$	the set of all positive real numbers
$n(S)$	the number of elements in set S
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty set, or null set
$\mathbf{U}$	the universal set
$\cup$	union
$\cap$	intersection
$\lvert x\rvert$	the modulus of x, or the absolute value of x $\lvert x \rvert = x$ if $x \geqslant 0$ or $-x$ if $x \leqslant 0$
u_n	the nth term of a sequence
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	$u_1 + u_2 + u_3 + \ldots\ldots + u_n$ the sum of the first n terms of a sequence
S_∞	the sum to infinity of a sequence
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + u_3 + \ldots\ldots + u_n$
$\binom{n}{r}$	the binomial coefficient of the $(r+1)$th term in the expansion of $(a+b)^n$
$f : x \mapsto y$	f is the function where x goes to y
$f(x)$	the image of x operated on by f
$f^{-1}(x)$	the inverse function of $f(x)$
$f \circ g$	the composite function of f and g
$\lim\limits_{x\to a} f(x)$	the limit of $f(x)$ as x tends to a
$\dfrac{dy}{dx}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\int y\,dx$	the indefinite integral of y with respect to x
$\int_a^b y\,dx$	the definite integral of y with respect to x from $x = a$ to $x = b$
e^x	the exponential function
$\log_a x$	the logarithm of x, in base a
$\ln x$	the natural logarithm of x, $\log_e x$
$\sin x$, $\cos x$ and $\tan x$	the circular functions
$\mathrm{P}(x, y)$	point P with coordinates x and y
$\measuredangle\mathrm{A}$	the angle at A
$\measuredangle\mathrm{PQR}$	the angle between QP and QR
$\Delta\mathrm{PQR}$	the triangle with vertices P, Q and R
$\mathbf{v}$	vector $\mathbf{v}$
$\overrightarrow{\mathrm{AB}}$	the vector from point A to point B
$\mathbf{a}$	the position vector of A, $\overrightarrow{\mathrm{OA}}$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the direction of the x-, y- and z-axis respectively
$\lvert\mathbf{a}\rvert$	the magnitude of $\mathbf{a}$
$\mathbf{a} \bullet \mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{A}^{-1}$	the inverse of matrix $\mathbf{A}$
$\det \mathbf{A}$	the determinant of matrix $\mathbf{A}$
$\mathbf{I}$	the identity matrix under $\times$
P(A)	the probability of event A
P(A′)	the probability of event ‘not A’
P(A/B)	the probability of A occurring given that B has occurred
$\mathrm{N}(\mu, \sigma^2)$	the normal distribution with mean μ and variance σ^2
μ	population mean
σ^2	population variance
σ	population standard deviation
$\overline{x}$	sample mean
s_n^2	sample variance
s_n	sample standard deviation

BACKGROUND KNOWLEDGE

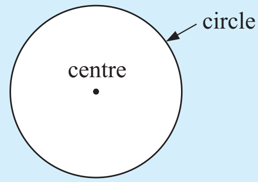
Before starting this course you can make sure that you have a good understanding of the necessary background knowledge.

Click on the icon alongside to obtain a printable set of exercises and answers on this background knowledge.

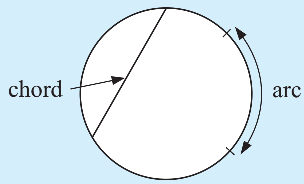
BACKGROUND KNOWLEDGE

SUMMARY OF CIRCLE PROPERTIES

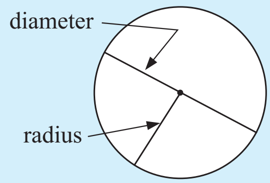
- A **circle** is a set of points which are equidistant from a fixed point, which is called its **centre**.



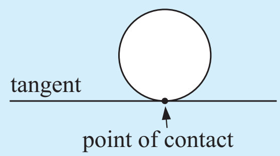
- The **circumference** is the distance around the entire circle boundary.
- An **arc** of a circle is any continuous part of the circle.
- A **chord** of a circle is a line segment joining any two points of a circle.



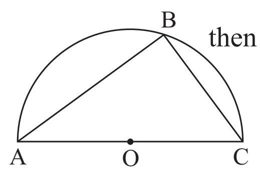
- A **semi-circle** is a half of a circle.
- A **diameter** of a circle is any chord passing through its centre.
- A **radius** of a circle is any line segment joining its centre to any point on the circle.

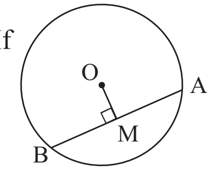
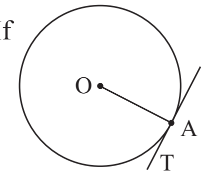
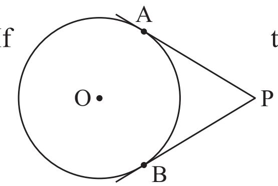
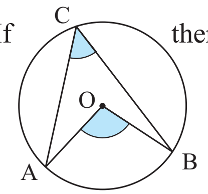
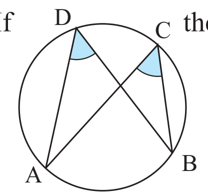
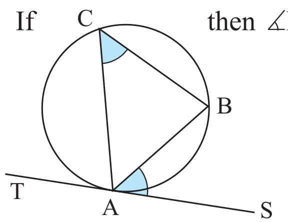


- A **tangent** to a circle is any line which touches the circle in exactly one point.



Below is a summary of well known results called theorems. Click on the appropriate icon to revisit them.

Name of theorem	Statement	Diagram
Angle in a semi-circle	The angle in a semi-circle is a right angle.	If  then $\angle ABC = 90^\circ$. GEOMETRY PACKAGE

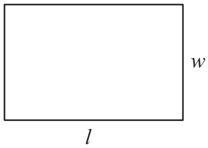
Name of theorem	Statement	Diagram
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	If  then AM = BM. GEOMETRY PACKAGE
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	If  then $\measuredangle OAT = 90°$. GEOMETRY PACKAGE
Tangents from an external point	Tangents from an external point are equal in length.	If  then AP = BP. GEOMETRY PACKAGE
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	If  then $\measuredangle AOB = 2\measuredangle ACB$. GEOMETRY PACKAGE
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	If  then $\measuredangle ADB = \measuredangle ACB$. GEOMETRY PACKAGE
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	If  then $\measuredangle BAS = \measuredangle BCA$. GEOMETRY PACKAGE

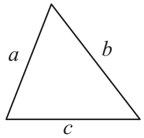
SUMMARY OF MEASUREMENT FACTS

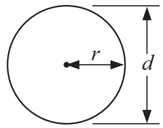
PERIMETER FORMULAE

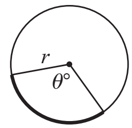
The distance around a closed figure is its **perimeter**.

For some shapes we can derive a formula for perimeter. The formulae for the most common shapes are given below:





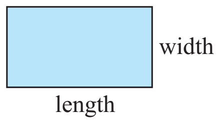
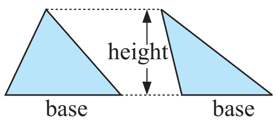
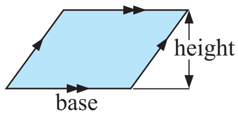
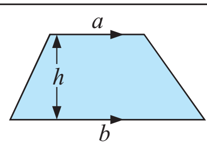
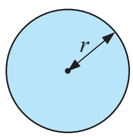
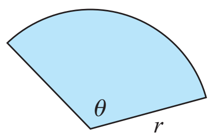




square	rectangle	triangle	circle	arc
$P = 4l$	$P = 2(l + w)$	$P = a + b + c$	$C = 2\pi r$ or $C = \pi d$	$l = \left(\frac{\theta}{360}\right) 2\pi r$

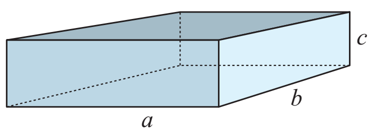
The length of an arc is a fraction of the circumference of a circle.

AREA FORMULAE

Shape	*Figure*	*Formula*
Rectangle		**Area = length × width**
Triangle		**Area =** $\frac{1}{2}$**base × height**
Parallelogram		**Area = base × height**
Trapezium or **Trapezoid**		**Area =** $\left(\frac{a+b}{2}\right) \times h$
Circle		**Area =** πr^2
Sector		**Area =** $\left(\frac{\theta}{360}\right) \times \pi r^2$

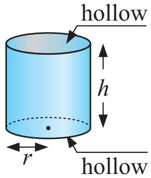
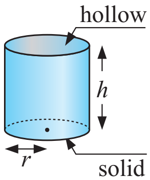
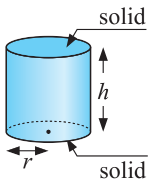
SURFACE AREA FORMULAE

RECTANGULAR PRISM

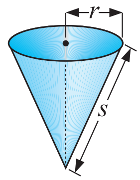
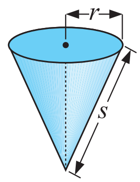


$$A = 2(ab + bc + ac)$$

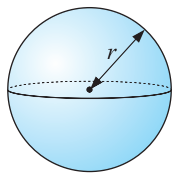
CYLINDER

<table>
<tr><th>Object</th><th>Outer surface area</th></tr>
<tr><td>Hollow cylinder

</td><td>$A = 2\pi rh$
(no ends)</td></tr>
<tr><td>Open can

</td><td>$A = 2\pi rh + \pi r^2$
(one end)</td></tr>
<tr><td>Solid cylinder

</td><td>$A = 2\pi rh + 2\pi r^2$
(two ends)</td></tr>
</table>

CONE

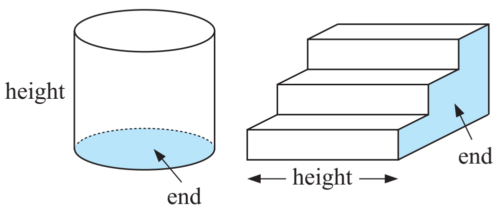
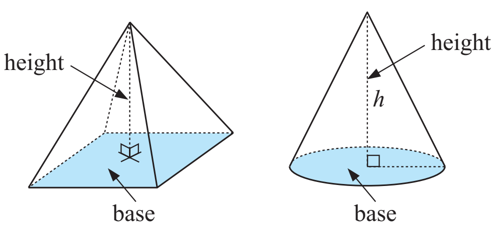
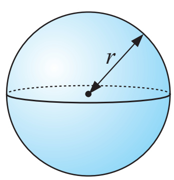
<table>
<tr><th>Object</th><th>Outer surface area</th></tr>
<tr><td>Open cone

</td><td>$A = \pi rs$
(no base)</td></tr>
<tr><td>Solid cone

</td><td>$A = \pi rs + \pi r^2$
(solid)</td></tr>
</table>

SPHERE



Area,
$A = 4\pi r^2$

VOLUME FORMULAE

<table>
<tr><th>Object</th><th>Figure</th><th>Volume</th></tr>
<tr><td>Solids of uniform cross-section</td><td>
</td><td>Volume of uniform solid
$=$ area of end $\times$ length</td></tr>
<tr><td>Pyramids and cones</td><td>
</td><td>Volume of a pyramid or cone
$= \frac{1}{3}$(area of base $\times$ height)</td></tr>
<tr><td>Spheres</td><td>
</td><td>Volume of a sphere
$= \frac{4}{3}\pi r^3$</td></tr>
</table>

Chapter 1

Functions

Contents:

A Relations and functions

B Interval notation, domain and range

C Function notation

Investigation: Fluid filling functions

D Composite functions, $f \circ g$

E The reciprocal function $x \mapsto \frac{1}{x}$

F Inverse functions

G The identity function

Review set 1A

Review set 1B

A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an Airport are given in the table shown alongside.

There is an obvious relationship between time spent and the cost. The cost is dependent on the length of time the car is parked.

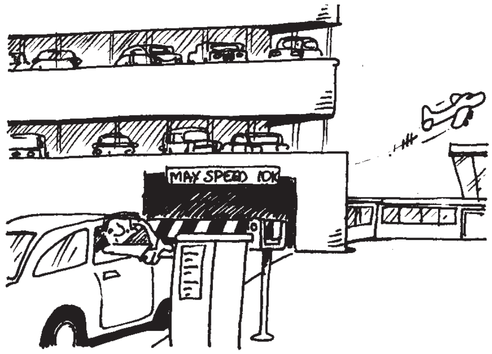
Looking at this table we might ask: How much would be charged for exactly one hour? Would it be \$5 or \$9?

To make the situation clear, and to avoid confusion, we could adjust the table and draw a graph. We need to indicate that 2-3 hours really means for time over 2 hours up to and including 3 hours i.e., $2 < t \leqslant 3$.

Car park charges	
Period (h)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00

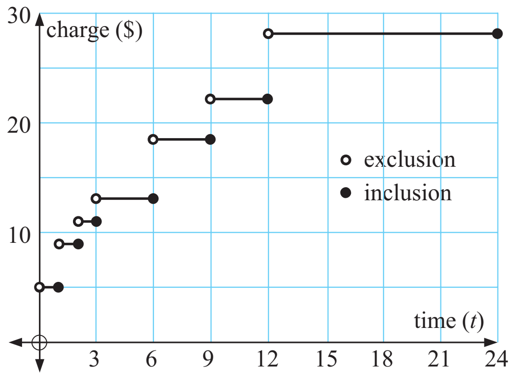
So, we now have

Car park charges	
Period	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, because we have a relationship between two variables, time and cost, the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ or an infinite number of ordered pairs.



The parking charges example is clearly the latter as any real value of time (t hours) in the interval $0 < t \leqslant 24$ is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- $\{t: \ 0 < t \leqslant 24\}$ is the domain for the car park relation
- $\{-2, 1, 4\}$ is the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$.

The set which describes the possible y-values is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

RELATIONS

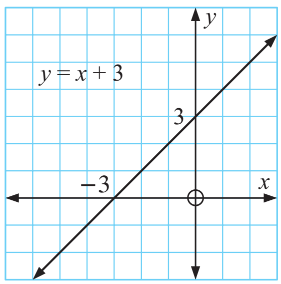
A **relation** is any set of points on the Cartesian plane.

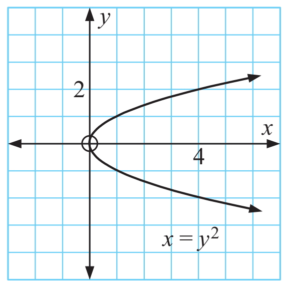
A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

For example $y = x + 3$ and $x = y^2$ are the equations of two relations.

These equations generate sets of ordered pairs.

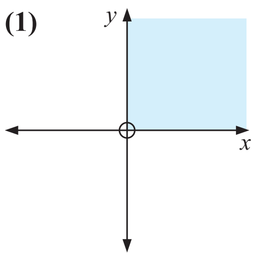
Their graphs are:





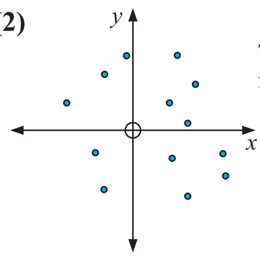
However, a relation may not be able to be defined by an equation. Below are two examples which show this:

(1)



All points in the first quadrant are a relation.
$x > 0$, $y > 0$

(2)



These 13 points form a relation.

FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same x-coordinate (first member).

We can see from the above definition that a function is a special type of relation.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, we have a function.

For example:
- $y = 3x - 1$ is a function, as for any value of x there is only one value of y
- $x = y^2$ is not a function since if $x = 4$, say, then $y = \pm 2$.

Geometric Test ("Vertical Line Test"):

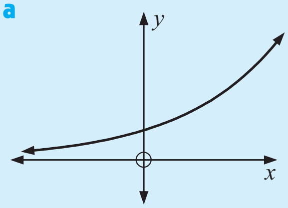
If we draw all possible vertical lines on the graph of a relation, the relation:
- is a function if each line cuts the graph no more than once
- is not a function if one line cuts the graph more than once.

DEMO

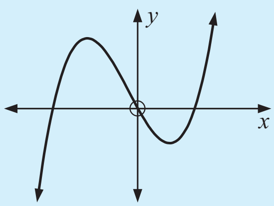
Example 1

Which of the following relations are functions?

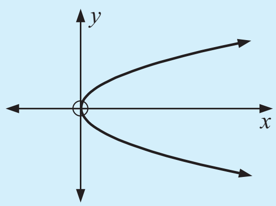
a



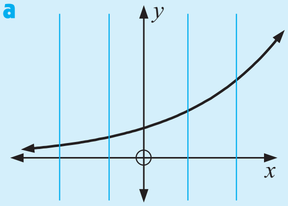
b



c

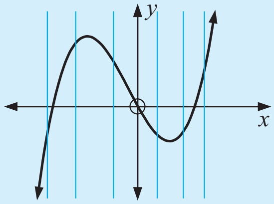


a



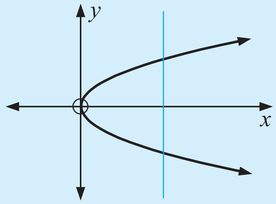
a function

b



a function

c



not a function

GRAPHICAL NOTE

- If a graph contains a small **open circle** end point such as , the end point is **not included**.
- If a graph contains a small **filled-in circle** end point such as , the end point **is included**.
- If a graph contains an **arrow head** at an end such as then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 1A

1 Which of the following sets of ordered pairs are functions? Give reasons.

a (1, 3), (2, 4), (3, 5), (4, 6)

b (1, 3), (3, 2), (1, 7), (−1, 4)

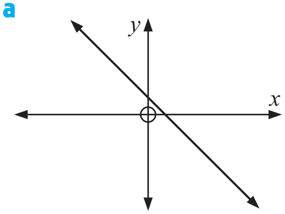
c (2, −1), (2, 0), (2, 3), (2, 11)

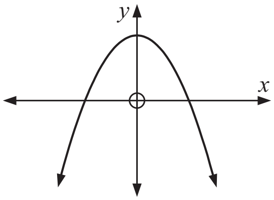
d (7, 6), (5, 6), (3, 6), (−4, 6)

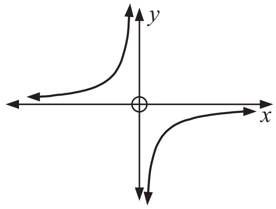
e (0, 0), (1, 0), (3, 0), (5, 0)

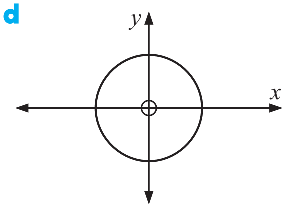
f (0, 0), (0, −2), (0, 2), (0, 4)

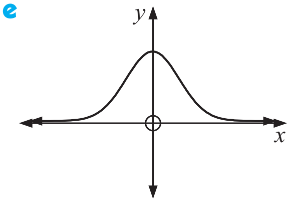
2 Use the vertical line test to determine which of the following relations are functions:

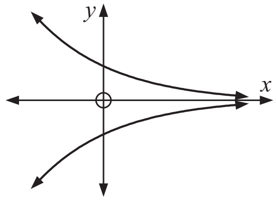
a


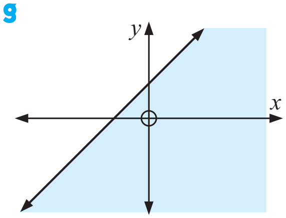
b


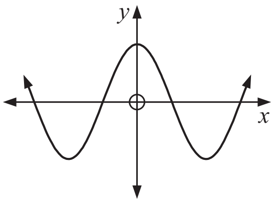
c


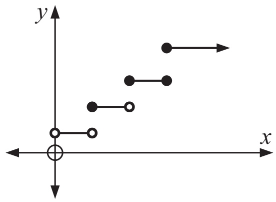
d


e


f


g


h


i


3 Will the graph of a straight line always be a function? Give evidence.

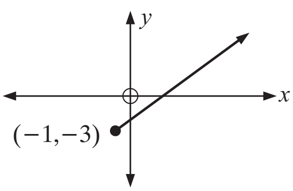
4 Give algebraic evidence to show that the relation $x^2 + y^2 = 9$ is not a function.

B INTERVAL NOTATION, DOMAIN AND RANGE

DOMAIN AND RANGE

The **domain** of a relation is the set of permissible values that x may have.
The **range** of a relation is the set of permissible values that y may have.

For example:

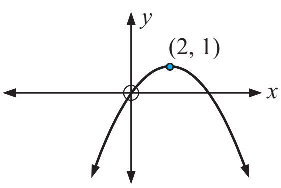
(1)


All values of $x \geqslant -1$ are permissible.

So, the domain is $\{x: x \geqslant -1\}$.

All values of $y \geqslant -3$ are permissible.

So, the range is $\{y: y \geqslant -3\}$.

(2)


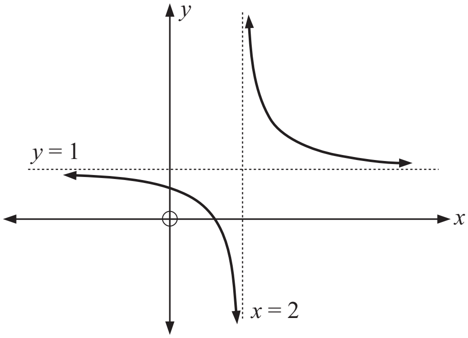
x can take any value.

So, the domain is $\{x: x \text{ is in } \mathcal{R}\}$.

y cannot be > 1

$\therefore$ range is $\{y: y \leqslant 1\}$.

(3)



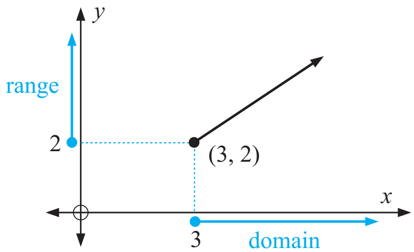
x can take all values except $x = 2$.

So, the domain is $\{x: x \neq 2\}$.

Likewise, the range is $\{y: y \neq 1\}$.

The domain and range of a relation are best described where appropriate using **interval notation**.

For example:

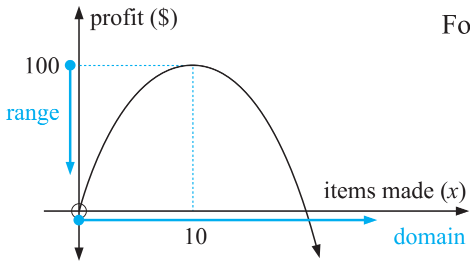


The domain consists of all real x such that $x \geqslant 3$ and we write this as

$$\{x : x \geqslant 3\}$$

the set of all   such that

Likewise the range would be $\{y: y \geqslant 2\}$.



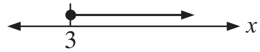
For this profit function:

- the domain is $\{x: x \geqslant 0\}$
- the range is $\{y: y \leqslant 100\}$.

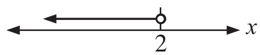
Intervals have corresponding graphs.

For example:

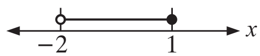
$\{x: x \geqslant 3\}$ or $[3, \infty[$ is read "the set of all x such that x is greater than or equal to 3" and has number line graph



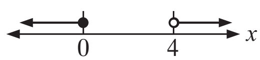
$\{x: x < 2\}$ or $]-\infty, 2[$ has number line graph



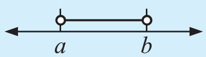
$\{x: -2 < x \leqslant 1\}$ or $]-2, 1]$ has number line graph



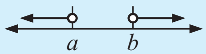
$\{x: x \leqslant 0$ or $x > 4\}$
i.e., $]-\infty, 0]$ or $]4, \infty[$ has number line graph



Note:



for numbers *between* a and b we write $a < x < b$ or $]a, b[$.

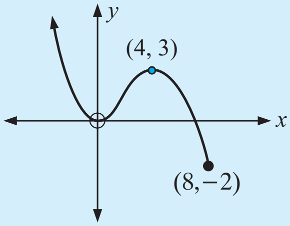


for numbers '*outside*' a and b we write $x < a$ or $x > b$
i.e., $]-\infty, a[$ or $]b, \infty[$.

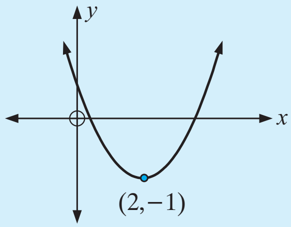
Example 2

For each of the following graphs state the domain and range:

a



b



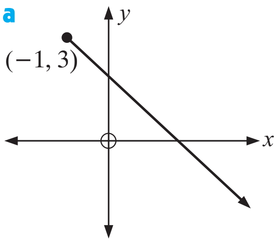
a Domain is $\{x: x \leqslant 8\}$.
Range is $\{y: y \geqslant -2\}$.

b Domain is $\{x: x \text{ is in } \mathcal{R}\}$.
Range is $\{y: y \geqslant -1\}$.

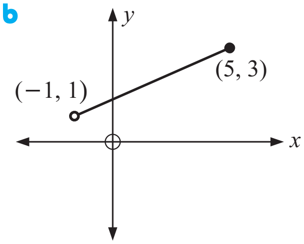
EXERCISE 1B

1 For each of the following graphs find the domain and range:

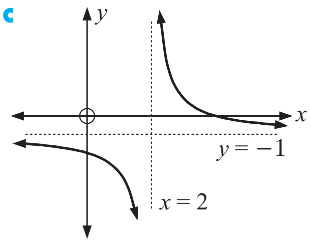
a



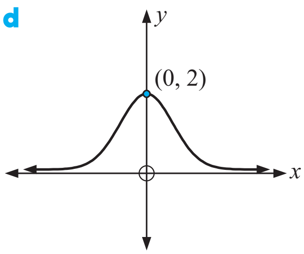
b



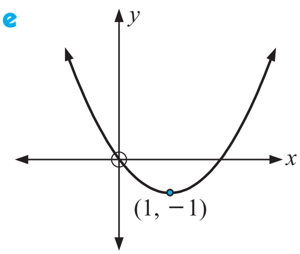
c



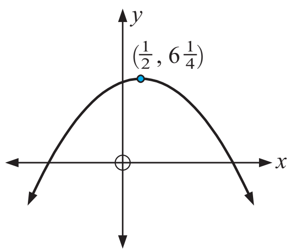
d



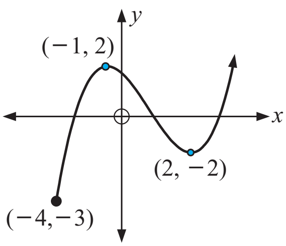
e



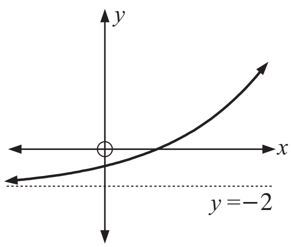
f



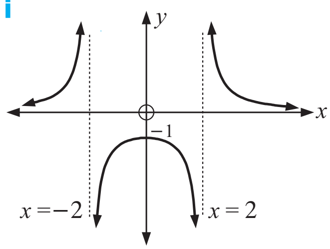
g



h



i



2 Use a graphics calculator to help sketch carefully the graphs of the following functions and find the domain and range of each:

a $f(x) = \sqrt{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \sqrt{4-x}$

d $y = x^2 - 7x + 10$ **e** $y = 5x - 3x^2$ **f** $y = x + \dfrac{1}{x}$

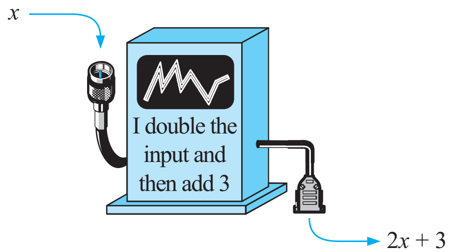
g $y = \dfrac{x+4}{x-2}$ **h** $y = x^3 - 3x^2 - 9x + 10$ **i** $y = \dfrac{3x-9}{x^2-x-2}$

j $y = x^2 + x^{-2}$ **k** $y = x^3 + \dfrac{1}{x^3}$ **l** $y = x^4 + 4x^3 - 16x + 3$

C FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.

For example:



So, if 4 is fed into the machine, $2(4) + 3 = 11$ comes out.

The above 'machine' has been programmed to perform a particular function.
If f is used to represent that particular function we can write:

f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

This function can be written as:

$$\boldsymbol{f : \underbrace{x \longmapsto 2x + 3}}$$

function f such that x is converted into $2x + 3$

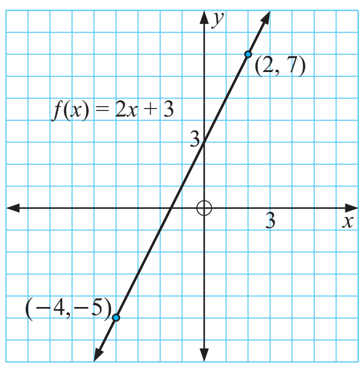
Two other equivalent forms we use are: $f(x) = 2x + 3$ or $y = 2x + 3$

So, $f(x)$ is the value of y for a given value of x, i.e., $y = f(x)$.

Notice that for $f(x) = 2x + 3$, $f(2) = 2(2) + 3 = 7$ and
$f(-4) = 2(-4) + 3 = -5$.

Consequently, $f(2) = 7$ indicates that the point $(2, 7)$ lies on the graph of the function.

Likewise $f(-4) = -5$ indicates that the point $(-4, -5)$ also lies on the graph.



Note:
- $f(x)$ is read as "f of x" and is the value of the function at any value of x.
- If (x, y) is any point on the graph then $y = f(x)$.
- f is the function which converts x into $f(x)$, i.e., $f : x \longmapsto f(x)$.
- $f(x)$ is sometimes called the **image** of x.

Example 3

If $f : x \longmapsto 2x^2 - 3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x by (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x by (-4)}
$= 2(16) + 12$
$= 44$

EXERCISE 1C

1 If $f : x \longmapsto 3x + 2$, find the value of:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $g : x \longmapsto x - \dfrac{4}{x}$, find the value of:

a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 If $f : x \longmapsto 3x - x^2 + 2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

Example 4

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x by $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x by $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$

6 If $G(x) = \dfrac{2x+3}{x-4}$:

a evaluate **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b find a value of x where $G(x)$ does not exist

c find $G(x+2)$ in simplest form

d find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 If $f(x) = 2^x$, show that $f(a)f(b) = f(a+b)$.

9 Given $f(x) = x^2$ find in simplest form:

a $\dfrac{f(x) - f(3)}{x-3}$ **b** $\dfrac{f(2+h) - f(2)}{h}$

10 If the value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ dollars:

a find $V(4)$ and state what $V(4)$ means

b find t when $V(t) = 5780$ and explain what this represents

c find the original purchase price of the photocopier.

11 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

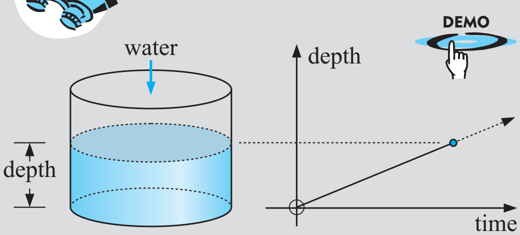
12 Find $f(x) = ax + b$, a linear function, in which $f(2) = 1$ and $f(-3) = 11$.

13 Find constants a and b where $f(x) = ax + \dfrac{b}{x}$ and $f(1) = 1$, $f(2) = 5$.

14 Given $T(x) = ax^2 + bx + c$, find a, b and c if $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$.

INVESTIGATION FLUID FILLING FUNCTIONS

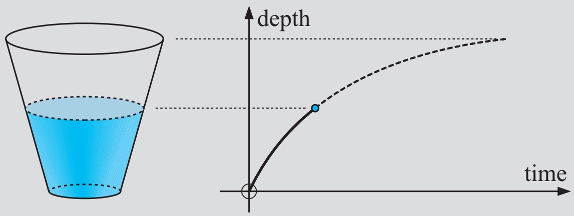
When water is added at a **constant rate** to a cylindrical container the depth of water in the container is a function of time.



This is because the volume of water added is directly proportional to the time taken to add it. If water was not added at a constant rate the direct proportionality would not exist.

The depth-time graph for the case of a cylinder would be as shown alongside.

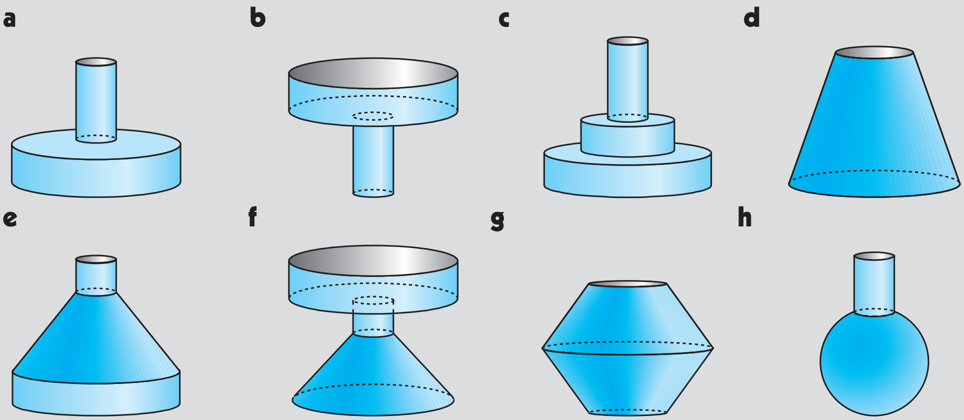
The question arises: ‘What changes in appearance of the graph occur for different shaped containers?’ Consider a vase of conical shape.



DEMO

What to do:

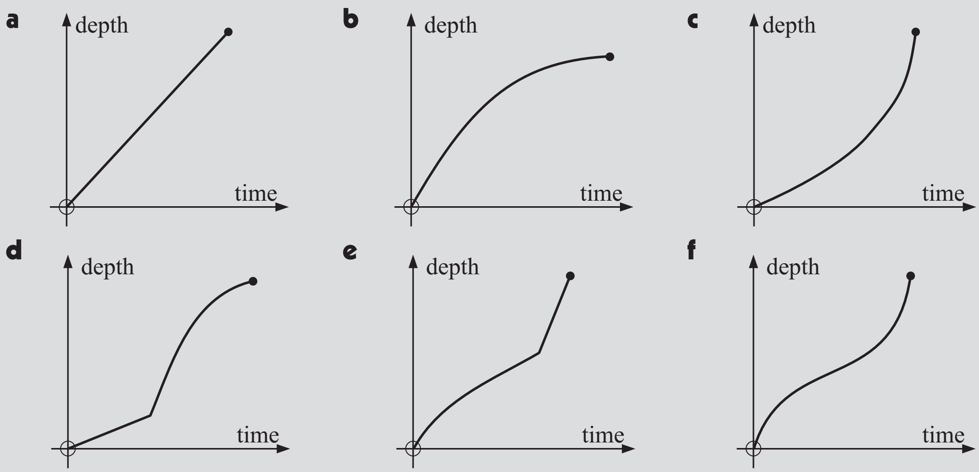
1 For each of the following containers, draw a ‘depth v time’ graph as water is added:



2 Use the water filling demonstration to check your answers to question **1**.

3 Write a brief report on the connection between the shape of a vessel and the corresponding shape of its depth-time graph. You may wish to discuss this in parts. For example, first examine cylindrical containers, then conical, then other shapes. Slopes of curves must be included in your report.

4 Draw possible containers as in question **1** which have the following ‘depth v time’ graphs:



D COMPOSITE FUNCTIONS, $f \circ g$

Given $f: x \mapsto f(x)$ and $g: x \mapsto g(x)$, then the **composite function** of f and g will convert x into $f(g(x))$.

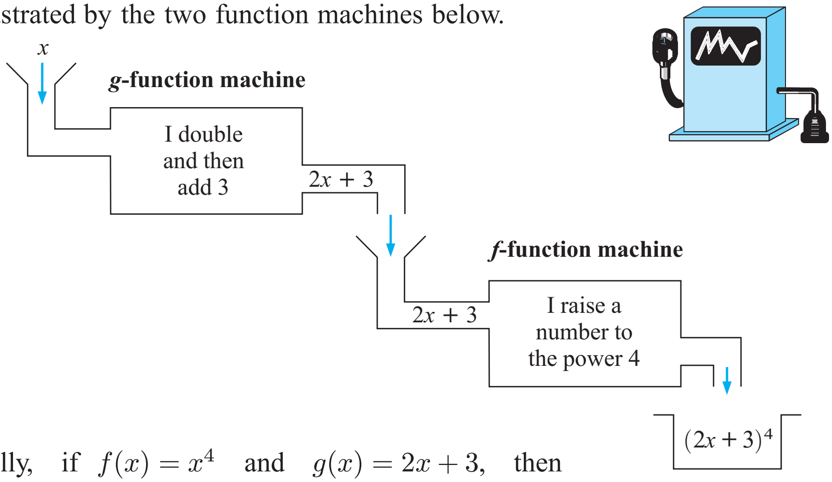
$f \circ g$ is used to represent the composite function of f and g.
$f \circ g$ means f following g and $(f \circ g)(x) = f(g(x))$, i.e., $f \circ g: x \mapsto f(g(x))$.

Consider $f: x \mapsto x^4$ and $g: x \mapsto 2x+3$.

$f \circ g$ means that g converts x to $2x+3$ and then
f converts $(2x+3)$ to $(2x+3)^4$.

DEMO

This is illustrated by the two function machines below.



Algebraically, if $f(x) = x^4$ and $g(x) = 2x+3$, then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) \quad \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 \quad \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

Likewise,
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) \quad \{f \text{ operates on } x \text{ first}\} \\ &= 2(x^4)+3 \quad \{g \text{ operates on } f(x) \text{ next}\} \\ &= 2x^4+3\end{aligned}$$

So, in general, $f(g(x)) \neq g(f(x))$.

The ability to break down functions into composite functions is useful in **differential calculus**.

Example 5

Given $f: x \mapsto 2x+1$ and $g: x \mapsto 3-4x$ find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

a $\therefore \quad (f \circ g)(x) = f(g(x))$
$= f(3 - 4x)$
$= 2(3 - 4x) + 1$
$= 6 - 8x + 1$
$= 7 - 8x$

b $(g \circ f)(x) = g(f(x))$
$= g(2x + 1)$
$= 3 - 4(2x + 1)$
$= 3 - 8x - 4$
$= -8x - 1$

Note: If $f(x) = 2x + 1$ then $f(\Delta) = 2(\Delta) + 1$
$f(*) = 2(*) + 1$
and $f(3 - 4x) = 2(3 - 4x) + 1$

EXERCISE 1D

1 Given $f : x \mapsto 2x + 3$ and $g : x \mapsto 1 - x$, find in simplest form:
a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f : x \mapsto x^2$ and $g : x \mapsto 2 - x$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

3 Given $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$, find in simplest form:
a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** x if $(g \circ f)(x) = f(x)$

4 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
(**Hint:** If it is true for all x, it is true for $x = 0$ and $x = 1$.)

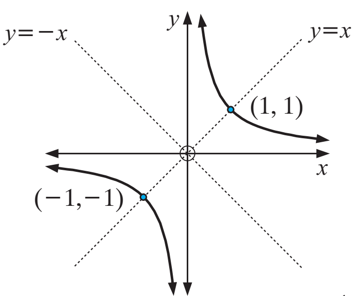
b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

E THE RECIPROCAL FUNCTION $x \mapsto \frac{1}{x}$

$x \mapsto \frac{1}{x}$, i.e., $f(x) = \frac{1}{x}$ is defined as the **reciprocal function**.

It has graph:



The two branches of $y = \frac{1}{x}$

Notice that:

- $f(x) = \frac{1}{x}$ is meaningless when $x = 0$
- The graph of $f(x) = \frac{1}{x}$ exists in the first and third quadrants only.
- $f(x) = \frac{1}{x}$ is symmetric about $y = x$ and $y = -x$

GRAPHING PACKAGE

- $f(x) = \dfrac{1}{x}$ is **asymptotic** (approaches) to the x-axis and to the y-axis.

- as $x \to \infty$, $f(x) \to 0$ (above)
  as $x \to -\infty$, $f(x) \to 0$ (below)
  as $y \to \infty$, $x \to 0$ (right)
  as $y \to -\infty$, $x \to 0$ (left)
  $\to$ reads *approaches* or *tends to*

EXERCISE 1E

1 Sketch the graph of $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{2}{x}$, $h(x) = \dfrac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

2 Sketch the graphs of $f(x) = -\dfrac{1}{x}$, $g(x) = -\dfrac{2}{x}$, $h(x) = -\dfrac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

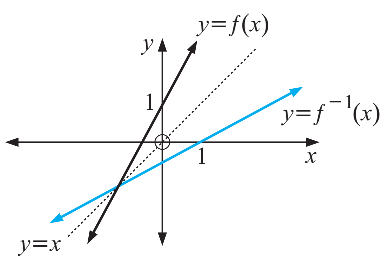
F INVERSE FUNCTIONS

A function $y = f(x)$ *may or may not* have an inverse function.

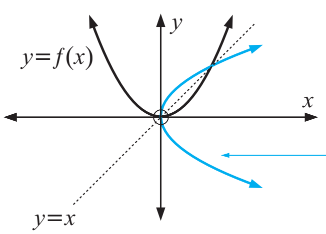
If $y = f(x)$ has an **inverse function**, this new function
- must indeed be a function, i.e., satisfy the vertical line test and it
- must be the reflection of $y = f(x)$ in the line $y = x$.

The inverse function of $y = f(x)$ is denoted by $y = f^{-1}(x)$.
If (x, y) lies on f, then (y, x) lies on f^{-1}. So reflecting the function in $y = x$ has the algebraic effect of interchanging x and y,
e.g., $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$.

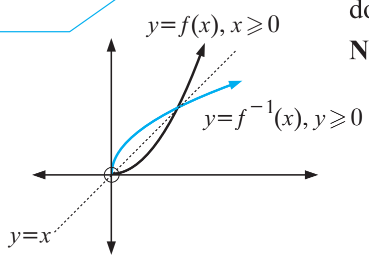
For example,



$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as
- it is also a function
- it is the reflection of $y = f(x)$ in the oblique line $y = x$.



This is the reflection of $y = f(x)$ in $y = x$, but it is not the inverse function of $y = f(x)$ as it fails the vertical line test.
We say that the function $y = f(x)$ does not have an inverse.



Note: $y = f(x)$ subject to $x \geqslant 0$ does have an inverse function.

Also, $y = f(x)$ subject to $x \leqslant 0$ does have an inverse function.

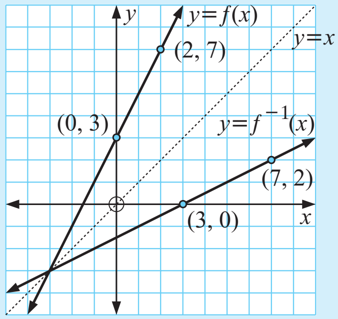
Example 6

Consider $f: x \longmapsto 2x+3$.

a On the same axes, graph f and its inverse function f^{-1}.

b Find $f^{-1}(x)$ using **i** coordinate geometry and the slope of $f^{-1}(x)$ from **a**
ii variable interchange.

a $f(x)=2x+3$ passes through (0, 3) and (2, 7).
$\therefore$ $f^{-1}(x)$ passes through (3, 0) and (7, 2).



b **i** This line has slope $\dfrac{2-0}{7-3}=\dfrac{1}{2}$.

So, its equation is

$$\frac{y-0}{x-3}=\frac{1}{2}$$

i.e., $y=\dfrac{x-3}{2}$

i.e., $f^{-1}(x)=\dfrac{x-3}{2}$

ii f is $y=2x+3$, so f^{-1} is $x=2y+3$

$\therefore$ $x-3=2y$

$\therefore$ $\dfrac{x-3}{2}=y$ i.e., $f^{-1}(x)=\dfrac{x-3}{2}$

Note: If f includes point (a, b) then f^{-1} includes point (b, a),
i.e., the point obtained by interchanging the coordinates.

EXERCISE 1F

1 Consider $f: x \longmapsto 3x+1$.

a On the same axes graph $y=x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry and **a**.

c Find $f^{-1}(x)$ using variable interchange.

2 Consider $f: x \longmapsto \dfrac{x+2}{4}$.

a On the same set of axes graph $y=x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry and **a**.

c Find $f^{-1}(x)$ using variable interchange.

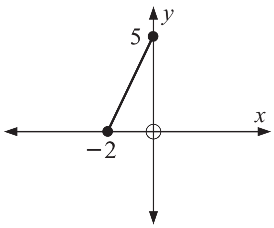
3 For each of the following functions f

i find $f^{-1}(x)$ **ii** sketch $y=f(x)$, $y=f^{-1}(x)$ and $y=x$ on the same axes:

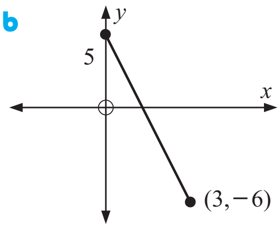
a $f: x \longmapsto 2x+5$ **b** $f: x \longmapsto \dfrac{3-2x}{4}$ **c** $f: x \longmapsto x+3$

4 Copy the graphs of the following functions and in each case include the graphs of $y = x$ and $y = f^{-1}(x)$.

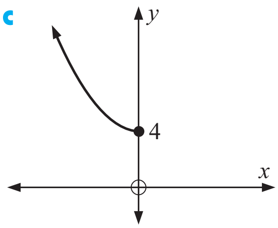
a



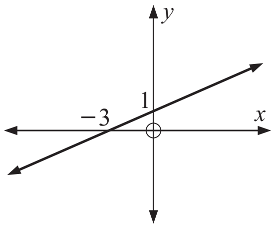
b



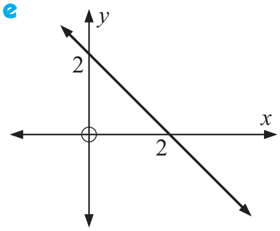
c



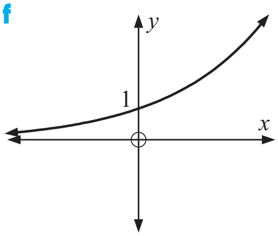
d



e



f



5 **a** Sketch the graph of $f : x \longmapsto x^2 - 4$ and reflect it in the line $y = x$.

b Does f have an inverse function?

c Does f where $x \geqslant 0$ have an inverse function?

6 Sketch the graph of $f : x \longmapsto x^3$ and its inverse function $f^{-1}(x)$.

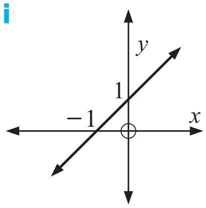
7 The '**horizontal line test**' says that:

for a function to have an inverse function, no horizontal line can cut it more than once.

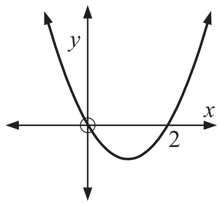
a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

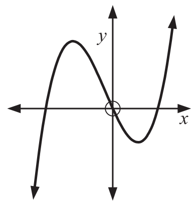
i



ii



iii



Example 7

Consider $f : x \longmapsto x^2$ where $x \geqslant 0$.

a Find $f^{-1}(x)$.

b Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

a f is defined by $y = x^2$, $x \geqslant 0$

$\therefore$ f^{-1} is defined by $x = y^2$, $y \geqslant 0$

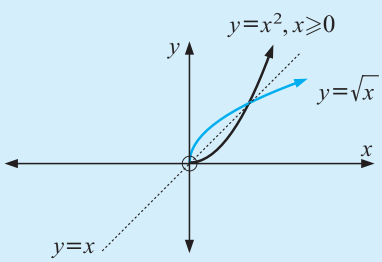
$\therefore$ $y = \pm\sqrt{x}$, $y \geqslant 0$

i.e., $y = \sqrt{x}$

{as $-\sqrt{x}$ is $\leqslant 0$}

So, $f^{-1}(x) = \sqrt{x}$

b



8 Consider $f: x \longmapsto x^2$ where $x \leqslant 0$.

a Find $f^{-1}(x)$.

b Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

9 **a** Explain why $f: x \longmapsto x^2 - 4x + 3$ is a function but does not have an inverse function.

b Explain why f for $x \geqslant 2$ has an inverse function.

c Show that the inverse function of the function in **b** is $f^{-1}(x) = 2 + \sqrt{1 + x}$.

d If the domain of f is restricted to $x \geqslant 2$, state the domain and range of

i f **ii** f^{-1}.

10 Consider $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$.

b Find **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

11 Given $f: x \longmapsto (x+1)^2 + 3$ where $x \geqslant -1$,

a find the defining equation of f^{-1}

b sketch, using technology, the graphs of $y = f(x)$, $y = x$ and $y = f^{-1}(x)$

c state the domain and range of **i** f **ii** f^{-1}.

12 Consider the functions $f: x \longmapsto 2x + 5$ and $g: x \longmapsto \dfrac{8 - x}{2}$.

a Find $g^{-1}(-1)$. **b** Solve for x the equation $(f \circ g^{-1})(x) = 9$.

13 Given $f: x \longmapsto 5^x$ and $g: x \longmapsto \sqrt{x}$,

a find **i** $f(2)$ **ii** $g^{-1}(4)$

b solve the equation $(g^{-1} \circ f)(x) = 25$.

14 Given $f: x \longmapsto 2x$ and $g: x \longmapsto 4x - 3$ show that

$(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

15 Which of these functions are their own inverses, that is $f^{-1}(x) = f(x)$?

a $f(x) = 2x$ **b** $f(x) = x$ **c** $f(x) = -x$ **d** $f(x) = \dfrac{1}{x}$ **e** $f(x) = -\dfrac{6}{x}$

G THE IDENTITY FUNCTION

In question **10** of the previous exercise we considered $f(x) = \frac{1}{2}x - 1$.

We found that $f^{-1}(x) = 2x + 2$ and that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$e(x) = x$ is called the **identity function** of function $y = f(x)$

It is the unique solution of $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = e(x)$.

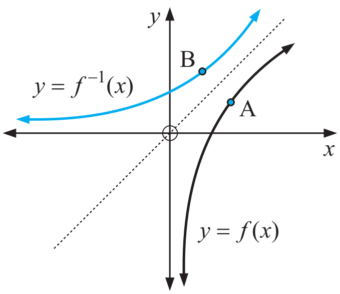
EXERCISE 1G

1 For $f(x) = 3x + 1$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

2 For $f(x) = \dfrac{x+3}{4}$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

3 For $f(x) = \sqrt{x}$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

4



a B is the image of A under a reflection in the line $y = x$.
If A is $(x, f(x))$, what are the coordinates of B under the reflection?

b Substitute your result from **a** into $y = f^{-1}(x)$.
What result do you obtain?

c Explain how to establish that $f(f^{-1}(x)) = x$ also.

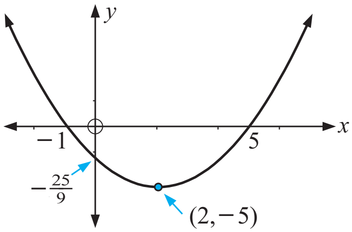
REVIEW SET 1A

1 Draw a graph to show what happens in the following jar-water-golf ball situation:
Water is added to an empty jar at a constant rate for two minutes and then one golf ball is added. After one minute another golf ball is added. Two minutes later both golf balls are removed. Half the water is then removed at a constant rate over a two minute period.

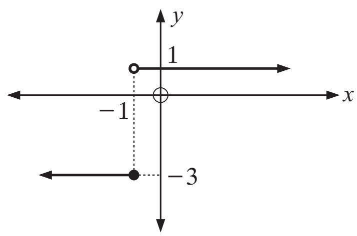
2 If $f(x) = 2x - x^2$ find: **a** $f(2)$ **b** $f(-3)$ **c** $f(-\frac{1}{2})$

3 For the following graphs determine:
i the range and domain **ii** the x and y-intercepts **iii** whether it is a function.

a

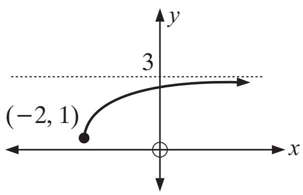


b

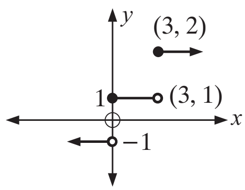


4 For each of the following graphs find the domain and range:

a



b



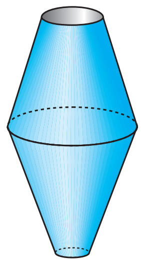
5 If $h(x) = 7 - 3x$:
a find in simplest form $h(2x - 1)$ **b** find x if $h(2x - 1) = -2$

6 If $f(x) = ax + b$ where a and b are constants, find a and b for $f(1) = 7$ and $f(3) = -5$.

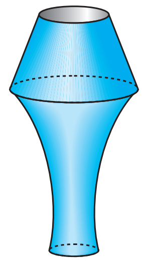
7 Find a, b and c if $f(0)=5$, $f(-2)=21$ and $f(3)=-4$ for $f(x)=ax^2+bx+c$.

8 For each of the following containers draw a 'depth v time' graph as water is added.

a



b



9 Consider $f(x)=\dfrac{1}{x^2}$.

a For what value of x is $f(x)$ meaningless?
b Sketch the graph of this function using technology.
c State the domain and range of the function.

10 If $f(x)=2x-3$ and $g(x)=x^2+2$, find in simplest form:

a $f(g(x))$ **b** $g(f(x))$

11 If $f(x)=1-2x$ and $g(x)=\sqrt{x}$, find in simplest form:

a $(f\circ g)(x)$ **b** $(g\circ f)(x)$

12 Find an f and a g function given that:

a $f(g(x))=\sqrt{1-x^2}$ **b** $g(f(x))=\left(\dfrac{x-2}{x+1}\right)^2$

REVIEW SET 1B

1 If $f(x)=5-2x$, find **a** $f(0)$ **b** $f(5)$ **c** $f(-3)$ **d** $f(\frac{1}{2})$

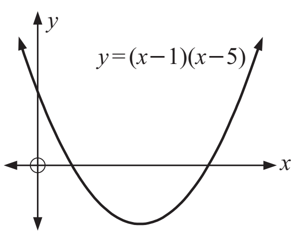
2 If $g(x)=x^2-3x$, find in simplest form **a** $g(x+1)$ **b** $g(x^2-2)$

3 For each of the following functions $f(x)$ find $f^{-1}(x)$:

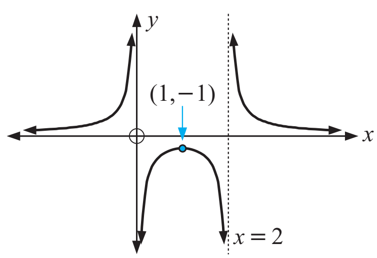
a $f(x)=7-4x$ **b** $f(x)=\dfrac{3+2x}{5}$

4 For each of the following graphs, find the domain and range.

a

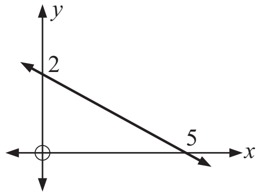


b

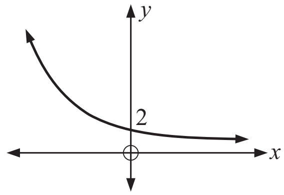


5 Copy the following graphs and draw the graph of each inverse function:

a



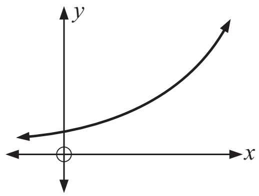
b



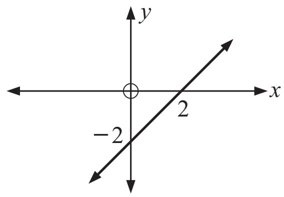
6 Find $f^{-1}(x)$ given that $f(x)$ is: **a** $4x+2$ **b** $\dfrac{3-5x}{4}$

7 Copy the following graphs and draw the graph of each inverse function:

a



b



8 Given $f(x)=2x+11$ and $g(x)=x^2$, find $(g\circ f^{-1})(3)$.

9 Consider $x \longmapsto 2x-7$.

a On the same set of axes graph $y=x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry.

c Find $f^{-1}(x)$ using variable interchange.

10 **a** Sketch the graph of $g: x \longmapsto x^2+6x+7$.

b Explain why g for $x \leqslant -3$ has an inverse function g^{-1}.

c Find algebraically, the equation of g^{-1}.

d Sketch the graph of g^{-1}.

11 Given $h: x \longmapsto (x-4)^2+3$ where $x \geqslant 4$, find the defining equation of h^{-1}.

12 Given $f: x \longmapsto 3x+6$ and $h: x \longmapsto \dfrac{x}{3}$, show that $(f^{-1}\circ h^{-1})(x) = (h\circ f)^{-1}(x)$.

Chapter 2

Sequences and series

Contents:

A Number patterns
B Sequences of numbers
C Arithmetic sequences
D Geometric sequences
E Series
F Sigma notation
Investigation: Von Koch's snowflake curve

Review set 2A
Review set 2B
Review set 2C

A NUMBER PATTERNS

An important skill in mathematics is to be able to

- **recognise** patterns in sets of numbers,
- **describe** the patterns in words, and
- **continue** the patterns.

A list of numbers where there is a pattern is called a **number sequence**.
The members (numbers) of a sequence are said to be its **terms**.

For example, 3, 7, 11, 15, ... form a number sequence.

The first term is 3, the second term is 7, the third term is 11, etc.

We describe this pattern in words:

"*The sequence starts at 3 and each term is 4 more than the previous one.*"

Thus, the fifth term is 19, and the sixth term is 23, etc.

Example 1

Describe the sequence: 14, 17, 20, 23, ... and write down the next two terms.

The sequence starts at 14 and each term is 3 more than the previous term.
The next two terms are 26 and 29.

EXERCISE 2A

1 Write down the first four terms of the sequences described by the following:

a Start with 4 and add 9 each time.
b Start with 45 and subtract 6 each time.
c Start with 2 and multiply by 3 each time.
d Start with 96 and divide by 2 each time.

2 For each of the following write down a description of the sequence and find the next two terms:

a 8, 16, 24, 32, ...
b 2, 5, 8, 11, ...
c 36, 31, 26, 21, ...
d 96, 89, 82, 75, ...
e 1, 4, 16, 64, ...
f 2, 6, 18, 54, ...
g 480, 240, 120, 60, ...
h 243, 81, 27, 9, ...
i 50000, 10000, 2000, 400, ...

3 Find the next two terms of:

a 95, 91, 87, 83, ...
b 5, 20, 80, 320, ...
c 45, 54, 63, 72, ...

4 Describe the following number patterns and write down the next three terms:

a 1, 4, 9, 16, ...
b 1, 8, 27, 64, ...
c 2, 6, 12, 20, ...

[**Hint:** In **c** $2 = 1 \times 2$ and $6 = 2 \times 3$.]

5 Find the next two terms of:

a 1, 16, 81, 256, ...
b 1, 1, 2, 3, 5, 8, ...
c 6, 8, 7, 9, 8, 10, ...
d 2, 3, 5, 7, 11, ...
e 2, 4, 7, 11, ...
f 3, 4, 6, 8, 12, ...

SPREADSHEET "NUMBER PATTERNS"

A spreadsheet is a computer software program that enables you to do calculations, write messages, draw graphs and do "what if" calculations.

This exercise will get you using a spreadsheet to construct and investigate number patterns.

SPREADSHEET

What to do:

1 To form a number pattern with a spreadsheet like "start with 7 and add 4 each time" follow the given steps.

Step 1: Open a new spreadsheet.

Step 2: In cell A1,
type the **label** 'Value'

Step 3: In cell A2,
type the **number** 7

Step 4: In cell A3,
type the **formula** =A2 + 4

	A	B
1	Value	
2	7	
3	=A2+4	

Step 5: Press **ENTER**. Your spreadsheet should look like this:

	A	B
1	Value	
2	7	
3	11	

Step 6: Highlight cell A3 and position your cursor on the right hand bottom corner until it changes to a **+**. Click the left mouse key and "drag" the cursor down to Row 10.
This is called **filling down**.

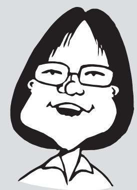
	A	B
1	Value	
2	7	
3	11	
4		

Step 7: You should have generated the first nine members of the number sequence as shown:

	A	B
1	Value	
2	7	
3	11	
4	15	
5	19	
6	23	
7	27	
8	31	
9	35	
10	39	

	A	B	C
1			
2			
3			
4			
5			

A spreadsheet consists of a series of rectangles called **cells** and each cell has a position according to the **column** and **row** it is in. Cell **B2** is shaded.
All formulae start with =



Filling down copies the formula from A3 to A4 and so on.

Step 8: Use the **fill down** process to answer the following questions:

a What is the first member of the sequence greater than 100?

b Is 409 a member of the sequence?

2 Now that you are familiar with a basic spreadsheet, try to generate the first 20 members of the following number patterns:

a Start with 132 and subtract 6 each time. Type:

Value in B1, 132 in B2, =B2 – 6 in B3, then fill down to Row 21.

b Start with 3 and multiply by 2 each time. Type:

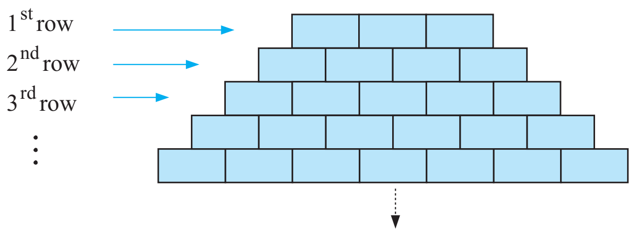
Value in C1, 3 in C2, =C2*2 in C3, then fill down to Row 21.

c Start with 1 000 000 and divide by 5 each time. Type:

Value in D1, 1 000 000 in D2, =D2/5 in D3, then fill down to Row 21.

3 Find out how to use a **graphics calculator** to generate **sequences** of numbers such as those above.

B SEQUENCES OF NUMBERS



Consider the illustrated tower of bricks. The top row, or first row, has three bricks. The second row has four bricks. The third row has five, etc.

If u_n represents the number of bricks in row n (from the top) then

$$u_1 = 3, \quad u_2 = 4, \quad u_3 = 5, \quad u_4 = 6, \quad ......$$

The number pattern: 3, 4, 5, 6, ... is called a **sequence** of numbers.

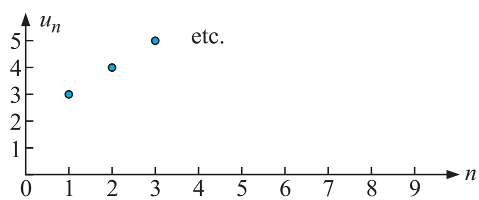
This sequence can be specified by:

- **Using words** — The top row has three bricks and each successive row under it has one more brick.
- **Using an explicit formula** — $u_n = n + 2$ is the **general term** (or **nth term**) formula for $n = 1, 2, 3, 4, 5,$ ...... etc.

Check: $u_1 = 1 + 2 = 3$ ✓
$u_2 = 2 + 2 = 4$ ✓
$u_3 = 3 + 2 = 5$ ✓ etc.

Early members of a sequence can be graphed. Each term is represented by a dot.

The dots *must not* be joined.



ARITHMETIC SEQUENCES

Number patterns like the one above where we **add** (or **subtract**) the same fixed number to get the next number are called **arithmetic sequences**.

Further examples where arithmetic sequence models apply are:

- Simple interest accumulated amounts at the end of each period.

  For example: on a \$1000 investment at 7% simple interest p.a. (per annum) the value of the investment at the end of successive years is:
  \$1000, \$1070, \$1140, \$1210, \$1280, ...

- The amount still owed to a friend when repaying a personal loan with fixed weekly repayments.

  For example: if repaying \$75 each week to repay a \$1000 personal loan the amounts still owing are: \$1000, \$925, \$850, \$775, ...

GEOMETRIC SEQUENCES

Instead of adding (or subtracting) a fixed number to get the next number in a sequence we sometimes **multiply** (or **divide**) by a fixed number.

When we do this we create **geometric sequences**.

Consider investing \$6000 at a fixed rate of 7% p.a. compound interest over a lengthy period. The initial investment of \$6000 is called the principal.

After 1 year, its value is $\$6000 \times 1.07$ {to increase by 7% we multiply to 107%}

After 2 years, its value is $(\$6000 \times 1.07) \times 1.07$
$= \$6000 \times (1.07)^2$

After 3 years, its value is $\$6000 \times (1.07)^3$, etc.

The amounts \$6000, $\$6000 \times 1.07$, $\$6000 \times (1.07)^2$, $\$6000 \times (1.07)^3$, etc. form a geometric sequence where each term is multiplied by 1.07 which is called the **common ratio**.

Once again we can specify the sequence by:

- **Using words** The initial value is \$6000 and after each successive year the increase is 7%.
- **Using an explicit formula** $u_n = 6000 \times (1.07)^{n-1}$ for $n = 1, 2, 3, 4, ...$

  Check: $u_1 = 6000 \times (1.07)^0 = 6000$ ✓
  $u_2 = 6000 \times (1.07)^1$ ✓
  $u_3 = 6000 \times (1.07)^2$ ✓ etc.

Notice that u_n is the amount after $n - 1$ years.

Other examples where geometric models occur are:

- Problems involving depreciation.

  For example: The value of a \$12 000 photocopier may decrease by 20% p.a.
  i.e., \$12 000, $\$12\,000 \times 0.8$, $\$12\,000 \times (0.8)^2$, ... etc.

- In fractals, as we shall see later in the chapter on page **58**.

OPENING PROBLEM

A circular stadium consists of sections as illustrated, with aisles in between. The diagram shows the tiers of concrete steps for the final section, **Section K**. Seats are to be placed along every step, with each seat being 0.45 m wide. AB, the arc at the front of the first row is 14.4 m long, while CD, the arc at the back of the back row is 20.25 m long.

For you to consider:

1 How wide is each concrete step?

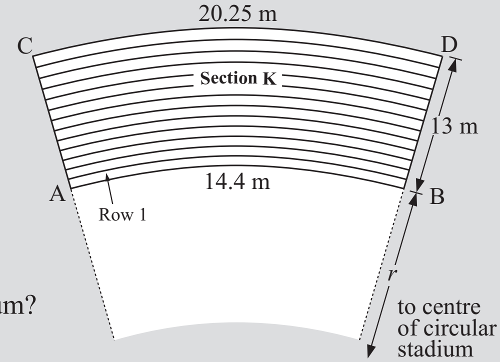
2 What is the length of the arc of the back of Row 1, Row 2, Row 3, etc?

3 How many seats are there in Row 1, Row 2, Row 3, ... Row 13?

4 How many sections are there in the stadium?

5 What is the total seating capacity of the stadium?

6 What is the radius of the 'playing surface'?



To solve problems like the **Opening Problem** and many others, a detailed study of **sequences** and their sums (called **series**) is required.

NUMBER SEQUENCES

A **number sequence** is a set of numbers defined by a rule for positive integers.

Sequences may be defined in one of the following ways:

- by using a formula which represents the **general term** (called the n**th term**)
- by giving a description in words
- by listing the first few terms and assuming that the pattern represented continues indefinitely.

THE GENERAL TERM

u_n, T_n, t_n, A_n, etc. can all be used to represent the **general term** (or n**th term**) of a sequence and are defined for $n = 1, 2, 3, 4, 5, 6, ...$

$\{u_n\}$ represents the sequence that can be generated by using u_n as the n**th term**.

For example, $\{2n+1\}$ generates the sequence $3, 5, 7, 9, 11, ...$

EXERCISE 2B

1 List the first *five* terms of the sequence:

a $\{2n\}$ **b** $\{2n+2\}$ **c** $\{2n-1\}$

d $\{2n-3\}$ **e** $\{2n+3\}$ **f** $\{2n+11\}$

g $\{3n+1\}$ **h** $\{4n-3\}$ **i** $\{5n+4\}$

2 List the first *five* terms of the sequence:

a $\{2^n\}$ **b** $\{3 \times 2^n\}$ **c** $\{6 \times (\frac{1}{2})^n\}$ **d** $\{(-2)^n\}$

3 List the first *five* terms of the sequence $\{15 - (-2)^n\}$.

C ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number.

For example: 2, 5, 8, 11, 14, ... is arithmetic as $5 - 2 = 8 - 5 = 11 - 8 = 14 - 11$, etc.

Likewise, 31, 27, 23, 19, ... is arithmetic as $27 - 31 = 23 - 27 = 19 - 23$, etc.

ALGEBRAIC DEFINITION

$\boldsymbol{\{u_n\}}$ is **arithmetic** $\Leftrightarrow$ $\boldsymbol{u_{n+1} - u_n = d}$ for all positive integers $\boldsymbol{n}$ where $\boldsymbol{d}$ is a constant (the **common difference**).

Note:
- $\Leftrightarrow$ is read as 'if and only if'
- If $\{u_n\}$ is arithmetic then $u_{n+1} - u_n$ is a constant *and* if $u_{n+1} - u_n$ is a constant then $\{u_n\}$ is arithmetic.

THE 'ARITHMETIC' NAME

If a, b and c are any consecutive terms of an arithmetic sequence then

$$b - a = c - b \quad \{\text{equating common differences}\}$$
$$\therefore \quad 2b = a + c$$
$$\therefore \quad b = \frac{a+c}{2}$$

i.e., middle term = arithmetic mean (average) of terms on each side of it.
Hence the name *arithmetic sequence*.

THE GENERAL TERM FORMULA

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$ $\therefore$ $u_3 = u_1 + 2d$ $\therefore$ $u_4 = u_1 + 3d$ etc.

then, $u_n = u_1 + \underbrace{(n-1)}d$

The coefficient of d is one less than the subscript.

So, for an **arithmetic sequence** with **first term** $\boldsymbol{u_1}$ and **common difference** $\boldsymbol{d}$ the **general term** (or nth **term**) is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 2

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find the formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a member of the sequence?

a
$$\begin{aligned} 9-2&=7\\ 16-9&=7\\ 23-16&=7\\ 30-23&=7 \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 7

$\therefore$ the sequence is arithmetic with $u_1 = 2$, $d = 7$.

b $u_n = u_1 + (n-1)d$ $\therefore$ $u_n = 2 + 7(n-1)$ i.e., $u_n = 7n - 5$

c If $n = 100$, $u_{100} = 7(100) - 5 = 695$.

d **i** Let $u_n = 828$

$\therefore$ $7n - 5 = 828$

$\therefore$ $7n = 833$

$\therefore$ $n = 119$

$\therefore$ 828 is a term of the sequence.

In fact it is the 119th term.

ii Let $u_n = 2341$

$\therefore$ $7n - 5 = 2341$

$\therefore$ $7n = 2346$

$\therefore$ $n = 335\frac{1}{7}$

which is not possible as n is an integer. $\therefore$ 2341 cannot be a term.

EXERCISE 2C

1 Consider the sequence 6, 17, 28, 39, 50, ...

a Show that the sequence is arithmetic.
b Find the formula for its general term.
c Find its 50th term.
d Is 325 a member?
e Is 761 a member?

2 Consider the sequence 87, 83, 79, 75, ...

a Show that the sequence is arithmetic.
b Find the formula for the general term.
c Find the 40th term.
d Is -143 a member?

3 A sequence is defined by $u_n = 3n - 2$.

a Prove that the sequence is arithmetic. (**Hint:** Find $u_{n+1} - u_n$.)
b Find u_1 and d.
c Find the 57th term.
d What is the least term of the sequence which is greater than 450?

4 A sequence is defined by $u_n = \dfrac{71 - 7n}{2}$.

a Prove that the sequence is arithmetic.
b Find u_1 and d.
c Find u_{75}.
d For what values of n are the terms of the sequence less than -200?

Example 3

Find k given that $3k+1$, k and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive,

$$\begin{aligned} k-(3k+1) &= -3-k \qquad \{\text{equating common differences}\} \\ \therefore\ k-3k-1 &= -3-k \\ \therefore\ -2k-1 &= -3-k \\ \therefore\ -1+3 &= -k+2k \\ \therefore\ 2 &= k \end{aligned}$$

or $$k = \frac{(3k+1)+(-3)}{2} \qquad \{\text{middle term is average of other two}\}$$

$$\therefore\ k = \frac{3k-2}{2}$$ which when solved gives $k=2$.

5 Find k given the consecutive arithmetic terms:

a $32,\ k,\ 3$ **b** $k,\ 7,\ 10$ **c** $k+1,\ 2k+1,\ 13$

d $k-1,\ 2k+3,\ 7-k$ **e** $k,\ k^2,\ k^2+6$ **f** $5,\ k,\ k^2-8$

Example 4

Find the general term u_n for an arithmetic sequence given that $u_3 = 8$ and $u_8 = -17$.

$u_3 = 8 \quad \therefore\ u_1 + 2d = 8 \quad ....(1) \qquad \{u_n = u_1 + (n-1)d\}$

$u_8 = -17 \quad \therefore\ u_1 + 7d = -17 \quad ....(2)$

We now solve (1) and (2) simultaneously

$$\begin{aligned} -u_1 - 2d &= -8 \\ u_1 + 7d &= -17 \\ \hline \therefore\ 5d &= -25 \qquad \{\text{adding the equations}\} \\ \therefore\ d &= -5 \end{aligned}$$

So in (1)
$$\begin{aligned} u_1 + 2(-5) &= 8 \\ \therefore\ u_1 - 10 &= 8 \\ \therefore\ u_1 &= 18 \end{aligned}$$

Now
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 18 - 5(n-1) \\ \therefore\ u_n &= 18 - 5n + 5 \\ \therefore\ u_n &= 23 - 5n \end{aligned}$$

Check:
$$\begin{aligned} u_3 &= 23 - 5(3) \\ &= 23 - 15 \\ &= 8 \ \checkmark \end{aligned}$$
$$\begin{aligned} u_8 &= 23 - 5(8) \\ &= 23 - 40 \\ &= -17 \ \checkmark \end{aligned}$$

6 Find the general term u_n for an arithmetic sequence given that:

a $u_7 = 41$ and $u_{13} = 77$ **b** $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$

c the seventh term is 1 and the fifteenth term is -39

d the eleventh and eighth terms are -16 and $-11\frac{1}{2}$ respectively.

Example 5

Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

If the numbers are 3, $3+d$, $3+2d$, $3+3d$, $3+4d$, 12

$$\text{then} \quad 3+5d=12$$
$$\therefore \quad 5d=9$$
$$\therefore \quad d=\tfrac{9}{5}=1.8$$

So we have 3, 4.8, 6.6, 8.4, 10.2, 12.

7 **a** Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

b Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

8 Consider the finite arithmetic sequence $36, 35\frac{1}{3}, 34\frac{2}{3}, ..., -30$.

a Find u_1 and d. **b** How many terms does the sequence have?

9 An arithmetic sequence starts 23, 36, 49, 62, ... What is the first term of the sequence to exceed 100 000?

D GEOMETRIC SEQUENCES

A sequence is **geometric** if each term can be obtained from the previous one by multiplying by the same non-zero constant.

For example: 2, 10, 50, 250, ... is a geometric sequence as

$2\times 5=10$ and $10\times 5=50$ and $50\times 5=250$.

Notice that $\frac{10}{2}=\frac{50}{10}=\frac{250}{50}=5$, i.e., each term divided by the previous one is constant.

Algebraic definition:

$\{u_n\}$ is **geometric** $\Leftrightarrow$ $\dfrac{u_{n+1}}{u_n}=r$ for all positive integers n

where r is a **constant** (the **common ratio**).

Notice:
- 2, 10, 50, 250, ... is geometric with $r=5$.
- 2, -10, 50, -250, ... is geometric with $r=-5$.

THE 'GEOMETRIC NAME'

If a, b and c are any consecutive terms of a geometric sequence then

$$\frac{b}{a} = \frac{c}{b} \qquad \{\text{equating common ratios}\}$$

$\therefore \quad b^2 = ac$ and so $b = \pm\sqrt{ac}$ where $\sqrt{ac}$ is the **geometric mean** of a and c.

THE GENERAL TERM

Suppose the first term of a geometric sequence is u_1 and the common ratio is r.

Then $u_2 = u_1 r \quad \therefore \quad u_3 = u_1 r^2 \quad \therefore \quad u_4 = u_1 r^3$ etc.

then $u_n = u_1 r^{\overbrace{n-1}}$

The power of r is one less than the subscript.

So, for a **geometric sequence** with **first term u_1** and **common ratio r**, the **general term** (or **nth term**) is $\boldsymbol{u_n = u_1 r^{n-1}}$.

Example 6

For the sequence $8, 4, 2, 1, \frac{1}{2}, \ldots\ldots$

a Show that the sequence is geometric.

b Find the general term u_n.

c Hence, find the 12th term as a fraction.

a $\frac{4}{8} = \frac{1}{2}$, $\frac{2}{4} = \frac{1}{2}$, $\frac{1}{2} = \frac{1}{2}$, $\frac{\frac{1}{2}}{1} = \frac{1}{2}$

So, assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1} \quad \therefore \quad u_n = 8\left(\frac{1}{2}\right)^{n-1}$ *or*

$$\begin{aligned} u_n &= 2^3 \times (2^{-1})^{n-1} \\ &= 2^3 \times 2^{-n+1} \\ &= 2^{3+(-n+1)} \\ &= 2^{4-n} \end{aligned}$$

c

$$\begin{aligned} u_{12} &= 8 \times \left(\tfrac{1}{2}\right)^{11} \\ &= \frac{8}{2^{11}} \\ &= \frac{1}{256} \end{aligned}$$

(See chapter **3** for exponent simplification.)

EXERCISE 2D

1 For the geometric sequence with first two terms given, find b and c:

a $2, 6, b, c, ...$ **b** $10, 5, b, c, ...$ **c** $12, -6, b, c, ...$

2 **a** Show that the sequence $5, 10, 20, 40, ...$ is geometric.

b Find u_n and hence find the 15th term.

3 **a** Show that the sequence $12, -6, 3, -1.5, ...$ is geometric.

b Find u_n and hence find the 13th term (as a fraction).

4 Show that the sequence $8, -6, 4.5, -3.375, ...$ is geometric and hence find the 10th term as a decimal.

5 Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$ is geometric and hence find, in simplest form, the general term u_n.

Example 7

$k-1$, $2k$ and $21-k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\dfrac{2k}{k-1} = \dfrac{21-k}{2k}$ {equating r's}

$$\therefore \quad 4k^2 = (21-k)(k-1)$$
$$\therefore \quad 4k^2 = 21k - 21 - k^2 + k$$
$$\therefore \quad 5k^2 - 22k + 21 = 0$$
$$\therefore \quad (5k-7)(k-3) = 0$$
$$\therefore \quad k = \tfrac{7}{5} \quad \text{or} \quad 3$$

Check: If $k = \frac{7}{5}$ terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ terms are: $2, 6, 18$. ✓ $\{r = 3\}$

6 Find k given that the following are consecutive terms of a geometric sequence:

a $7, k, 28$ **b** $k, 3k, 20-k$ **c** $k, k+8, 9k$

Example 8

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.

$u_2 = u_1 r = -6$ .... (1) {using $u_n = u_1 r^{n-1}$ with $n = 2$}

and $u_5 = u_1 r^4 = 162$ .... (2)

So, $\dfrac{u_1 r^4}{u_1 r} = \dfrac{162}{-6}$ {(2) ÷ (1)}

$$\therefore \quad r^3 = -27$$
$$\therefore \quad r = \sqrt[3]{-27}$$
$$\therefore \quad r = -3$$

and so in (1) $\quad u_1(-3) = -6$

$$\therefore \quad u_1 = 2$$

Thus $\quad u_n = 2 \times (-3)^{n-1}$.

Note: $(-3)^{n-1} \neq -3^{n-1}$
as we do not know the value of n.
If n is odd, then $\quad (-3)^{n-1} = 3^{n-1}$
If n is even, then $\quad (-3)^{n-1} = -3^{n-1}$

7 Find the general term u_n, of the geometric sequence which has:

a $u_4 = 24$ and $u_7 = 192$

b $u_3 = 8$ and $u_6 = -1$

c $u_7 = 24$ and $u_{15} = 384$

d $u_3 = 5$ and $u_7 = \frac{5}{4}$

Example 9

Find the first term of the geometric sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, ...$ which exceeds 1400.

First we find u_n:

Now $\quad u_1 = 6 \quad$ and $\quad r = \sqrt{2}$

so as $\quad u_n = u_1 r^{n-1} \quad$ then $\quad u_n = 6 \times (\sqrt{2})^{n-1}$.

Next we need to find n such that $u_n > 1400$.

Using a graphics calculator with $\mathrm{Y}_1 = 6 \times (\sqrt{2})\hat{}(n-1)$, we view a *table of values:*

X	Y1
15	768
16	1086.1
17	1536
18	2172.2
19	3072
20	4344.5
21	6144

X=15

So, the first term to exceed 1400 is u_{17} where $u_{17} = 1536$.

Note: Later we can solve problems like this one using logarithms.

8 **a** Find the first term of the sequence $2, 6, 18, 54, ...$ which exceeds 10 000.

b Find the first term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ which exceeds 4800.

c Find the first term of the sequence $12, 6, 3, 1.5, ...$ which is less than 0.0001.

COMPOUND INTEREST

Consider the following: You invest \$1000 in the bank.
You leave the money in the bank for 3 years.
You are paid an interest rate of 10% p.a.
The interest is added to your investment each year.

An interest rate of 10% p.a. is paid, *increasing the value* of your investment yearly.

Your percentage increase each year is 10%, i.e., $100\% + 10\% = 110\%$ of the value at the start of the year, which corresponds to a *multiplier* of 1.1 .

$$\begin{aligned}
\text{After one year your investment is worth} \quad & \$1000 \times 1.1 = \$1100 \\
\text{After two years it is worth} \quad & \$1100 \times 1.1 \\
& = \$1000 \times 1.1 \times 1.1 \\
& = \$1000 \times (1.1)^2 = \$1210 \\
\text{After three years it is worth} \quad & \$1210 \times 1.1 \\
& = \$1000 \times (1.1)^2 \times 1.1 \\
& = \$1000 \times (1.1)^3
\end{aligned}$$

This suggests that if the money is left in your account for n years it would amount to $\$1000 \times (1.1)^n$.

Note:

$$\begin{aligned}
u_1 &= \$1000 && = \text{initial investment} \\
u_2 &= u_1 \times 1.1 && = \text{amount after 1 year} \\
u_3 &= u_1 \times (1.1)^2 && = \text{amount after 2 years} \\
u_4 &= u_1 \times (1.1)^3 && = \text{amount after 3 years} \\
&\vdots \\
u_{15} &= u_1 \times (1.1)^{14} && = \text{amount after 14 years} \\
&\vdots \\
u_n &= u_1 \times (1.1)^{n-1} && = \text{amount after } (n-1) \text{ years} \\
u_{n+1} &= u_1 \times (1.1)^n && = \text{amount after } n \text{ years}
\end{aligned}$$

In general, $u_{n+1} = u_1 \times r^n$ is used for compound growth

$$\begin{aligned}
u_1 &= \text{initial investment} \\
r &= \text{growth multiplier} \\
n &= \text{number of years} \\
u_{n+1} &= \text{amount after } n \text{ years}
\end{aligned}$$

Example 10

\$5000 is invested for 4 years at 7% p.a. compound interest.
What will it amount to at the end of this period?

$$\begin{aligned}
u_5 &= u_1 \times r^4 && \text{is the amount after 4 years} \\
&= 5000 \times (1.07)^4 && \{\text{for a 7\% increase 100\% becomes 107\%}\} \\
&\doteqdot 6553.98 && \{5000 \ \boxed{\times}\ 1.07 \ \boxed{\wedge}\ 4 \ \boxed{=}\ \}
\end{aligned}$$

So, it amounts to \$6553.98 .

9 **a** What will an investment of \$3000 at 10% p.a. compound interest amount to after 3 years?
b What part of this is interest?

10 How much compound interest is earned by investing 20 000 Euro at 12% p.a. if the investment is over a 4 year period?

11 **a** What will an investment of 30 000 yen at 10% p.a. compound interest amount to after 4 years?
b What part of this is interest?

12 How much compound interest is earned by investing \$80 000 at 9% p.a., if the investment is over a 3 year period?

13 What will an investment of 100 000 yen amount to after 5 years if it earns 8% p.a. compounded semi-annually?

14 What will an investment of £45 000 amount to after 21 months if it earns 7.5% p.a. compounded quarterly?

Example 11

How much should I invest now if I want the maturing value to be \$10 000 in 4 years' time, if I am able to invest at 8.5% p.a. compounded annually?

$u_1 = ?,\quad u_5 = 10\,000,\quad r = 1.085$

$$u_5 = u_1 \times r^4 \qquad \{\text{using } u_{n+1} = u_1 \times r^n\}$$

$$\therefore\ 10\,000 = u_1 \times (1.085)^4$$

$$\therefore\ u_1 = \frac{10\,000}{(1.085)^4}$$

$$\therefore\ u_1 \doteqdot 7215.74 \qquad \{10\,000\ \boxed{\div}\ 1.085\ \boxed{\wedge}\ 4\ \boxed{=}\}$$

So, you should invest \$7215.74 now.

15 How much money must be invested now if you require \$20 000 for a holiday in 4 years' time and the money can be invested at a fixed rate of 7.5% p.a. compounded annually?

16 What initial investment is required to produce a maturing amount of £15 000 in 60 months' time given that a fixed rate of 5.5% p.a. compounded annually is guaranteed?

17 How much should I invest now if I want a maturing amount of 25 000 Euro in 3 years' time and the money can be invested at a fixed rate of 8% p.a. compounded quarterly?

18 What initial investment is required to produce a maturing amount of 40 000 yen in 8 years' time if your money can be invested at 9% p.a., compounded monthly?

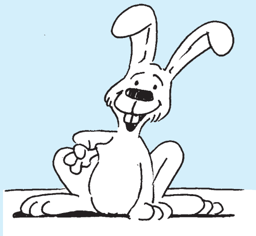
Example 12

The initial population of rabbits on a farm was 50.
The population increased by 7% each week.

a How many rabbits were present after:

i 15 weeks **ii** 30 weeks?

b How long would it take for the population to reach 500?



We notice that $u_1 = 50$ and $r = 1.07$

$u_2 = 50 \times 1.07 =$ the population after 1 week

a **i**

$$u_{n+1} = u_1 \times r^n$$
$$\therefore \quad u_{16} = 50 \times (1.07)^{15}$$
$$\doteqdot 137.95....$$

i.e., 138 rabbits

ii and

$$u_{31} = 50 \times (1.07)^{30}$$
$$\doteqdot 380.61....$$

i.e., 381 rabbits

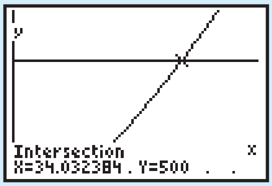
b $u_{n+1} = u_1 \times (1.07)^n$ after n weeks

So, we need to find when $50 \times (1.07)^n = 500$.

Trial and error on your calculator gives $n \doteqdot 34$ weeks

or using the **Equation Solver** gives $n \doteqdot 34.03$

or by finding the **point of intersection** of $\text{Y}_1 = 50 \times 1.07\hat{\ }\text{X}$ and $\text{Y}_2 = 500$ on a graphics calculator, the solution is $\doteqdot 34.03$ weeks.

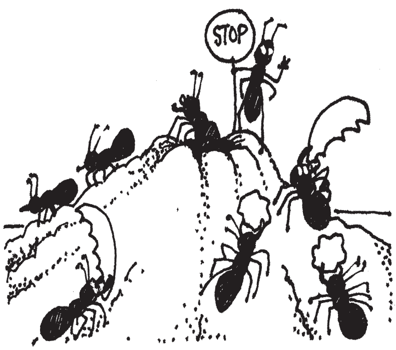


19 A nest of ants initially consists of 500 ants.
The population is increasing by 12% each week.

a How many ants will there be after

i 10 weeks **ii** 20 weeks?

b Use technology to find how many weeks it will take for the ant population to reach 2000.



20 The animal *Eraticus* is endangered. Since 1985 there has only been one colony remaining and in 1985 the population of the colony was 555. Since then the population has been steadily decreasing at 4.5% per year.

Find:

a the population in the year 2000

b the year in which we would expect the population to have declined to 50.

E SERIES

A **series** is the addition of the terms of a sequence,

i.e., $u_1 + u_2 + u_3 + ... + u_n$ is a series.

The **sum** of a series is the result when all terms of the series are added.

Notation: $S_n = u_1 + u_2 + u_3 + ... + u_n$ is the sum of the first n terms.

Example 13

For the sequence 1, 4, 9, 16, 25, ...

a Write down an expression for S_n.

b Find S_n for $n = 1, 2, 3, 4$ and 5.

a $S_n = 1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$ {all terms are perfect squares}

b $S_1 = 1$
$S_2 = 1 + 4 = 5$
$S_3 = 1 + 4 + 9 = 14$
$S_4 = 1 + 4 + 9 + 16 = 30$
$S_5 = 1 + 4 + 9 + 16 + 25 = 55$

EXERCISE 2E.1

1 For the following sequences:

i write down an expression for S_n **ii** find S_5.

a 3, 11, 19, 27, ...
b 42, 37, 32, 27, ...
c 12, 6, 3, $1\frac{1}{2}$, ...
d 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ...
e 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...
f 1, 8, 27, 64, ...

ARITHMETIC SERIES

An **arithmetic series** is the addition of successive terms of an arithmetic sequence.

For example: 21, 23, 25, 27, ..., 49 is an arithmetic sequence.

So, $21 + 23 + 25 + 27 + ... + 49$ is an arithmetic series.

SUM OF AN ARITHMETIC SERIES

Recall that if the first term is u_1 and the common difference is d, then the terms are: u_1, $u_1 + d$, $u_1 + 2d$, $u_1 + 3d$, etc.

Suppose that u_n is the last or final term of an arithmetic series.

Then, $S_n = u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_n - 2d) + (u_n - d) + u_n$

but, $S_n = u_n + (u_n - d) + (u_n - 2d) + ... + (u_1 + 2d) + (u_1 + d) + u_1$ {reversing them}

Adding these two expressions vertically we get

$$2S_n = \underbrace{(u_1+u_n)+(u_1+u_n)+(u_1+u_n)+ .... +(u_1+u_n)+(u_1+u_n)+(u_1+u_n)}_{n \text{ of these}}$$

$\therefore \quad 2S_n = n(u_1+u_n)$

i.e., $S_n = \dfrac{n}{2}(u_1+u_n)$ where $u_n = u_1+(n-1)d$

so $\boldsymbol{S_n = \dfrac{n}{2}(u_1+u_n)}$ *or* $\boldsymbol{S_n = \dfrac{n}{2}(2u_1+(n-1)d)}$

Example 14

Find the sum of $4+7+10+13+ ....$ to 50 terms.

The series is arithmetic with $u_1 = 4$, $d = 3$ and $n = 50$.

So, $S_{50} = \frac{50}{2}(2\times 4 + 49\times 3)$ $\left\{\text{Using } S_n = \dfrac{n}{2}(2u_1+(n-1)d)\right\}$

$= 25(8+147)$

$= 3875$

EXERCISE 2E.2

1 Find the sum of:

a $3+7+11+15+ ....$ to 20 terms

b $\frac{1}{2}+3+5\frac{1}{2}+8+ ....$ to 50 terms

c $100+93+86+79+ ....$ to 40 terms

d $50+48\frac{1}{2}+47+45\frac{1}{2}+ ....$ to 80 terms

Example 15

Find the sum of $-6+1+8+15+ .... +141$.

The series is arithmetic with $u_1 = -6$, $d = 7$ and $u_n = 141$.

First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 141 \\ \therefore \quad u_1+(n-1)d &= 141 \\ \therefore \quad -6+7(n-1) &= 141 \\ \therefore \quad 7(n-1) &= 147 \\ \therefore \quad n-1 &= 21 \\ \therefore \quad n &= 22 \end{aligned}$$

Using $S_n = \frac{n}{2}(u_1+u_n)$ *or* $S_n = \frac{n}{2}(2u_1+(n-1)d)$

$\therefore \quad S_{22} = \frac{22}{2}(-6+141)$

$= 11\times 135$

$= 1485$

$S_{22} = \frac{22}{2}(2\times -6 + 21\times 7)$

$= 11\times 135$

$= 1485$

2 Find the sum of:

a $5+8+11+14+....+101$

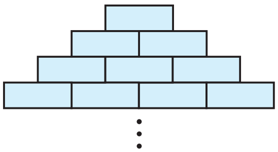
b $50+49\frac{1}{2}+49+48\frac{1}{2}+....+(-20)$

c $8+10\frac{1}{2}+13+15\frac{1}{2}+....+83$

3 An arithmetic series has seven terms. The first term is 5 and the last term is 53. Find the sum of the series.

4 An arithmetic series has eleven terms. The first term is 6 and the last term is -27. Find the sum of the series.

5



A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?

6 Each section of a soccer stadium has 44 rows with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there

a in row 44

b in a section

c at a stadium which has 25 sections?

7 Find the sum of:

a the first 50 multiples of 11

b the multiples of 7 between 0 and 1000

c the integers between 1 and 100 which are not divisible by 3.

8 Prove that the sum of the first n positive integers is $\dfrac{n(n+1)}{2}$,

i.e., show that $1+2+3+4+....+n=\dfrac{n(n+1)}{2}$.

9 Consider the series of odd numbers $1+3+5+7+....$

a What is the nth odd number, that is, u_n?

b Prove that "the sum of the first n odd numbers is n^2".

c Check your answer to **b** by finding S_1, S_2, S_3 and S_4.

10 Find the first two terms of an arithmetic sequence where the sixth term is 21 and the sum of the first seventeen terms is 0.

11 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms. (**Hint:** Let the terms be $x-d$, x and $x+d$.)

12 Five consecutive terms of an arithmetic sequence have a sum of 40. The product of the middle and the two end terms is 224. Find the terms of the sequence.

GEOMETRIC SERIES

A **geometric series** is the addition of successive terms of a geometric sequence.

For example, 1, 2, 4, 8, 16, ..., 1024 is a geometric sequence.

So, $1 + 2 + 4 + 8 + 16 + ... + 1024$ is a geometric series.

SUM OF A GEOMETRIC SERIES

Recall that if the first term is u_1 and the common ratio is r, then the terms are:

$u_1, u_1r, u_1r^2, u_1r^3, ...$ etc.

So, $$S_n = u_1 + \underbrace{u_1r}_{u_2} + \underbrace{u_1r^2}_{u_3} + \underbrace{u_1r^3}_{u_4} + \; .... \; + \underbrace{u_1r^{n-2}}_{u_{n-1}} + \underbrace{u_1r^{n-1}}_{u_n}$$

and **for** $\boldsymbol{r \neq 1}$, $$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \quad \textit{or} \quad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

Proof: If $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1}$ .... (1)

then $rS_n = (u_1r + u_1r^2 + u_1r^3 + u_1r^4 + ... + u_1r^{n-1}) + u_1r^n$

$\therefore \; rS_n = (S_n - u_1) + u_1r^n$ {from (1)}

$\therefore \; rS_n - S_n = u_1r^n - u_1$

$\therefore \; S_n(r - 1) = u_1(r^n - 1)$

$\therefore \; S_n = \dfrac{u_1(r^n - 1)}{r - 1}$ or $\dfrac{u_1(1 - r^n)}{1 - r}$ p.v. $r \neq 1$.

Example 16

Find the sum of $2 + 6 + 18 + 54 + ...$ to 12 terms.

The series is geometric with $u_1 = 2$, $r = 3$ and $n = 12$.

So, $S_{12} = \dfrac{2(3^{12} - 1)}{3 - 1}$ {Using $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$}

$= \dfrac{2(3^{12} - 1)}{2}$

$= 531\,440$

EXERCISE 2E.3

1 Find the sum of the following series:

a $12 + 6 + 3 + 1.5 + ...$ to 10 terms

b $\sqrt{7} + 7 + 7\sqrt{7} + 49 + ...$ to 12 terms

c $6 - 3 + 1\frac{1}{2} - \frac{3}{4} + ...$ to 15 terms

d $1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + ...$ to 20 terms

Example 17

Find a formula for S_n for $9 - 3 + 1 - \frac{1}{3} + ...$ to n terms.

The series is geometric with $u_1 = 9$, $r = -\frac{1}{3}$, "n" $= n$.

$$\text{So,} \quad S_n = \frac{u_1(1 - r^n)}{1 - r}$$

$$= \frac{9(1 - (-\frac{1}{3})^n)}{\frac{4}{3}}$$

$$\therefore \quad S_n = \tfrac{27}{4}(1 - (-\tfrac{1}{3})^n)$$

Note: This answer cannot be simplified as we do not know if n is odd or even.

2 Find a formula for S_n for:

a $\sqrt{3} + 3 + 3\sqrt{3} + 9 + ...$ to n terms

b $12 + 6 + 3 + 1\frac{1}{2} + ...$ to n terms

c $0.9 + 0.09 + 0.009 + 0.0009 + ...$ to n terms

d $20 - 10 + 5 - 2\frac{1}{2} + ...$ to n terms

3 Each year a sales-person is paid a bonus of \$2000 which is banked into the same account which earns a fixed rate of interest of 6% p.a. with interest being paid annually. The amount at the end of each year in the account is calculated as follows:

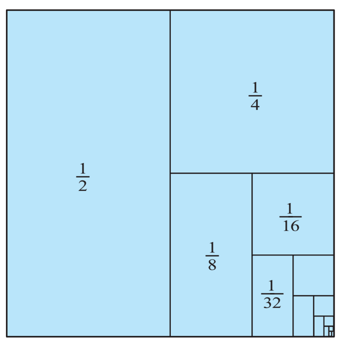
$$A_0 = 2000$$
$$A_1 = A_0 \times 1.06 + 2000$$
$$A_2 = A_1 \times 1.06 + 2000 \quad \text{etc.}$$

a Show that $A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$.

b Show that $A_3 = 2000[1 + 1.06 + (1.06)^2 + (1.06)^3]$.

c Hence find the total bank balance after 10 years. (Assume no fees and charges.)

4 Consider $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \dfrac{1}{2^n}$.



a Find S_1, S_2, S_3, S_4 and S_5 in fractional form.

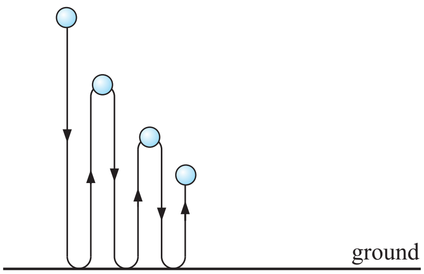
b From **a** guess the formula for S_n.

c Find S_n using $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$.

d Comment on S_n as n gets very large.

e What is the relationship between the given diagram and **d**?

5



A ball takes 1 second to hit the ground when dropped. It then takes 90% of this time to rebound to its new height and this continues until the ball comes to rest.

a Show that the total time of motion is given by $1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \ldots..$

b Find S_n for the series in **a**.

c How long does it take for the ball to come to rest?

Note: This diagram is inaccurate as the motion is really up and down on the same spot. It has been separated out to help us visualise what is happening.

SUM TO INFINITY OF GEOMETRIC SERIES

Sometimes it is necessary to consider $S_n = \dfrac{u_1(1-r^n)}{1-r}$ when n gets very large.

What happens to S_n in this situation?

If $-1 < r < 1$, i.e., $|r| < 1$, then r^n approaches 0 for very large n.

This means that S_n will get closer and closer to $\dfrac{u_1}{1-r}$.

We say that the series **converges** and has a sum to infinity of $\dfrac{u_1}{1-r}$.

We write $\boxed{\boldsymbol{S_\infty = \dfrac{u_1}{1-r} \quad \text{for} \quad |r| < 1.}}$

This result can be used to find the value of recurring decimals.

Example 18

Write $0.\overline{7}$ as a rational number.

$0.\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10\,000} + \ldots.$

which is a geometric series with infinitely many terms

$\therefore \quad S_\infty = \dfrac{u_1}{1-r} = \dfrac{\frac{7}{10}}{1-\frac{1}{10}}$ which simplifies to $\frac{7}{9}$

so, $0.\overline{7} = \frac{7}{9}$

6 Consider $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots.$ which is an infinite geometric series.

a What are **i** u_1 and **ii** r?

b Using **a**, show that $0.\overline{3} = \frac{1}{3}$.

7 Use $S_\infty = \dfrac{u_1}{1-r}$ to check your answers to **4d** and **5c**.

F SIGMA NOTATION

$u_1 + u_2 + u_3 + u_4 + ... + u_n$ can be written more compactly using **sigma notation**.

$\sum$, which is called **sigma**, is the equivalent of capital S in the Greek alphabet.

We write $u_1 + u_2 + u_3 + u_4 + ... + u_n$ as $\sum_{r=1}^{n} u_r$.

So, $\sum_{r=1}^{n} u_r$ reads "the **sum of all numbers** of the form u_r where $r = 1, 2, 3, ...,$ up to n".

Example 19

Expand and find the sum of: **a** $\sum_{r=1}^{7} (r+1)$ **b** $\sum_{r=1}^{5} \frac{1}{2^r}$

a $\sum_{r=1}^{7} (r+1)$

$= 2+3+4+5+6+7+8$

which has a sum of 35

b $\sum_{r=1}^{5} \frac{1}{2^r}$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

which has a sum of $\frac{31}{32}$

EXERCISE 2F

1 Expand and find the sum of:

a $\sum_{r=1}^{4} (3r-5)$ **b** $\sum_{r=1}^{5} (11-2r)$ **c** $\sum_{r=1}^{7} r(r+1)$ **d** $\sum_{i=1}^{5} 10 \times 2^{i-1}$

2 For $u_n = 3n - 1$, list $u_1 + u_2 + u_3 + ... + u_{20}$ and find its sum.

3 Find the sum of these arithmetic series:

a $\sum_{r=1}^{10} (2r+5)$ **b** $\sum_{r=1}^{15} (r-50)$ **c** $\sum_{r=1}^{20} \left(\frac{r+3}{2}\right)$

Hint: List the first 3 terms and the last term.

4 Find the sum of these geometric series:

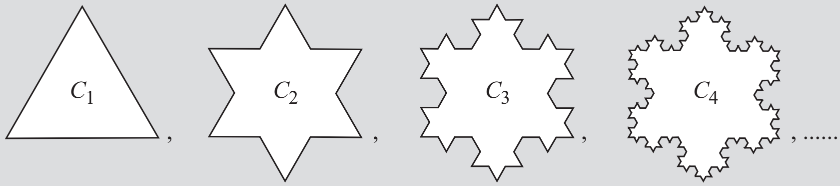
a $\sum_{r=1}^{10} 3 \times 2^{r-1}$ **b** $\sum_{r=1}^{12} (\frac{1}{2})^{r-2}$ **c** $\sum_{r=1}^{25} 6 \times (-2)^r$

Hint: List the first 3 terms and the last term.

5 Find the sum of: **a** $\sum_{k=1}^{5} k(k+1)(k+2)$ **b** $\sum_{k=6}^{12} 100(1.2)^{k-3}$

INVESTIGATION VON KOCH'S SNOWFLAKE CURVE

PORTFOLIO OPPORTUNITY



To draw **Von Koch's Snowflake curve** we

- start with an equilateral triangle, C_1
- then divide each side into 3 equal parts
- then on each middle part draw an equilateral triangle
- then delete the side of the smaller triangle which lies on C_1.

DEMO

The resulting curve is C_2, and C_3, C_4, C_5, ... are found by 'pushing out' equilateral triangles on each edge of the previous curve as we did with C_1 to get C_2.

We get a sequence of special curves C_1, C_2, C_3, C_4, ... and Von Koch's curve is the limiting case, i.e., when n is infinitely large for this sequence.

Your task is to investigate the perimeter and area of Von Koch's curve.

What to do:

1 Suppose C_1 has a perimeter of 3 units. Find the perimeter of C_2, C_3, C_4 and C_5.

(**Hint:** becomes i.e., 3 parts become 4 parts.)

Remembering that Von Koch's curve is C_n, where n is infinitely large, find the perimeter of Von Koch's curve.

2 Suppose the area of C_1 is 1 unit^2. Explain why the areas of C_2, C_3, C_4 and C_5 are

$A_2 = 1 + \frac{1}{3}$ units^2 $\qquad$ $A_3 = 1 + \frac{1}{3}[1 + \frac{4}{9}]$ units^2

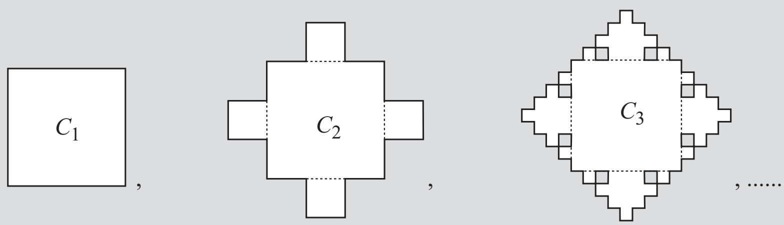
$A_4 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2]$ units^2 $\qquad$ $A_5 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3]$ units^2.

Use your calculator to find A_n where $n = 1, 2, 3, 4, 5, 6, 7$, etc., giving answers which are as accurate as your calculator permits.

What do you think will be the area within Von Koch's snowflake curve?

3 Similarly, investigate the sequence of curves obtained by *pushing out* squares on successive curves from the middle third of each side,

i.e., the curves C_1, C_2, C_3, C_4, etc.



Region contains 8 holes.

REVIEW SET 2A

1 List the first four members of the following sequences defined by:

a $u_n = 3^{n-2}$ **b** $u_n = \dfrac{3n+2}{n+3}$ **c** $u_n = 2^n - (-3)^n$

2 A sequence is defined by $u_n = 68 - 5n$.

a Prove that the sequence is arithmetic.

b Find u_1 and d.

c Find the 37th term.

d What is the first term of the sequence less than -200?

3 **a** Show that the sequence 3, 12, 48, 192, ... is geometric.

b Find u_n and hence find u_9.

4 Find k if $3k$, $k-2$ and $k+7$ are consecutive terms of an arithmetic sequence.

5 Find the general term of an arithmetic sequence given that $u_7 = 31$ and $u_{15} = -17$. Hence, find the value of u_{34}.

6 A sequence is defined by $u_n = 6(\frac{1}{2})^{n-1}$.

a Prove that the sequence is geometric.

b Find u_1 and r.

c Find the 16th term to 3 significant figures.

7 Show that 28, 23, 18, 13, ... is arithmetic and hence find u_n and the sum S_n of the first n terms in simplest form.

8 Find k given that 4, k and $k^2 - 1$ are consecutive geometric terms.

9 Determine the general term of a geometric sequence given that its sixth term is $\frac{16}{3}$ and its tenth term is $\frac{256}{3}$.

REVIEW SET 2B

1 **a** Determine the number of terms in the sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, ..., -36$.

b Find the value of u_{35} for the sequence in **a**.

c Find the sum of the terms of the sequence in **a**.

2 Insert six numbers between 23 and 9 so that all eight numbers are in arithmetic sequence.

3 Find the formula for u_n, the general term of:

a 86, 83, 80, 77, ... **b** $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}, ...$ **c** 100, 90, 81, 72.9, ...

[**Note:** One of these sequences is neither arithmetic nor geometric.]

4 Write down the expansion of:

a $\displaystyle\sum_{r=1}^{7} r^2$ **b** $\displaystyle\sum_{r=1}^{8} \frac{r+3}{r+2}$

5 Write in the form $\sum_{r=1}^{n}(.....)$:

a $4+11+18+25+....$ for n terms **b** $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+....$ for n terms.

6 Find the sum of:

a $3+9+15+21+....$ to 23 terms **b** $24+12+6+3+....$ to 12 terms.

7 Find the sum of **a** $\sum_{r=1}^{8}\left(\frac{31-3r}{2}\right)$ **b** $\sum_{r=1}^{15}50(0.8)^{r-1}$

8 Find the first term of the sequence 5, 10, 20, 40, ... which exceeds 10 000.

9 What will an investment of 6000 Euro at 7% p.a. compound interest amount to after 5 years if the interest is compounded:

a annually **b** quarterly **c** monthly?

REVIEW SET 2C

1 A geometric sequence has $u_6 = 24$ and $u_{11} = 768$. Determine the general term of the sequence and hence find:

a u_{17} **b** the sum of the first 15 terms.

2 How many terms of the series $11+16+21+26+....$ are needed to exceed a sum of 450?

3 Find the first term of the sequence $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ which is less than 0.001.

4 **a** Determine the number of terms in the sequence $128, 64, 32, 16, ..., \frac{1}{512}$.

b Find the sum of these terms.

5 $12 500 is invested in an account which pays 8.25% p.a. compounded. Find the value of the investment after 5 years if the interest is compounded:

a half-yearly **b** monthly.

6 How much should be invested at a fixed rate of 9% p.a. compounded interest if you wish it to amount to $20 000 after 4 years with interest paid monthly?

7 In 1998 there were 3000 koalas on Koala Island. Since then, the population of koalas on the island has increased by 5% each year.

a How many koalas were on the island in 2001?

b In what year will the population first exceed 5000?

Chapter 3

Exponents

Contents:

A Index notation
B Negative bases
C Index laws
D Rational indices
E Algebraic expansion
F Exponential equations
G Graphs of exponential functions
Investigation: Exponential graphs
H Growth
I Decay

Review set 3A
Review set 3B
Review set 3C
Review set 3D

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices** or **exponents** to easily represent such expressions. For example, $5 \times 5 \times 5 = 5^3$.

Indices have many applications in areas such as finance, engineering, physics, biology, electonics and computer science.

Problems encountered in these areas may involve situations where quantities increase or decrease over time. Such problems are often examples of **exponential growth** or **decay**.

OPENING PROBLEM

In 1995, a research establishment started testing the rabbit calicivirus on an island in an attempt to eradicate rabbits. The island was relatively isolated and overrun by rabbits and it thus provided an excellent test site.

The disease was found to be highly contagious and the introduction of the virus had a dramatic impact on the island's rabbit population.

Scientists monitored rabbit numbers over a series of weeks and found that the number of rabbits R, could be predicted by the formula

$R = 8000 \times (0.837)^t$ where t is the number of weeks after the calicivirus was released.

Consider the following questions:

1 If we let $t = 0$ weeks, how many rabbits were on the island?

2 If we let $t = 3\frac{1}{2}$ weeks, we get $R = 8000 \times (0.837)^{3.5}$.
Discuss 'to the power of 3.5'.

3 How long would it take to reduce the rabbit numbers to 80?

4 Will all rabbits ever be eradicated?

5 What would the graph of rabbit numbers plotted against the time after the release of the virus look like?

After studying the concepts of this chapter, you should be able to investigate the questions above.

A INDEX NOTATION

Rather than write $2 \times 2 \times 2 \times 2 \times 2$, we write such a product as 2^5.

2^5 reads "two to the power of five" or "two with index five".

Thus $5^3 = 5 \times 5 \times 5$ and $3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$.

$$2^5$$

power, index or exponent: 5

base: 2

If n is a positive integer, then a^n is the product of n factors of a

i.e., $$a^n = \underbrace{a \times a \times a \times a \times \dots \times a}_{n \text{ factors}}$$

EXERCISE 3A

1 Copy and complete the values of these common powers. Try to become familiar with them.

a $2^1 = ...,\quad 2^2 = ...,\quad 2^3 = ...,\quad 2^4 = ...,\quad 2^5 = ...,\quad 2^6 = ...$

b $3^1 = ...,\quad 3^2 = ...,\quad 3^3 = ...,\quad 3^4 = ...$

c $5^1 = ...,\quad 5^2 = ...,\quad 5^3 = ...,\quad 5^4 = ...$

d $7^1 = ...,\quad 7^2 = ...,\quad 7^3 = ...$

HISTORICAL NOTE

Nicomachus who lived around 100 AD discovered an interesting number pattern involving cubes and sums of odd numbers.

$$1 = 1^3$$
$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3 \quad \text{etc.}$$

B NEGATIVE BASES

So far we have only considered **positive** bases raised to a power.
We will now briefly look at **negative** bases. Consider the statements below:

$(-1)^1 = -1$

$(-1)^2 = -1 \times -1 = 1$

$(-1)^3 = -1 \times -1 \times -1 = -1$

$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$

$(-2)^1 = -2$

$(-2)^2 = -2 \times -2 = 4$

$(-2)^3 = -2 \times -2 \times -2 = -8$

$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

In the pattern above it can be seen that:

> A **negative** base raised to an **odd** power is **negative**; whereas a **negative** base raised to an **even** power is **positive**.

Example 1

Evaluate:

a $(-2)^4$ **b** -2^4 **c** $(-2)^5$ **d** $-(-2)^5$

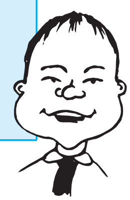
Notice the effect of the brackets in these examples.

a $(-2)^4$
$= 16$

b -2^4
$= -1 \times 2^4$
$= -16$

c $(-2)^5$
$= -32$

d $-(-2)^5$
$= -1 \times (-2)^5$
$= -1 \times -32$
$= 32$



EXERCISE 3B

1 Simplify:

a $(-1)^3$ **b** $(-1)^4$ **c** $(-1)^{12}$ **d** $(-1)^{17}$

e $(-1)^6$ **f** -1^6 **g** $-(-1)^6$ **h** $(-2)^3$

i -2^3 **j** $-(-2)^3$ **k** $-(-5)^2$ **l** $-(-5)^3$

CALCULATOR USE

Although different calculators vary in the appearance of keys, they all perform operations of raising to powers in a similar manner.

Power keys $\boxed{x^2}$ squares the number in the display.

$\boxed{\wedge}$ 3 raises the number in the display to the power 3.

$\boxed{\wedge}$ 5 raises the number in the display to the power 5.

$\boxed{\wedge}$ $\boxed{(-)}$ 4 raises the number in the display to the power -4.

Example 2

Find, using your calculator:

a 6^5 **b** $(-5)^4$ **c** -7^4

		Answer
a	Press: 6 $\boxed{\wedge}$ 5 $\boxed{\text{ENTER}}$	7776
b	Press: $\boxed{(}$ $\boxed{(-)}$ 5 $\boxed{)}$ $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$	625
c	Press: $\boxed{(-)}$ 7 $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$	-2401

Note: You will need to check if your calculator uses the same key sequence as in the examples. If not, work out the sequence which gives you the correct answers.

2 Use your calculator to find the value of the following, recording the entire display:

a 2^9 **b** $(-5)^5$ **c** -3^5 **d** 7^5 **e** 8^3

f $(-9)^4$ **g** -9^4 **h** 1.16^{11} **i** -0.981^{14} **j** $(-1.14)^{23}$

Example 3

Find using your calculator, and comment on: **a** 5^{-2} **b** $\frac{1}{5^2}$

		Answer
a	Press: 5 $\boxed{\wedge}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	0.04
b	Press: 1 $\boxed{\div}$ 5 $\boxed{\wedge}$ 2 $\boxed{\text{ENTER}}$	0.04

The answers indicate that $5^{-2} = \frac{1}{5^2}$.

3 Use your calculator to find the values of the following:

a 7^{-1} **b** $\frac{1}{7^1}$ **c** 3^{-2} **d** $\frac{1}{3^2}$

e 4^{-3} **f** $\frac{1}{4^3}$ **g** 13^0 **h** 172^0

What do you notice?

4 By considering $3^1, 3^2, 3^3, 3^4, 3^5$ .... and looking for a pattern, find the last digit of 3^{33}.

5 What is the last digit of 7^{77}?

C INDEX LAWS

Recall the following **index laws** where the bases a and b are both positive and the indices m and n are integers.

$\boldsymbol{a^m \times a^n = a^{m+n}}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\boldsymbol{\dfrac{a^m}{a^n} = a^{m-n}}$	To **divide** numbers with the same base, keep the base and **subtract** the indices.
$\boldsymbol{(a^m)^n = a^{m \times n}}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$\boldsymbol{(ab)^n = a^n b^n}$	The power of a product is the product of the powers.
$\boldsymbol{\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}}$	The power of a quotient is the quotient of the powers.
$\boldsymbol{a^o = 1, \quad a \neq 0}$	Any non-zero number raised to the power of zero is **1**.
$\boldsymbol{a^{-n} = \dfrac{1}{a^n}}$ and	$\boldsymbol{\dfrac{1}{a^{-n}} = a^n}$ and in particular $\boldsymbol{a^{-1} = \dfrac{1}{a}}$.

Example 4

Simplify using $a^m \times a^n = a^{m+n}$:

a $11^5 \times 11^3$ **b** $a^4 \times a^5$ **c** $x^4 \times x^a$

a $11^5 \times 11^3$
$= 11^{5+3}$
$= 11^8$

b $a^4 \times a^5$
$= a^{4+5}$
$= a^9$

c $x^4 \times x^a$
$= x^{4+a}$
$(= x^{a+4})$

EXERCISE 3C

1 Simplify using $a^m \times a^n = a^{m+n}$:

a $7^3 \times 7^2$ **b** $5^4 \times 5^3$ **c** $a^7 \times a^2$ **d** $a^4 \times a$

e $b^8 \times b^5$ **f** $a^3 \times a^n$ **g** $b^7 \times b^m$ **h** $m^4 \times m^2 \times m^3$

Example 5

Simplify using $\dfrac{a^m}{a^n} = a^{m-n}$: **a** $\dfrac{7^8}{7^5}$ **b** $\dfrac{b^6}{b^m}$

a $\dfrac{7^8}{7^5}$
$= 7^{8-5}$
$= 7^3$

b $= \dfrac{b^6}{b^m}$
$= b^{6-m}$

2 Simplify using $\dfrac{a^m}{a^n} = a^{m-n}$:

a $\dfrac{5^9}{5^2}$ **b** $\dfrac{11^{13}}{11^9}$ **c** $7^7 \div 7^4$ **d** $\dfrac{a^6}{a^2}$

e $\dfrac{b^{10}}{b^7}$ **f** $\dfrac{p^5}{p^m}$ **g** $\dfrac{y^a}{y^5}$ **h** $b^{2x} \div b$

Example 6

Simplify using $(a^m)^n = a^{m\times n}$:
a $(2^4)^3$ **b** $(x^3)^5$ **c** $(b^7)^m$

a $(2^4)^3$
$= 2^{4\times 3}$
$= 2^{12}$

b $(x^3)^5$
$= x^{3\times 5}$
$= x^{15}$

c $(b^7)^m$
$= b^{7\times m}$
$= b^{7m}$

3 Simplify using $(a^m)^n = a^{m\times n}$:

a $(3^2)^4$ **b** $(5^3)^5$ **c** $(2^4)^7$ **d** $(a^5)^2$

e $(p^4)^5$ **f** $(b^5)^n$ **g** $(x^y)^3$ **h** $(a^{2x})^5$

Example 7

Express in simplest form with a prime number base:

a 9^4 **b** 4×2^p **c** $\dfrac{3^x}{9^y}$ **d** 25^{x-1}

a 9^4
$= (3^2)^4$
$= 3^{2\times 4}$
$= 3^8$

b 4×2^p
$= 2^2 \times 2^p$
$= 2^{2+p}$

c $\dfrac{3^x}{9^y}$
$= \dfrac{3^x}{(3^2)^y}$
$= \dfrac{3^x}{3^{2y}}$
$= 3^{x-2y}$

d 25^{x-1}
$= (5^2)^{x-1}$
$= 5^{2(x-1)}$
$= 5^{2x-2}$

4 Express in simplest form with a prime number base:

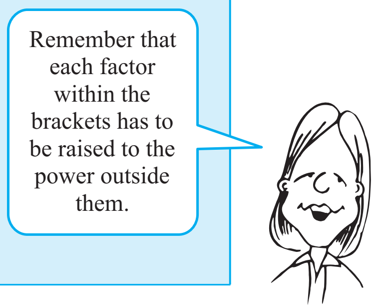
a 8	**b** 25	**c** 27	**d** 4^3
e 9^2	**f** $3^a \times 9$	**g** $5^t \div 5$	**h** $3^n \times 9^n$
i $\dfrac{16}{2^x}$	**j** $\dfrac{3^{x+1}}{3^{x-1}}$	**k** $(5^4)^{x-1}$	**l** $2^x \times 2^{2-x}$
m $\dfrac{2^y}{4^x}$	**n** $\dfrac{4^y}{8^x}$	**o** $\dfrac{3^{x+1}}{3^{1-x}}$	**p** $\dfrac{2^t \times 4^t}{8^{t-1}}$

Example 8

Remove the brackets of: **a** $(2x)^3$ **b** $\left(\dfrac{3c}{b}\right)^4$

a
$$\begin{aligned}&(2x)^3\\ &= 2^3 \times x^3\\ &= 8x^3\end{aligned}$$

b
$$\begin{aligned}&\left(\frac{3c}{b}\right)^4\\ &= \frac{3^4 \times c^4}{b^4}\\ &= \frac{81c^4}{b^4}\end{aligned}$$



5 Remove the brackets of:

a $(ab)^3$	**b** $(ac)^4$	**c** $(bc)^5$	**d** $(abc)^3$
e $(2a)^4$	**f** $(5b)^2$	**g** $(3n)^4$	**h** $(2bc)^3$
i $(4ab)^3$	**j** $\left(\dfrac{a}{b}\right)^3$	**k** $\left(\dfrac{m}{n}\right)^4$	**l** $\left(\dfrac{2c}{d}\right)^5$

Example 9

Express the following in simplest form, without brackets:

a $(3a^3b)^4$ **b** $\left(\dfrac{x^2}{2y}\right)^3$

a
$$\begin{aligned}&(3a^3b)^4\\ &= 3^4 \times (a^3)^4 \times b^4\\ &= 81 \times a^{3\times 4} \times b^4\\ &= 81a^{12}b^4\end{aligned}$$

b
$$\begin{aligned}&\left(\frac{x^2}{2y}\right)^3\\ &= \frac{(x^2)^3}{2^3 \times y^3}\\ &= \frac{x^6}{8y^3}\end{aligned}$$

6 Express the following in simplest form, without brackets:

a $(2b^4)^3$ **b** $\left(\frac{3}{x^2y}\right)^2$ **c** $(5a^4b)^2$ **d** $\left(\frac{m^3}{2n^2}\right)^4$

e $\left(\frac{3a^3}{b^5}\right)^3$ **f** $(2m^3n^2)^5$ **g** $\left(\frac{4a^4}{b^2}\right)^2$ **h** $(5x^2y^3)^3$

Example 10

Write in simplest form, without brackets:

a $(-3a^2)^4$ **b** $\left(-\frac{2a^2}{b}\right)^3$

a
$$\begin{aligned}&(-3a^2)^4\\&= (-3)^4 \times (a^2)^4\\&= 81 \times a^{2\times 4}\\&= 81a^8\end{aligned}$$

b
$$\begin{aligned}&\left(-\frac{2a^2}{b}\right)^3\\&= \frac{(-2)^3 \times (a^2)^3}{b^3}\\&= \frac{-8a^6}{b^3}\end{aligned}$$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$ **b** $(-6b^2)^2$ **c** $(-2a)^3$ **d** $(-3m^2n^2)^3$

e $(-2ab^4)^4$ **f** $\left(\frac{-2a^2}{b^2}\right)^3$ **g** $\left(\frac{-4a^3}{b}\right)^2$ **h** $\left(\frac{-3p^2}{q^3}\right)^2$

Example 11

Simplify using the index laws:

a $3x^2 \times 5x^5$ **b** $\frac{20a^9}{4a^6}$ **c** $\frac{b^3 \times b^7}{(b^2)^4}$

a
$$\begin{aligned}&3x^2 \times 5x^5\\&= 3 \times 5 \times x^2 \times x^5\\&= 15 \times x^{2+5}\\&= 15x^7\end{aligned}$$

b
$$\begin{aligned}&\frac{20a^9}{4a^6}\\&= \tfrac{20}{4} \times a^{9-6}\\&= 5a^3\end{aligned}$$

c
$$\begin{aligned}&\frac{b^3 \times b^7}{(b^2)^4}\\&= \frac{b^{10}}{b^8}\\&= b^{10-8}\\&= b^2\end{aligned}$$

8 Simplify the following expressions using one or more of the index laws:

a $\frac{a^3}{a}$ **b** $4b^2 \times 2b^3$ **c** $\frac{m^5n^4}{m^2n^3}$

d $\dfrac{14a^7}{2a^2}$ **e** $\dfrac{12a^2b^3}{3ab}$ **f** $\dfrac{18m^7a^3}{4m^4a^3}$

g $10hk^3 \times 4h^4$ **h** $\dfrac{m^{11}}{(m^2)^8}$ **i** $\dfrac{p^2 \times p^7}{(p^3)^2}$

Example 12

Simplify, giving answers in simplest rational form:

a 7^0 **b** 3^{-2} **c** $3^0 - 3^{-1}$ **d** $\left(\frac{5}{3}\right)^{-2}$

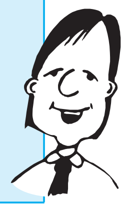
a 7^0
$= 1$

b 3^{-2}
$= \dfrac{1}{3^2}$
$= \frac{1}{9}$

c $3^0 - 3^{-1}$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

d $\left(\frac{5}{3}\right)^{-2}$
$= \left(\frac{3}{5}\right)^2$
$= \frac{9}{25}$

Notice that $\left(\dfrac{a}{b}\right)^{-2} = \left(\dfrac{b}{a}\right)^2$



9 Simplify, giving answers in simplest rational form:

a 5^0 **b** 3^{-1} **c** 6^{-1} **d** 8^0
e 2^2 **f** 2^{-2} **g** 2^3 **h** 2^{-3}
i 5^2 **j** 5^{-2} **k** 10^2 **l** 10^{-2}

10 Simplify, giving answers in simplest rational form:

a $\left(\frac{2}{3}\right)^0$ **b** $\dfrac{4^3}{4^3}$ **c** $3y^0$ **d** $(3y)^0$
e 2×3^0 **f** 6^0 **g** $\dfrac{5^2}{5^4}$ **h** $\dfrac{2^{10}}{2^{15}}$
i $\left(\frac{1}{3}\right)^{-1}$ **j** $\left(\frac{2}{5}\right)^{-1}$ **k** $\left(\frac{4}{3}\right)^{-1}$ **l** $\left(\frac{1}{12}\right)^{-1}$
m $\left(\frac{2}{3}\right)^{-2}$ **n** $5^0 - 5^{-1}$ **o** $7^{-1} + 7^0$ **p** $2^0 + 2^1 + 2^{-1}$

Example 13

Write the following without brackets or negative indices:

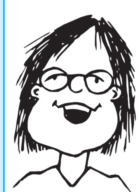
a $(5x)^{-1}$ **b** $5x^{-1}$ **c** $(3b^2)^{-2}$

a $(5x)^{-1}$
$= \dfrac{1}{5x}$

b $5x^{-1}$
$= \dfrac{5}{x}$

c $(3b^2)^{-2}$
$= \dfrac{1}{(3b^2)^2}$
$= \dfrac{1}{3^2b^4}$
$= \dfrac{1}{9b^4}$

In $5x^{-1}$ the index -1 refers to the x only.



11 Write the following without brackets or negative indices:

a $(2a)^{-1}$ **b** $2a^{-1}$ **c** $3b^{-1}$ **d** $(3b)^{-1}$

e $(\frac{2}{b})^{-2}$ **f** $(2b)^{-2}$ **g** $(3n)^{-2}$ **h** $(3n^{-2})^{-1}$

i ab^{-1} **j** $(ab)^{-1}$ **k** ab^{-2} **l** $(ab)^{-2}$

m $(2ab)^{-1}$ **n** $2(ab)^{-1}$ **o** $2ab^{-1}$ **p** $\dfrac{(ab)^2}{b^{-1}}$

Example 14

Write the following as powers of 2, 3 and/or 5:

a $\dfrac{1}{8}$ **b** $\dfrac{1}{9^n}$ **c** $\dfrac{25}{5^4}$

a $\dfrac{1}{8}$
$= \dfrac{1}{2^3}$
$= 2^{-3}$

b $\dfrac{1}{9^n}$
$= \dfrac{1}{(3^2)^n}$
$= \dfrac{1}{3^{2n}}$
$= 3^{-2n}$

c $\dfrac{25}{5^4}$
$= \dfrac{5^2}{5^4}$
$= 5^{2-4}$
$= 5^{-2}$

12 Write the following as powers of 2, 3 and/or 5:

a $\dfrac{1}{3}$ **b** $\dfrac{1}{2}$ **c** $\dfrac{1}{5}$ **d** $\dfrac{1}{4}$

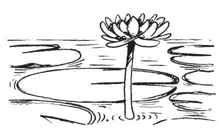
e $\dfrac{1}{27}$ **f** $\dfrac{1}{25}$ **g** $\dfrac{1}{8^x}$ **h** $\dfrac{1}{16^y}$

i $\dfrac{1}{81^a}$ **j** $\dfrac{9}{3^4}$ **k** 25×5^{-4} **l** $\dfrac{5^{-1}}{5^2}$

m $2 \div 2^{-3}$ **n** 1 **o** 6^{-3} **p** 4×10^2

13 The water lily *Growerosa Veryfasterosa* doubles its size every day. From the time it was planted until it completely covered the pond took 26 days.

How many days did it take to cover half the pond?



14 Suppose you have the following six coins in your pocket: 5 cents, 10 cents, 20 cents, 50 cents, \$1, \$2. How many different sums of money can you make?

(**Hint:** Simplify the problem to a smaller number of coins and look for a pattern.)

15 Read about Nicomachus' pattern on page **63** and find the sequence of odd numbers for:

a 5^3 **b** 7^3 **c** 12^3

16 Find the smaller of 2^{175} and 5^{75} without a calculator.

D RATIONAL INDICES

Since $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$, {for index laws to be obeyed}

and $\sqrt{a} \times \sqrt{a} = a$ also, then

$$\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$,

compared with $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$

Thus in general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ { $\sqrt[n]{a}$ reads "the nth root of a"}

Notice also that $a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^2$

i.e., $\left(a^{\frac{2}{3}}\right)^3 = a^2$ {if $(a^m)^n = a^{mn}$ is to be used}

$\therefore \quad a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

In general, $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$

Example 15

Write as a single power of 2: **a** $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2 \times \frac{1}{5}}$
$= 2^{\frac{2}{5}}$

EXERCISE 3D

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

Example 16

Use your calculator to evaluate to 3 decimal places **a** $2^{\frac{7}{5}}$ **b** $9^{-\frac{3}{4}}$

		Calculator:	*Answer:*
a	$2^{\frac{7}{5}}$	2 [^] [(] 7 [÷] 5 [)] [ENTER]	2.639
b	$9^{-\frac{3}{4}}$	9 [^] [(] [(-)] 3 [÷] 4 [)] [ENTER]	0.192

4 Use your calculator to evaluate to 3 decimal places:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$

Example 17

Use your calculator to evaluate to 3 decimal places: **a** $\sqrt[5]{4}$ **b** $\dfrac{1}{\sqrt[6]{11}}$

		Calculator:	*Answer:*
a	$\sqrt[5]{4}$ $= 4^{\frac{1}{5}}$	4 [^] [(] 1 [÷] 5 [)] [ENTER]	1.320
b	$\dfrac{1}{\sqrt[6]{11}}$ $= 11^{-\frac{1}{6}}$	11 [^] [(] [(-)] 1 [÷] 6 [)] [ENTER]	0.671

5 Use your calculator to find to 3 decimal places:

a $\sqrt{9}$ **b** $\sqrt[4]{8}$ **c** $\sqrt[5]{27}$ **d** $\dfrac{1}{\sqrt[3]{7}}$

Example 18

Without using a calculator, write in simplest rational form: **a** $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a
$$\begin{aligned} & 8^{\frac{4}{3}} \\ &= (2^3)^{\frac{4}{3}} \\ &= 2^{3\times\frac{4}{3}} \qquad \{(a^m)^n = a^{mn}\} \\ &= 2^4 \\ &= 16 \end{aligned}$$

b
$$\begin{aligned} & 27^{-\frac{2}{3}} \\ &= (3^3)^{-\frac{2}{3}} \\ &= 3^{3\times-\frac{2}{3}} \\ &= 3^{-2} \\ &= \tfrac{1}{9} \end{aligned}$$

6 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

E ALGEBRAIC EXPANSION

Recall the expansion laws:

$(a+b)(c+d) = ac+ad+bc+bd$ {sometimes called 'FOIL'}
$(a+b)(a-b) = a^2 - b^2$ {difference of squares}
$(a+b)^2 = a^2+2ab+b^2$ {perfect squares}
$(a-b)^2 = a^2-2ab+b^2$

Example 19

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$

$$x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$$
$$= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}} \quad \{\text{each term is} \times \text{by } x^{-\frac{1}{2}}\}$$
$$= x^1 + 2x^0 - 3x^{-1} \quad \{\text{adding indices}\}$$
$$= x + 2 - \frac{3}{x}$$

EXERCISE 3E

1 Expand and simplify:

a $x^2(x^3+2x^2+1)$ **b** $2^x(2^x+1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$

d $e^x(e^x+2)$ **e** $3^x(2-3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$

g $2^{-x}(2^x+5)$ **h** $5^{-x}(5^{2x}+5^x)$ **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

Example 20

Expand and simplify: **a** $(2^x+3)(2^x+1)$ **b** $(e^x+e^{-x})^2$

a
$$(2^x+3)(2^x+1)$$
$$= 2^x \times 2^x + 2^x + 3 \times 2^x + 3 \quad \{\text{using FOIL}\}$$
$$= 2^{2x} + 4 \times 2^x + 3 \quad \{\text{the 2 middle terms are 'like'}\}$$
$$= 4^x + 2^{2+x} + 3$$

b
$$(e^x+e^{-x})^2$$
$$= (e^x)^2 + 2e^x \times e^{-x} + (e^{-x})^2 \quad \{\text{using } (a+b)^2 = a^2+2ab+b^2\}$$
$$= e^{2x} + 2e^0 + e^{-2x}$$
$$= e^{2x} + 2 + e^{-2x}$$

2 Expand and simplify:

a $(2^x+1)(2^x+3)$ **b** $(3^x+2)(3^x+5)$ **c** $(5^x-2)(5^x-4)$

d $(2^x+3)^2$ **e** $(3^x-1)^2$ **f** $(4^x+7)^2$

g $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$ **h** $(2^x+3)(2^x-3)$ **i** $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$

j $(x+\dfrac{2}{x})^2$ **k** $(e^x-e^{-x})^2$ **l** $(5-2^{-x})^2$

F EXPONENTIAL EQUATIONS

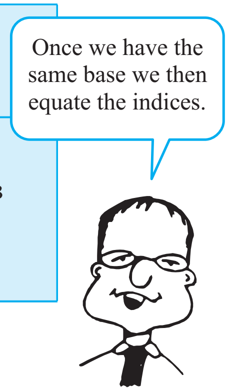
An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent. For example: $2^x=8$ and $30\times 3^x=7$ are both exponential equations.

If $2^x=8$, then $2^x=2^3$. Thus $x=3$, and this is the only solution.

Hence: If $a^x=a^k$, then $x=k$,
i.e., if the base numbers are the same, we can **equate indices**.

Example 21

Solve for x: **a** $2^x=16$ **b** $3^{x+2}=\frac{1}{27}$



a
$$\begin{aligned} 2^x &= 16\\ \therefore\quad 2^x &= 2^4\\ \therefore\quad x &= 4 \end{aligned}$$

b
$$\begin{aligned} 3^{x+2} &= \tfrac{1}{27}\\ \therefore\quad 3^{x+2} &= 3^{-3}\\ \therefore\quad x+2 &= -3\\ \therefore\quad x &= -5 \end{aligned}$$

EXERCISE 3F

1 Solve for x:

a $2^x=2$ **b** $2^x=4$ **c** $3^x=27$ **d** $2^x=1$

e $2^x=\frac{1}{2}$ **f** $3^x=\frac{1}{3}$ **g** $2^x=\frac{1}{8}$ **h** $2^{x+1}=8$

i $2^{x-2}=\frac{1}{4}$ **j** $3^{x+1}=\frac{1}{27}$ **k** $2^{x+1}=64$ **l** $2^{1-2x}=\frac{1}{2}$

Example 22

Solve for x: **a** $4^x=8$ **b** $9^{x-2}=\frac{1}{3}$

a
$$\begin{aligned} 4^x &= 8\\ \therefore\quad (2^2)^x &= 2^3\\ \therefore\quad 2^{2x} &= 2^3\\ \therefore\quad 2x &= 3\\ \therefore\quad x &= \tfrac{3}{2} \end{aligned}$$

b
$$\begin{aligned} 9^{x-2} &= \tfrac{1}{3}\\ \therefore\quad (3^2)^{x-2} &= 3^{-1}\\ \therefore\quad 3^{2(x-2)} &= 3^{-1}\\ \therefore\quad 2x-4 &= -1\\ \therefore\quad 2x &= 3\\ \therefore\quad x &= \tfrac{3}{2} \end{aligned}$$

2 Solve for x:

a $4^x = 32$	**b** $8^x = \frac{1}{4}$	**c** $9^x = \frac{1}{3}$	**d** $49^x = \frac{1}{7}$
e $4^x = \frac{1}{8}$	**f** $25^x = \frac{1}{5}$	**g** $8^{x+2} = 32$	**h** $8^{1-x} = \frac{1}{4}$
i $4^{2x-1} = \frac{1}{2}$	**j** $9^{x-3} = 3$	**k** $(\frac{1}{2})^{x+1} = 2$	**l** $(\frac{1}{3})^{x+2} = 9$
m $4^x = 8^{-x}$	**n** $(\frac{1}{4})^{1-x} = 8$	**o** $(\frac{1}{7})^x = 49$	**p** $(\frac{1}{2})^{x+1} = 32$

3 Solve for x:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = (\frac{1}{3})^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

Example 23

Solve for x: **a** $5 \times 2^x = 40$ **b** $18 \times 3^x = 2$

a
$$\begin{aligned} 5 \times 2^x &= 40 \\ \therefore\ 2^x &= 8 && \{\div \text{ both sides by } 5\} \\ \therefore\ 2^x &= 2^3 \\ \therefore\ x &= 3 && \{\text{equating indices}\} \end{aligned}$$

b
$$\begin{aligned} 18 \times 3^x &= 2 \\ \therefore\ 3^x &= \tfrac{1}{9} && \{\div \text{ both sides by } 18\} \\ \therefore\ 3^x &= 3^{-2} \\ \therefore\ x &= -2 && \{\text{equating indices}\} \end{aligned}$$

4 Solve for x:

a $3 \times 2^x = 24$	**b** $7 \times 2^x = 56$	**c** $3 \times 2^{x+1} = 24$
d $12 \times 3^{-x} = \frac{4}{3}$	**e** $4 \times (\frac{1}{3})^x = 36$	**f** $5 \times (\frac{1}{2})^x = 20$

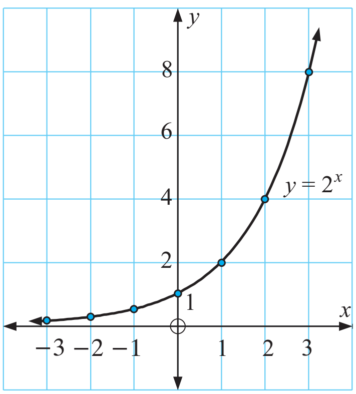
G GRAPHS OF EXPONENTIAL FUNCTIONS

The general **exponential function** has form $y = a^x$ where $a > 0$, $a \neq 1$.

For example, $y = 2^x$ is an exponential function.

Table of values:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



We notice that for $x = -10$, say, $y = 2^{-10} \doteqdot 0.001$

Also when $x = -50$, $y = 2^{-50} \doteqdot 8.88 \times 10^{-16}$

So, it appears that as x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above it. We say that $y = 2^x$ is **'asymptotic** to the x-axis', or '$y = 0$ is a **horizontal asymptote**'.

INVESTIGATION — EXPONENTIAL GRAPHS

PORTFOLIO OPPORTUNITY

We will investigate families of exponential functions.

TI C GRAPHING PACKAGE

What to do:

1 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 3^x$, $y = 10^x$, $y = (1.3)^x$.

b The functions in **a** are all members of the family $y = b^x$.

- **i** What effect does changing b values have on the shape of the graph?
- **ii** What is the y-intercept of each graph?
- **iii** What is the horizontal asymptote of each graph?

2 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 2^x + 1$, $y = 2^x - 2$.

b The functions in **a** are all members of the family $y = 2^x + d$ where d is a constant.

- **i** What effect does changing d values have on the position of the graph?
- **ii** What effect does changing d values have on the shape of the graph?
- **iii** What is the horizontal asymptote of each graph?
- **iv** What is the horizontal asymptote of $y = 2^x + d$?

c To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

3 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions: $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, $y = 2^{x-3}$

b The functions in **a** are all members of the family $y = 2^{x-c}$.

- **i** What effect does changing c values have on the position of the graph?
- **ii** What effect does changing c values have on the shape of the graph?
- **iii** What is the horizontal asymptote of each graph?

c To graph $y = 2^{x-c}$ from $y = 2^x$, what transformation is used?

4 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the functions $y = 2^x$ and $y = 2^{-x}$.

b
- **i** What is the y-intercept of each graph?
- **ii** What is the horizontal asymptote of each graph?
- **iii** What transformation moves $y = 2^x$ to $y = 2^{-x}$?

5 **a** On the same set of axes, use a **graphing package** or **graphics calculator** to graph the following functions:

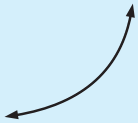
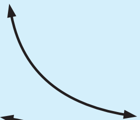
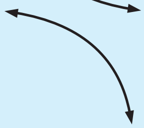
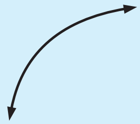
- **i** $y = 2^x$, $y = 3 \times 2^x$, $y = \frac{1}{2} \times 2^x$
- **ii** $y = -2^x$, $y = -3 \times 2^x$, $y = -\frac{1}{2} \times 2^x$

b The functions in **a** are all members of the family $y = a \times 2^x$ where a is a constant. Comment on the effect on the graph when **i** $a > 0$ **ii** $a < 0$.

c What is the horizontal asymptote of each graph?

From your investigation you should have discovered that:

for the general exponential function $y = a \times b^{x-c} + d$

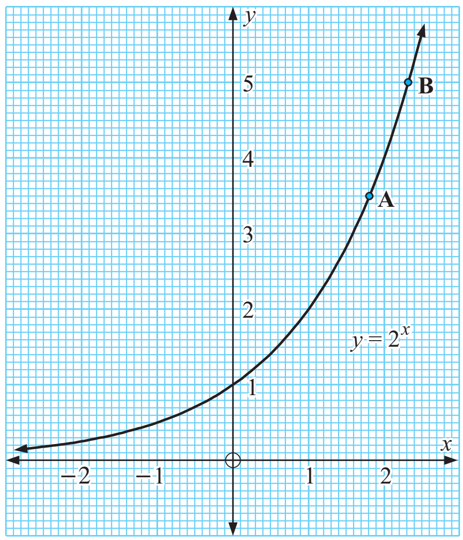
- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation and $y = d$ is the equation of the horizontal asymptote.
- • if $a > 0$, $b > 1$  i.e., increasing
  • if $a > 0$, $0 < b < 1$  i.e., decreasing
  • if $a < 0$, $b > 1$  i.e., decreasing
  • If $a < 0$, $0 < b < 1$  i.e., increasing

EXERCISE 3G

1 Given the graph of $y = 2^x$ we can find approximate values of 2^x for various x values.

For example:

- $2^{1.8} \doteqdot 3.5$ (see point A)
- $2^{2.3} \doteqdot 5$ (see point B)



Use the graph to determine approximate values of:

a $2^{\frac{1}{2}}$ (i.e., $\sqrt{2}$) **b** $2^{0.8}$

c $2^{1.5}$ **d** $2^{-1.6}$

2 Draw freehand sketches of the following pairs of graphs based on your observations from the previous investigation.

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2 \times 2^x$

GRAPHING PACKAGE

3 Check your answers to **2** using technology.

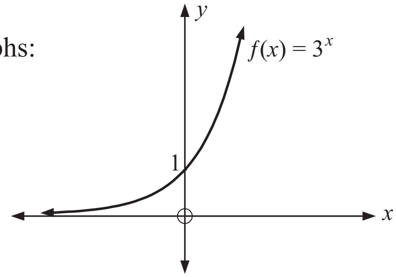
4 The graph of $y = 3^x$ is given alongside.
Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$

b $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$

d $y = 3^x$ and $y = 3^{x-1}$

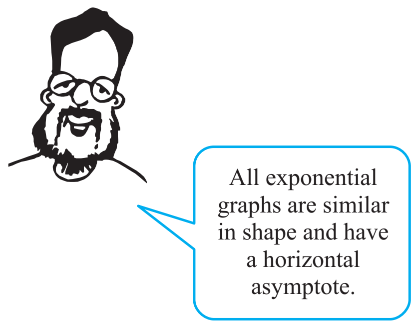


HORIZONTAL ASYMPTOTES

From the previous investigation we noted that for the general exponential function $y = a \times b^{x-c} + d$, $y = d$ is the **horizontal asymptote**.

We can actually obtain reasonably accurate sketch graphs of exponential functions using

- the horizontal asymptote
- the y-intercept
- two other points, say when $x = 2$, $x = -2$



Example 24

Sketch the graph of $y = 2^{-x} - 3$.

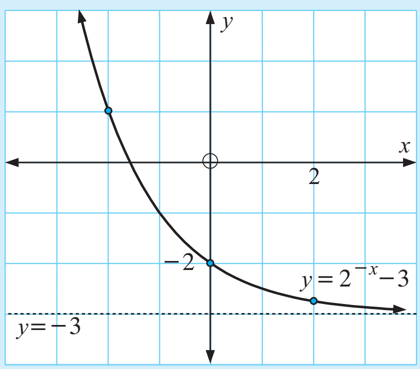
For $y = 2^{-x} - 3$
the horizontal asymptote is $y = -3$

when $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2

when $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

when $x = -2$, $y = 2^2 - 3$
$= 4 - 3$
$= 1$



5 Sketch the graphs of:

a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

H GROWTH

In this exercise we will examine situations where quantities are increasing exponentially (i.e., growth).

Populations of animals, people, bacteria, etc usually grow in an exponential way whereas radioactive substances and items that depreciate usually decay exponentially.

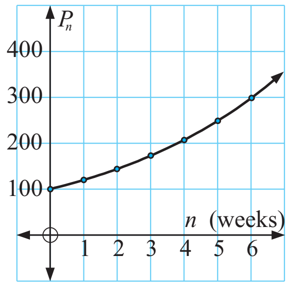
BIOLOGICAL GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week. To increase a quantity by 20%, we multiply it by 120% or 1.2.

So, if P_n is the population after n weeks, then

$P_0 = 100$ {the *original* population}
$P_1 = P_0 \times 1.2 = 100 \times 1.2$
$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$
$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, etc

and from this pattern we see that $P_n = 100 \times (1.2)^n$.



Alternatively:

This is an example of a *geometric sequence* and we could have found the rule to generate it. Clearly $r = 1.2$ and so as $P_n = P_0 r^n$, then $P_n = 100 \times (1.2)^n$ for $n = 0, 1, 2, 3, ...$

Example 25

An entomologist, monitoring a grasshopper plague, notices that the area affected by the grasshoppers is given by $A_n = 1000 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.
b Find the affected area after **i** 5 weeks **ii** 10 weeks.
c Find the affected area after 12 weeks.
d Draw the graph of A_n against n.

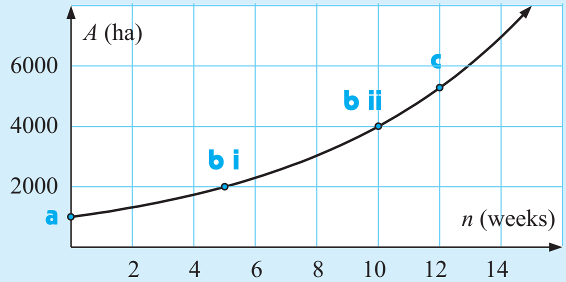
a $A_0 = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ $\therefore$ original area was 1000 ha.

b **i** $A_5 = 1000 \times 2^1$
$= 2000$
i.e., area is 2000 ha.

ii $A_{10} = 1000 \times 2^2$
$= 4000$
i.e., area is 4000 ha.

c $A_{12} = 1000 \times 2^{0.2\times 12}$
$= 1000 \times 2^{2.4}$ {*Press:* 1000 $\boxed{\times}$ 2 $\boxed{y^x}$ 2.4 $\boxed{=}$ }
$\doteq 5278$
$\therefore$ after 12 weeks, area affected is about 5300 ha.

d



EXERCISE 3H

1 The weight W_t grams, of bacteria in a culture t hours after establishment is given by $W_t = 100 \times 2^{0.1t}$ grams. Find:

a the initial weight

b the weight after

i 4 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of W_t against t using only **a** and **b** results.

d Use technology to graph $\mathrm{Y}_1 = 100 \times 2^{0.1\mathrm{X}}$ and check your answers to **a**, **b** and **c**.

GRAPHING PACKAGE

2 A breeding program to ensure the survival of pygmy possums was established with an initial population of 50 (25 pairs). From a previous program the expected population P_n in n years time is given by $P_n = P_0 \times 2^{0.3t}$.

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of P_n against n using only **a** and **b**.

d Use technology to graph $\mathrm{Y}_1 = 50 \times 2^{0.3\mathrm{X}}$ and use it to check your answers in **b**.

3 The speed V_t of a chemical reaction is given by $V_t = V_0 \times 2^{0.05t}$ where t is the temperature in ^{o}C. Find:

a the speed at 0^oC

b the speed at 20^oC

c the percentage increase in speed at 20^oC compared with the speed at 0^oC.

d Find $\left(\dfrac{V_{50} - V_{20}}{V_{20}}\right) \times 100\%$. What does this calculation represent?

4 A species of bear is introduced to a large island off Alaska where previously there were no bears. 6 pairs of bears were introduced in 1998. It is expected that the population will increase according to $B_t = B_0 \times 2^{0.18t}$ where t is the time since the introduction occurred.

a Find B_0.

b Find the expected bear population in 2018.

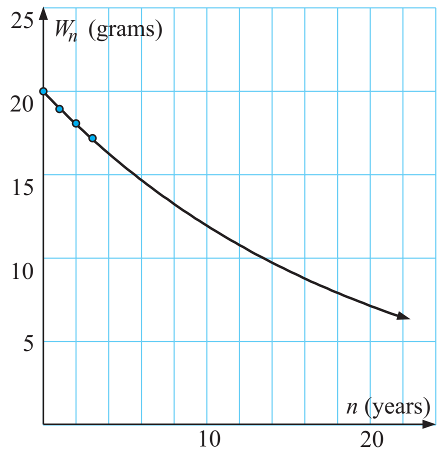
c Find the percentage increase from year 2008 to 2018.

I DECAY

Now consider a radioactive substance of original weight 20 grams which decays (reduces) by 5% each year. The multiplier is now 95% or 0.95.

So, if W_n is the weight after n years, then:

$$\begin{aligned} W_0 &= 20 \text{ grams} \\ W_1 &= W_0 \times 0.95 = 20 \times 0.95 \text{ grams} \\ W_2 &= W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams} \\ W_3 &= W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams} \\ &\vdots \quad \text{etc.} \\ W_{10} &= 20 \times (0.95)^{10} \doteqdot 12.0 \text{ grams} \\ &\vdots \\ W_{20} &= 20 \times (0.95)^{20} \doteqdot 7.2 \text{ grams} \\ &\vdots \\ W_{100} &= 20 \times (0.95)^{100} \doteqdot 0.1 \text{ grams} \end{aligned}$$



and from this we see that $W_n = 20 \times (0.95)^n$

and so $W_n = W_0 \times (0.95)^n$ if the original weight W_0 is unknown.

Alternatively:

Once again we have an example of a *geometric sequence* with $W_0 = 20$ and $r = 0.95$, and consequently $W_n = 20 \times (0.95)^n$ for $n = 0, 1, 2, 3, ...$

Example 26

When a CD player is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds.

a Find $I(t)$ when $t = 0, 1, 2$ and 3.

b What current flowed in the CD player at the instant when it was switched off?

c Plot the graph of $I(t)$ against t $(t \geqslant 0)$ using the information above.

d Use your graph and/or technology to find how long it takes for the current to reach 4 amps.

---

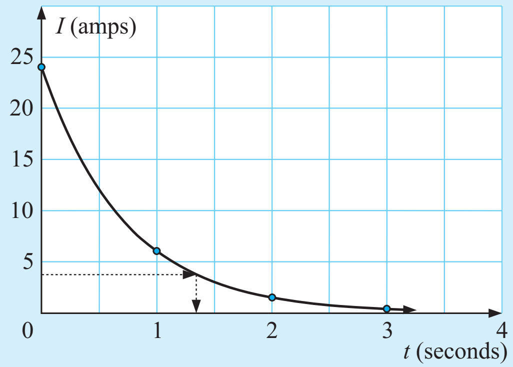
a $I(t) = 24 \times (0.25)^t$ amps

$$\begin{aligned} I(0) &= 24 \times (0.25)^0 \\ &= 24 \text{ amps} \end{aligned} \qquad \begin{aligned} I(1) &= 24 \times (0.25)^1 \\ &= 6 \text{ amps} \end{aligned} \qquad \begin{aligned} I(2) &= 24 \times (0.25)^2 \\ &= 1.5 \text{ amps} \end{aligned}$$

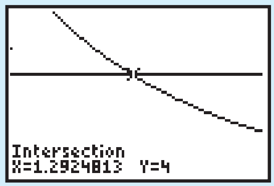
$$\begin{aligned} I(3) &= 24 \times (0.25)^3 \\ &= 0.375 \text{ amps} \end{aligned}$$

b When $t = 0$, $I(0) = 24$ $\therefore$ 24 amps of current flowed.

c



d From the graph above, the approximate time to reach 4 amps is 1.3 seconds.
(This solution can be refined by trial and error.) *or*
By finding the **point of intersection** of
$Y_1 = 24 \times (0.25)$^X and $Y_2 = 4$
on a graphics calculator.
The solution is $\doteqdot 1.29$ seconds.



EXERCISE 3I

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

- **a** How much radioactive substance was put aside?
- **b** Determine the weight of the substance after:
  i 400 years **ii** 800 years **iii** 1200 years.
- **c** Sketch the graph of W against t for $t \geqslant 0$, using the above information.
- **d** Use your graph or **graphics calculator** to find how long it takes for the substance to decay to 125 grams.

2 Revisit the **Opening Problem** on page **62** and answer the questions posed.

Example 27

The weight of radioactive material remaining after t years is given by $W_t = 11.7 \times 2^{-0.0067t}$ grams.

- **a** Find the original weight.
- **b** Find the weight after **i** 10 years **ii** 100 years **iii** 1000 years.
- **c** Graph W_t against t using **a** and **b** only.

$W_t = 11.7 \times 2^{-0.0067t}$

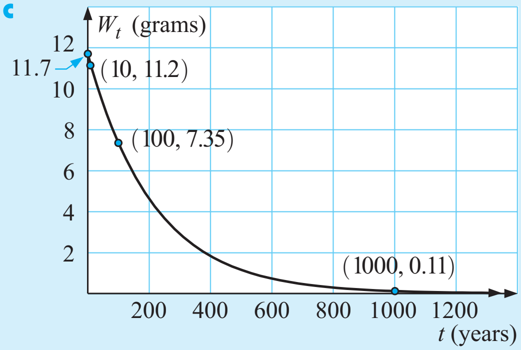
a When $t = 0$, $W_0 = 11.7 \times 2^0 = 11.7$ grams

b **i** W_{10}
$= 11.7 \times 2^{-0.067}$
$\doteq 11.2$ g

ii W_{100}
$= 11.7 \times 2^{-0.67}$
$\doteq 7.35$ g

iii W_{1000}
$= 11.7 \times 2^{-6.7}$
$\doteq 0.11$ g

c



3 The temperature $T_t(^oC)$ of a liquid which has been placed in a refrigerator is given by $T_t = 100 \times 2^{-0.02t}$ where t is the time in minutes. Find:

a the initial temperature

b the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of T_t against t using **a** and **b** only.

4 The weight W_t grams of radioactive substance remaining after t years is given by $W_t = 1000 \times 2^{-0.03t}$ grams. Find:

a the initial weight

b the weight after:

i 10 years **ii** 100 years **iii** 1000 years.

c Graph W_t against t using **a** and **b** only.

Example 28

The weight of radioactive material remaining after t years is given by $W_t = W_0 \times 2^{-0.001t}$ grams.

a Find the original weight.

b Find the percentage remaining after 200 years.

a When $t = 0$, $W_0 = W_0 \times 2^0 = W_0$

$\therefore$ W_0 is the original weight.

b When $t = 200$, $W_{200} = W_0 \times 2^{-0.001 \times 200}$

$= W_0 \times 2^{-0.2}$

$\doteq W_0 \times 0.8706$

$\doteq 87.06\%$ of W_0 $\therefore$ 87.1% remains.

5 The weight W_t of radioactive uranium remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find:

a the original weight **b** the percentage weight loss after 1000 years.

6 The current I_t amps, flowing in a transistor radio, t seconds after it is switched off is given by $I_t = I_0 \times 2^{-0.02t}$ amps. Find:

a the initial current

b the current after 1 second

c the percentage change in current after 1 second

d I_{50} and I_{100} and hence sketch the graph of I_t against t.

REVIEW SET 3A

1 Simplify:

a $-(-1)^{10}$ **b** $-(-3)^3$ **c** $3^0 - 3^{-1}$

2 Simplify using the index laws:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\dfrac{5(x^2y)^2}{(5x^2)^2}$

3 Write the following as a power of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

4 Write without brackets or negative indices:

a b^{-3} **b** $(ab)^{-1}$ **c** ab^{-1}

5 Find the value of x, without using your calculator:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$

6 Evaluate without using a calculator: **a** $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

7 Evaluate, correct to 3 significant figures, using your calculator:

a $3^{\frac{3}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

8 If $f(x) = 3 \times 2^x$, find the value of: **a** $f(0)$ **b** $f(3)$ **c** $f(-2)$

9 On the same set of axes draw the graphs of **a** $y = 2^x$ **b** $y = 2^x - 4$, stating the y-intercept and the equation of the horizontal asymptote.

10 The temperature of a liquid t minutes after it was heated is given by $T = 80 \times (0.913)^t$ ^{o}C. Find:

a the initial temperature of the liquid

b the temperature after **i** $t = 12$ **ii** $t = 24$ **ii** $t = 36$ minutes.

c Draw the graph of T against t, $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the temperature to reach 25^{o}C.

REVIEW SET 3B

1 Simplify:

a $-(-2)^3$ **b** $5^{-1} - 5^0$

2 Simplify using the index laws:

a $(a^7)^3$ **b** $pq^2 \times p^3q^4$ **c** $\dfrac{8ab^5}{2a^4b^4}$

3 Write as powers of 2:

a $\frac{1}{16}$ **b** $2^x \times 4$ **c** $4^x \div 8$

4 Write without brackets or negative indices:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

5 Solve for x without using a calculator:

a $2^{x+1} = 32$ **b** $4^{x+1} = \left(\frac{1}{8}\right)^x$

6 Write as powers of 3:

a 81 **b** 1 **c** $\frac{1}{27}$ **d** $\frac{1}{243}$

7 Write as a single power of 3:

a $\dfrac{27}{9^a}$ **b** $(\sqrt{3})^{1-x} \times 9^{1-2x}$

8 For $y = 3^x - 5$:

a find y when $x = 0, \pm 1, \pm 2$ **b** discuss y as $x \to \infty$ and as $x \to -\infty$

c sketch the graph of $y = 3^x - 5$ **d** state the equation of any asymptote.

9 Without using a calculator, solve for x:

a $27^x = 3$ **b** $9^{1-x} = 27^{x+2}$

10 Solve simultaneously for x and y: $4^x \times 2^y = 16$ and $8^x = 2^{\frac{y}{2}}$.

REVIEW SET 3C

1 Write as a power of 2:

a $\frac{1}{4}$ **b** 32 **c** $\dfrac{1}{\sqrt{2}}$ **d** $\sqrt{8}$

2 Write as a single power of 2:

a $4^a \times 8^b$ **b** $\dfrac{2^{1-x}}{4^{x+2}}$

3 For $y = 3 - 2^{-x}$:

a find y when $x = 0, \pm 1, \pm 2$ **b** discuss y as $x \to \infty$ and as $x \to -\infty$

c sketch the graph of $y = 3 - 2^{-x}$ **d** state the equation of any asymptote.

4 Without using a calculator, solve for x :

a $8^x = \dfrac{1}{\sqrt{2}}$ **b** $4^{2x+1} = \left(\frac{1}{2}\right)^{2-x}$

5 Write without negative indices:

a mn^{-2} **b** $(mn)^{-3}$ **c** $\dfrac{m^2n^{-1}}{p^{-2}}$ **d** $(4m^{-1}n)^2$

6 Write in the form a^x where a is a prime number and x is rational:

a $\sqrt[4]{5}$ **b** $\sqrt{27}$ **c** $\dfrac{1}{\sqrt[3]{16}}$ **d** $\dfrac{1}{\sqrt{125}}$

7 Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{\frac{4}{3}}$ **c** $32^{-\frac{2}{5}}$ **d** $25^{-\frac{3}{2}}$

8 On the same set of axes draw the graphs of:

a $y = 2^x$ and $y = 2^x + 2$ **b** $y = 2^x$ and $y = 2^{x+2}$

9 Sketch the graph of $y = 2^{-x} - 5$.

10 Solve simultaneously for x and y: $3^x \times 3^y = 9$ and $9^x = 3^{y-2}$.

REVIEW SET 3D

1 **a** Write 4×2^n as a power of 2. **b** Evaluate $7^{-1} - 7^0$.

c Write $(\frac{2}{3})^{-3}$ in simplest fractional form.

d Simplify $\left(\dfrac{2a^{-1}}{b^2}\right)^2$. Do not have negative indices or brackets in your answer.

2 **a** Write 288 as a product of prime numbers in index form.

b Simplify $\dfrac{2^{x+1}}{2^{1-x}}$.

3 Write as powers of 5 in simplest form:

a 1 **b** $5\sqrt{5}$ **c** $\dfrac{1}{\sqrt[4]{5}}$ **d** 25^{a+3}

4 Simplify:

a $-(-2)^2$ **b** $(-\frac{1}{2}a^{-3})^2$ **c** $(-3b^{-1})^{-3}$

5 Expand and simplify:

a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

6 Expand and simplify:

a $(3 - 2^a)^2$ **b** $(\sqrt{x} + 2)(\sqrt{x} - 2)$ **c** $2^{-x}(2^{2x} + 2^x)$

7 Solve for x: **a** $6 \times 2^x = 192$ **b** $4 \times (\frac{1}{3})^x = 324$.

8 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t, $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the weight to reduce to 100 grams.

Chapter 4

Logarithms

Contents:

A Introduction
B Logarithms in base 10
Investigation: Discovering the laws of logarithms
C Laws of logarithms
D Exponential equations (using logarithms)
E Growth and decay revisited
F Compound interest revisited
G The change of base rule

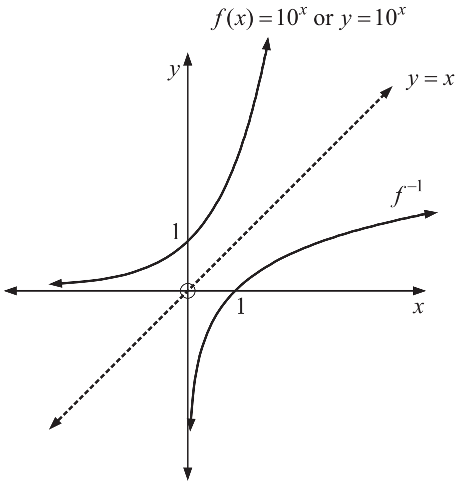
Review set 4A
Review set 4B

A INTRODUCTION

Consider the function $f : x \longmapsto 10^x$.

The defining equation of f is $f(x) = 10^x$ or $y = 10^x$.

Now consider the graph of f and its inverse function f^{-1}.



The question arises:

How can we write f^{-1} in functional form, i.e., what is the defining function of f^{-1}?

As f is defined by $y = 10^x$,
f^{-1} is defined by $x = 10^y$.
{interchanging x and y}

So, y is the exponent to which 10 (the base) is raised in order to get x,
and we write this as $y = \log_{10} x$
and say that "y is the logarithm of x in base 10."

So,
- if $f(x) = 10^x$, then $f^{-1}(x) = \log_{10} x$
- if $f(x) = 2^x$, then $f^{-1}(x) = \log_2 x$
- if $f(x) = b^x$, then $f^{-1}(x) = \log_b x$

LOGARITHMS IN BASE b

In general, if $A = b^n$ we say that n is the logarithm of A, in base b and that $A = b^n \Leftrightarrow n = \log_b A$, $A > 0$.

$A = b^n \Leftrightarrow n = \log_b A$ is a short way of writing
if $A = b^n$ then $n = \log_b A$, and if $n = \log_b A$ then $A = b^n$.

We say that $A = b^n$ and $n = \log_b A$ are equivalent or interchangeable.

For example:
- If $8 = 2^3$ we can immediately say that $3 = \log_2 8$ and vice versa.
- If $\log_5 25 = 2$ we can deduce that $5^2 = 25$ or $25 = 5^2$.

Example 1

a Write an equivalent exponential statement for $\log_{10} 1000 = 3$.
b Write an equivalent logarithmic statement for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 4A

1 Write an equivalent exponential statement for:

a $\log_{10} 10\,000 = 4$ **b** $\log_{10}(0.1) = -1$ **c** $\log_{10} \sqrt{10} = \frac{1}{2}$

d $\log_2 8 = 3$ **e** $\log_2(\frac{1}{4}) = -2$ **f** $\log_3 \sqrt{27} = 1.5$

2 Write an equivalent logarithmic statement for:

a $2^2 = 4$ **b** $2^{-3} = \frac{1}{8}$ **c** $10^{-2} = 0.01$

d $7^2 = 49$ **e** $2^6 = 64$ **f** $3^{-3} = \frac{1}{27}$

Example 2

Find: **a** $\log_{10} 100$ **b** $\log_2 32$ **c** $\log_5(0.2)$

a To find $\log_{10} 100$ we ask "What power must 10 be raised to, to get 100?"
As $10^2 = 100$, then $\log_{10} 100 = 2$.

b As $2^5 = 32$, then $\log_2 32 = 5$.

c As $5^{-1} = \frac{1}{5} = 0.2$, then $\log_5(0.2) = -1$.

3 Find:

a $\log_{10} 100\,000$ **b** $\log_{10}(0.01)$ **c** $\log_3 \sqrt{3}$ **d** $\log_2 8$

e $\log_2 64$ **f** $\log_2 128$ **g** $\log_5 25$ **h** $\log_5 125$

i $\log_2(0.125)$ **j** $\log_9 3$ **k** $\log_4 16$ **l** $\log_{36} 6$

m $\log_3 243$ **n** $\log_2 \sqrt[3]{2}$ **o** $\log_a a^n$ **p** $\log_8 2$

q $\log_t \left(\frac{1}{t}\right)$ **r** $\log_6 6\sqrt{6}$ **s** $\log_4 1$ **t** $\log_9 9$

4 Use your calculator to find:

a $\log_{10} 152$ **b** $\log_{10} 25$ **c** $\log_{10} 74$ **d** $\log_{10} 0.8$

5 Solve for x:

a $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_x 81 = 4$ **d** $\log_2(x - 6) = 3$

Example 3

In question **3o** of this exercise we observed that $\log_a a^n = n$.

Use this result to find: **a** $\log_2 16$ **b** $\log_{10} \sqrt[5]{100}$ **c** $\log_2 \left(\frac{1}{\sqrt{2}}\right)$

a
$$\begin{aligned} &\log_2 16 \\ &= \log_2 2^4 \quad \{\text{as } 16 = 2^4\} \\ &= 4 \end{aligned}$$

b
$$\begin{aligned} &\log_{10} \sqrt[5]{100} \\ &= \log_{10}(10^2)^{\frac{1}{5}} \\ &= \log_{10} 10^{\frac{2}{5}} \\ &= \tfrac{2}{5} \end{aligned}$$

c
$$\begin{aligned} &\log_2 \left(\tfrac{1}{\sqrt{2}}\right) \\ &= \log_2 2^{-\frac{1}{2}} \\ &= -\tfrac{1}{2} \end{aligned}$$

6 Use $\log_a a^n = n$ to find:

a $\log_2 4$ **b** $\log_3\left(\frac{1}{3}\right)$ **c** $\log_{10}(0.001)$ **d** $\log_3\left(\frac{1}{\sqrt{3}}\right)$

e $\log_{10}\sqrt[3]{100}$ **f** $\log_2(2\sqrt{2})$ **g** $\log_5(25\sqrt{5})$ **h** $\log_2\left(\frac{1}{\sqrt[3]{2}}\right)$

B LOGARITHMS IN BASE 10

Many positive numbers can be easily written in the form 10^x.

For example,
$$\begin{aligned} 10\,000 &= 10^4 \\ 1000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \quad \text{etc.} \end{aligned}$$

Also, numbers like $\sqrt{10}$, $10\sqrt{10}$ and $\dfrac{1}{\sqrt[5]{10}}$ can be written in the form 10^x.

$$\sqrt{10} = 10^{\frac{1}{2}} = 10^{0.5} \qquad 10\sqrt{10} = 10^1 \times 10^{0.5} = 10^{1.5} \qquad \frac{1}{\sqrt[5]{10}} = 10^{-\frac{1}{5}} = 10^{-0.2}$$

In fact, all positive numbers can be written in the form 10^x by introducing the concept of **logarithms**.

Definition: The **logarithm** of a positive number, in base 10, is its power of 10.

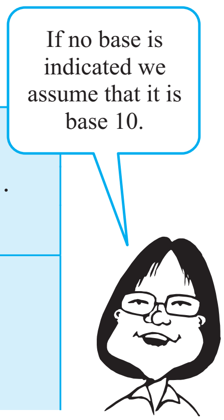
For example:

- Since $1000 = 10^3$, we write $\log_{10} 1000 = 3$ or $\log 1000 = 3$.
- Since $0.01 = 10^{-2}$, we write $\log_{10}(0.01) = -2$ or $\log(0.01) = -2$.

In algebraic form, $\boldsymbol{a = 10^{\log a}}$ for any $a > 0$.

Notice also that $\log 1000 = \log 10^3 = 3$ and $\log 0.01 = \log 10^{-2} = -2$

give us the useful alternative $\boldsymbol{\log 10^x = x}$



Example 4

a Without using a calculator, find: **i** $\log 100$ **ii** $\log(\sqrt[4]{10})$.

b What calculator steps can be used to check your answers in **a**?

a **i**
$$\begin{aligned} &\log 100 \\ &= \log 10^2 \\ &= 2 \end{aligned}$$

ii
$$\begin{aligned} &\log(\sqrt[4]{10}) \\ &= \log(10^{\frac{1}{4}}) \\ &= \tfrac{1}{4} \qquad \{\log 10^x = x\} \end{aligned}$$

b **i** press $\boxed{\log}$ 100 $\boxed{)}$ $\boxed{\text{ENTER}}$ *Answer:* 2

ii press $\boxed{\log}$ 10 $\boxed{\wedge}$ 0.25 $\boxed{)}$ $\boxed{\text{ENTER}}$ *Answer:* 0.25

EXERCISE 4B

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log 10$ **d** $\log 1$

e $\log \sqrt{10}$ **f** $\log(\sqrt[3]{10})$ **g** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\log 10\sqrt{10}$

i $\log \sqrt[3]{100}$ **j** $\log\left(\frac{100}{\sqrt{10}}\right)$ **k** $\log(10 \times \sqrt[3]{10})$ **l** $\log 1000\sqrt{10}$

m $\log 10^n$ **n** $\log(10^a \times 100)$ **o** $\log\left(\dfrac{10}{10^m}\right)$ **p** $\log\left(\dfrac{10^a}{10^b}\right)$

2 Find using a calculator:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log \sqrt{10}$ **d** $\log \sqrt[3]{10}$

e $\log \sqrt[3]{100}$ **f** $\log 10\sqrt{10}$ **g** $\log\left(\frac{1}{\sqrt{10}}\right)$ **h** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$

Example 5

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places: **a** 8 **b** 800 **c** 0.08

a
$$\begin{aligned} &8 \\ &= 10^{\log 8} \\ &\doteqdot 10^{0.9031} \end{aligned}$$
{ $\boxed{\log}$ 8 $\boxed{\text{ENTER}}$ }

b
$$\begin{aligned} &800 \\ &= 10^{\log 800} \\ &\doteqdot 10^{2.9031} \end{aligned}$$
{ $\boxed{\log}$ 800 $\boxed{\text{ENTER}}$ }

c
$$\begin{aligned} &0.08 \\ &= 10^{\log 0.08} \\ &\doteqdot 10^{-1.0969} \end{aligned}$$
{ $\boxed{\log}$ 0.08 $\boxed{\text{ENTER}}$ }

3 Use your calculator to write these in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 6

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a **i** $\log 2 \doteqdot 0.3010$

ii $\log 20 = 1.3010$ {calculator}

b
$$\begin{aligned} \log 20 &= \log(2 \times 10) \\ &\doteqdot \log(10^{0.3010} \times 10^1) \\ &\doteqdot \log 10^{1.3010} \quad \text{\{adding indices\}} \\ &\doteqdot 1.3010 \\ &\doteqdot \log 2 + 1 \end{aligned}$$

4 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$
b Explain why $\log 300 = \log 3 + 2$.

5 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log 0.05$
b Explain why $\log 0.05 = \log 5 - 2$.

Example 7

Find x if: **a** $\log x = 3$ **b** $\log x \doteqdot -0.271$

a As $x = 10^{\log x}$
$\therefore\ x = 10^3$
$\therefore\ x = 1000$

b As $x = 10^{\log x}$
$\therefore\ x \doteqdot 10^{-0.271}$
$\therefore\ x \doteqdot 0.536$

{Use $\boxed{\text{2nd}}$ $\boxed{10^x}$ $\boxed{(-)}$ 0.271 $\boxed{)}$ $\boxed{\text{ENTER}}$ }

6 Find x if:

a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$
d $\log x = -1$ **e** $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$
g $\log x = 4$ **h** $\log x = -5$ **i** $\log x \doteqdot 0.8351$
j $\log x \doteqdot 2.1457$ **k** $\log x \doteqdot -1.378$ **l** $\log x \doteqdot -3.1997$

INVESTIGATION DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 Use your calculator to find

a $\log 2 + \log 3$ **b** $\log 3 + \log 7$ **c** $\log 4 + \log 20$
d $\log 6$ **e** $\log 21$ **f** $\log 80$

From your answers, suggest a possible simplification for $\log a + \log b$.

2 Use your calculator to find

a $\log 6 - \log 2$ **b** $\log 12 - \log 3$ **c** $\log 3 - \log 5$
d $\log 3$ **e** $\log 4$ **f** $\log(0.6)$

From your answers, suggest a possible simplification for $\log a - \log b$.

3 Use your calculator to find

a $3 \log 2$ **b** $2 \log 5$ **c** $-4 \log 3$
d $\log(2^3)$ **e** $\log(5^2)$ **f** $\log(3^{-4})$

From your answers, suggest a possible simplification for $n \log a$.

C LAWS OF LOGARITHMS

There are 3 important **laws of logarithms**.

$$\log A + \log B = \log(AB)$$
$$\log A - \log B = \log\left(\frac{A}{B}\right), \quad B \neq 0$$
$$n \log A = \log(A^n)$$

These laws are easily established using index laws:

Since $A = 10^{\log A}$ and $B = 10^{\log B}$

- $AB = 10^{\log A} \times 10^{\log B} = 10^{\log A + \log B}$.
  But, $AB = 10^{\log(AB)}$
  $\therefore \log A + \log B = \log(AB)$.
- $\dfrac{A}{B} = \dfrac{10^{\log A}}{10^{\log B}} = 10^{\log A - \log B}$.
  But, $\dfrac{A}{B} = 10^{\log(\frac{A}{B})}$
  $\therefore \log A - \log B = \log\left(\dfrac{A}{B}\right)$.
- $A^n = (10^{\log A})^n = 10^{n \log A}$.
  But, $A^n = 10^{\log(A^n)}$
  $\therefore n \log A = \log(A^n)$.

Example 8

Use the laws of logarithms to write the following as a single logarithm:

a $\log 5 + \log 3$ **b** $\log 24 - \log 8$ **c** $\log 5 - 1$

a $\log 5 + \log 3$
$= \log(5 \times 3)$
$= \log 15$

b $\log 24 - \log 8$
$= \log\left(\frac{24}{8}\right)$
$= \log 3$

c $\log 5 - 1$
$= \log 5 - \log 10^1$
$= \log\left(\frac{5}{10}\right)$
$= \log\left(\frac{1}{2}\right)$

EXERCISE 4C

1 Write as a single logarithm:

a $\log 8 + \log 2$ **b** $\log 8 - \log 2$ **c** $\log 40 - \log 5$
d $\log 4 + \log 5$ **e** $\log 5 + \log(0.4)$ **f** $\log 2 + \log 3 + \log 4$
g $1 + \log 3$ **h** $\log 4 - 1$ **i** $\log 5 + \log 4 - \log 2$
j $2 + \log 2$ **k** $\log 40 - 2$ **l** $\log 6 - \log 2 - \log 3$
m $\log 50 - 4$ **n** $3 - \log 50$ **o** $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

Example 9

Use the laws of logarithms to simplify:

a $2\log 7 - 3\log 2$ **b** $2\log 3 - 1$

a $2\log 7 - 3\log 2$
$= \log(7^2) - \log(2^3)$
$= \log 49 - \log 8$
$= \log\left(\frac{49}{8}\right)$

b $2\log 3 - 1$
$= \log(3^2) - \log 10^1$
$= \log 9 - \log 10$
$= \log(0.9)$

2 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$

d $2\log 5 - 3\log 2$ **e** $\frac{1}{2}\log 4 + \log 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$

g $3 - \log 2 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log 4 - \log 5$

Example 10

Simplify, without using a calculator:

$$\frac{\log 8}{\log 4}$$

$$\frac{\log 8}{\log 4} = \frac{\log 2^3}{\log 2^2} = \frac{3\cancel{\log 2}}{2\cancel{\log 2}} = \frac{3}{2}$$

3 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log 27}{\log 9}$ **c** $\dfrac{\log 8}{\log 2}$

d $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log 25}{\log(0.2)}$ **f** $\dfrac{\log 8}{\log(0.25)}$

Check your answers using a calculator.

Example 11

Show that: **a** $\log\left(\frac{1}{9}\right) = -2\log 3$ **b** $\log 500 = 3 - \log 2$

a $\log\left(\frac{1}{9}\right) = \log(3^{-2})$
$= -2\log 3$

b $\log 500 = \log\left(\dfrac{1000}{2}\right)$
$= \log 1000 - \log 2$
$= \log 10^3 - \log 2$
$= 3 - \log 2$

4 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

Example 12

Write the following as logarithmic equations (in base 10):

a $y = a^2 b$ **b** $y = \dfrac{a}{b^3}$ **c** $P = \dfrac{20}{\sqrt{n}}$

a
$$\begin{aligned} y &= a^2 b \\ \therefore\quad \log y &= \log(a^2 b) \qquad \{\text{finding logs of both sides}\} \\ \therefore\quad \log y &= \log a^2 + \log b \\ \therefore\quad \log y &= 2\log a + \log b \end{aligned}$$

b
$$\begin{aligned} y &= \frac{a}{b^3} \\ \therefore\quad \log y &= \log\left(\frac{a}{b^3}\right) \qquad \{\text{finding logs of both sides}\} \\ \therefore\quad \log y &= \log a - \log b^3 \\ \therefore\quad \log y &= \log a - 3\log b \end{aligned}$$

c
$$\begin{aligned} P &= \left(\frac{20}{\sqrt{n}}\right) \\ \therefore\quad \log P &= \log\left(\frac{20}{\sqrt{n}}\right) \qquad \{\text{finding logs of both sides}\} \\ \therefore\quad \log P &= \log 20 - \log\sqrt{n} \end{aligned}$$

5 Write the following as logarithmic equations (in base 10):

a $y = 2^x$ **b** $y = 20b^3$ **c** $M = ad^4$ **d** $T = 5\sqrt{d}$

e $R = b\sqrt{l}$ **f** $Q = \dfrac{a}{b^n}$ **g** $y = ab^x$ **h** $F = \dfrac{20}{\sqrt{n}}$

i $L = \dfrac{ab}{c}$ **j** $N = \sqrt{\dfrac{a}{b}}$ **k** $S = 200 \times 2^t$ **l** $y = \dfrac{a^m}{b^n}$

Example 13

Write the following equations without logarithms:

a $\log A = \log b + 2\log c$ **b** $\log M = 3\log a - 1$

a
$$\begin{aligned} \log A &= \log b + 2\log c \\ \therefore\quad \log A &= \log b + \log c^2 \\ \therefore\quad \log A &= \log(bc^2) \\ \therefore\quad A &= bc^2 \end{aligned}$$

b
$$\begin{aligned} \log M &= 3\log a - 1 \\ \therefore\quad \log M &= \log a^3 - \log 10^1 \\ \therefore\quad \log M &= \log\left(\frac{a^3}{10}\right) \\ \therefore\quad M &= \frac{a^3}{10} \end{aligned}$$

6 Write the following equations without logarithms:

a $\log D = \log e + \log 2$
b $\log F = \log 5 - \log t$
c $\log P = \frac{1}{2}\log x$
d $\log M = 2\log b + \log c$
e $\log B = 3\log m - 2\log n$
f $\log N = -\frac{1}{3}\log p$
g $\log P = 3\log x + 1$
h $\log Q = 2 - \log x$

7 If $p = \log_b 2$, $q = \log_b 3$ and $r = \log_b 5$, write the following in terms of p and/or q and/or r:

a $\log_b 6$
b $\log_b 108$
c $\log_b 45$
d $\log_b \left(\frac{5\sqrt{3}}{2}\right)$
e $\log_b \left(\frac{5}{32}\right)$
f $\log_b (0.\overline{2})$

8 If $\log_2 P = x$, $\log_2 Q = y$ and $\log_2 R = z$, write the following in terms of x and/or y and/or z:

a $\log_2 (PR)$
b $\log_2 (RQ^2)$
c $\log_2 \left(\frac{PR}{Q}\right)$
d $\log_2 (P^2\sqrt{Q})$
e $\log_2 \left(\frac{Q^3}{\sqrt{R}}\right)$
f $\log_2 \left(\frac{R^2\sqrt{Q}}{P^3}\right)$

9 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$ find:

a $\log_t N$
b $\log_t (MN)$
c $\log_t \left(\frac{N^2}{\sqrt{M}}\right)$

10 Solve for x:

a $\log_3 27 + \log_3 (\frac{1}{3}) = \log_3 x$
b $\log_5 x = \log_5 8 - \log_5 (6 - x)$
c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$
d $\log_{20} x = 1 + \log_{20} 10$
e $\log x + \log(x + 1) = \log 30$
f $\log(x + 2) - \log(x - 2) = \log 5$

D EXPONENTIAL EQUATIONS (USING LOGARITHMS)

In earlier exercises we found solutions to simple exponential equations by equating indices after creating equal bases. However, when the bases cannot easily be made the same we find solutions using logarithms.

Example 14

Solve for x: $2^x = 30$, giving your answer to 3 significant figures.

$2^x = 30$

$\therefore \log 2^x = \log 30$ {find the logarithm of each side}

$\therefore x \log 2 = \log 30$ $\{\log a^n = n \log a\}$

$\therefore x = \dfrac{\log 30}{\log 2}$

$\therefore x \doteqdot 4.91$ (3 s.f.) {*Press:* **log** 30 **)** **÷** **log** 2 **)** **ENTER** }

EXERCISE 4D

1 Solve for x, giving your answer correct to 3 significant figures:

a $2^x = 10$ **b** $3^x = 20$ **c** $4^x = 100$

d $(1.2)^x = 1000$ **e** $2^x = 0.08$ **f** $3^x = 0.000\,25$

g $(\frac{1}{2})^x = 0.005$ **h** $(\frac{3}{4})^x = 10^{-4}$ **i** $(0.99)^x = 0.000\,01$

Example 15

Solve for t (to 3 s.f.) given that $200 \times 2^{0.04t} = 6$.

$$\begin{aligned} 200 \times 2^{0.04t} &= 6 \\ \therefore\ 2^{0.04t} &= \tfrac{6}{200} && \{\text{dividing both sides by } 200\} \\ \therefore\ 2^{0.04t} &= 0.03 \\ \therefore\ \log 2^{0.04t} &= \log 0.03 && \{\text{find the logarithm of each side}\} \\ \therefore\ 0.04t \times \log 2 &= \log 0.03 \\ \therefore\ t &= \frac{\log 0.03}{0.04 \times \log 2} \doteq -127 \quad \text{(3 s.f.)} \end{aligned}$$

{*Press:* log 0.03) ÷ (0.04 × log 2)) ENTER }

2 Find, correct to 3 s.f., the solution to:

a $200 \times 2^{0.25t} = 600$ **b** $20 \times 2^{0.06t} = 450$ **c** $30 \times 3^{-0.25t} = 3$

d $12 \times 2^{-0.05t} = 0.12$ **e** $50 \times 5^{-0.02t} = 1$ **f** $300 \times 2^{0.005t} = 1000$

E GROWTH AND DECAY REVISITED

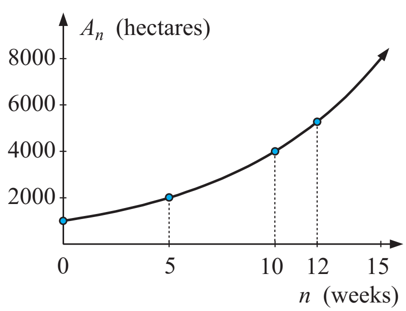
Earlier we considered growth and decay problems in which we were required to find the value of the dependent variable for a given value of the independent variable.

For example:

The grasshopper problem where the area of infestation was given by $A_n = 1000 \times 2^{0.2n}$ hectares (n is the number of weeks after initial observation).

We found A when $n = 0$, 5, 10 and 12 and drew a graph of the growth in area.

In this section we will consider the **reverse problem** of finding n (the independent variable) given values of A_n (the dependent variable).



We can do this by:

- reading from an accurate **graph** to get approximate solutions
- using **logarithms** to solve the appropriate equation
- using **technology** in the form of a graphics calculator or computer graphing package.

Example 16

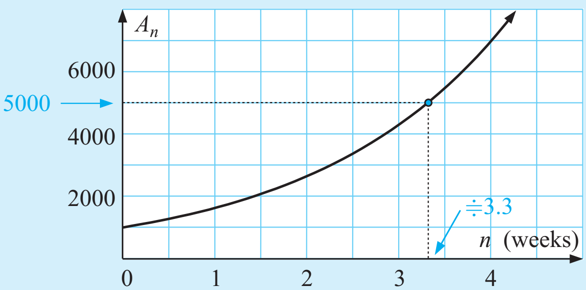
An entomologist, monitoring a grasshopper plague, notices that the area affected by the grasshoppers is given by $A_n = 100 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the infested area to reach 5000 ha.

b Find the answer to **a** using logarithms.

c Check your answer to **b** using suitable technology.

a



b When $A_n = 5000$,

$$1000 \times 2^{0.7n} = 5000$$
$$\therefore \quad 2^{0.7n} = 5$$
$$\therefore \quad \log 2^{0.7n} = \log 5$$
$$\therefore \quad 0.7n \log 2 = \log 5$$
$$\therefore \quad n = \frac{\log 5}{(0.7 \times \log 2)}$$
$$\therefore \quad n \doteqdot 3.32$$

i.e., it takes about 3 weeks and 2 more days.

c Go to the graphing package or graphics calculator icon to find the intersection of $y = 1000 \times 2^{0.7x}$ and $y = 5000$. $(n \doteqdot 3.32)$

As graphing by hand is rather tedious we will use logarithms and/or technology to solve problems of this kind.

TI
C
GRAPHING PACKAGE

EXERCISE 4E

1 The weight W_t grams, of bacteria in a culture t hours after establishment is given by $W_t = 20 \times 2^{0.15t}$. Find the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

2 The temperature $T(^o\text{C})$, of a liquid which has been placed in a refrigerator is given by $T = 100 \times 2^{-0.03t}$ where t is the time in minutes. Find the time required for the temperature to reach:

a 25^oC **b** 1^oC.

3 The weight W_t grams, of radioactive substance remaining after t years is given by $W_t = 1000 \times 2^{-0.04t}$ grams. Find the time taken for the weight to:

a halve **b** reach 20 grams **c** reach 1% of its original value.

Example 17

The weight of radioactive material remaining after t years is given by $W = W_0 \times 2^{-0.001t}$ grams. Find:

a the original weight

b how long it would take for the material to decay to a 'safe level' of 1% of its original value.

a When $t = 0$, $W = W_0 \times 2^0 = W_0$

So, W_0 is the original weight.

b For $W = 1\%$ of the original weight

$$\begin{aligned} \cancel{W_0} \times 2^{-0.001t} &= \tfrac{1}{100} \text{ of } \cancel{W_0} \\ \therefore \quad 2^{-0.001t} &= 0.01 \\ \therefore \quad \log 2^{-0.001t} &= \log 0.01 \\ \text{i.e.,} \quad -0.001t \log 2 &= \log 0.01 \\ \therefore \quad t &= \frac{\log 0.01}{(-0.001 \times \log 2)} \\ \therefore \quad t &\doteqdot 6644 \end{aligned}$$

$\therefore$ it would take 6644 years.

4 The weight W grams, of radioactive uranium remaining after t years is given by the formula $W = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find the time taken for the original weight to fall to:

a 25% of its original value **b** 0.1% of its original value.

5 The speed V, of a chemical reaction is given by $V = V_0 \times 2^{0.1t}$ where t is the temperature in ^{o}C. Find the temperature at which the speed is three times as fast as it was at 0^oC.

6 The current I amps, flowing in a transistor radio, t seconds after it is switched off is given by $I = I_0 \times 2^{-0.02t}$ amps. Find the time taken for the current to drop to 10% of its original value.

7 A man jumps from the basket of a stationary balloon and his speed of descent is given by $V = 50(1 - 2^{-0.2t})$ m/s where t is the time in seconds.

Find the time taken for his speed to reach 40 m/s.

F COMPOUND INTEREST REVISITED

Recall that $u_{n+1} = u_1 \times r^n$ is used to find the eventual value of an investment of u_1 at a rate of r% each compounding period for n periods. In order to find n, the **period** of the investment, we need to use **logarithms**.

Example 18

Iryna has \$5000 to invest in an account that pays 5.2% p.a. interest compounded annually. How long will it take for the value of her investment to reach \$20 000?

$u_{n+1} = 20\,000$ after n years
$u_1 = 5000$
$r = 105.2\% = 1.052$

Now $u_{n+1} = u_1 \times r^n$

$$\therefore \quad 20\,000 = 5000 \times (1.052)^n$$
$$\therefore \quad (1.052)^n = 4$$
$$\therefore \quad \log(1.052)^n = \log 4$$
$$\therefore \quad n \times \log 1.052 = \log 4$$
$$\therefore \quad n = \frac{\log 4}{\log 1.052} \doteqdot 27.3 \text{ years}$$

i.e., it will take at least 28 years.

EXERCISE 4F

1 A house is expected to increase in value at an average rate of 7.5% p.a. How long will it take for a \$160 000 house to be worth \$250 000?

2 Thabo has \$10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to \$15 000?

3 Dien invests \$15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches \$25 000, at which time he plans to travel. The formula $u_{n+1} = u_1 \times r^n$ can be used to calculate the time needed. $u_{n+1} = 25\,000$ after n months.

a Explain why $r = 1.007$. **b** After how many months can he withdraw the money?

G THE CHANGE OF BASE RULE

Let $\log_b A = x$, then $b^x = A$

$$\therefore \quad \log_c b^x = \log_c A \quad \{\text{taking logarithms in base } c\}$$
$$\therefore \quad x \log_c b = \log_c A \quad \{\text{power law of logarithms}\}$$
$$\therefore \quad x = \frac{\log_c A}{\log_c b}$$

So, $\boldsymbol{\log_b A = \dfrac{\log_c A}{\log_c b}}$

Example 19

Find $\log_2 9$ by: **a** letting $\log_2 9 = x$
b using the rule $\log_b A = \dfrac{\log_c A}{\log_c b}$ with $c = 10$.

a Let $\log_2 9 = x$
$$\therefore \quad 9 = 2^x$$
$$\therefore \quad \log 2^x = \log 9$$
$$\therefore \quad x \log 2 = \log 9$$
$$\therefore \quad x = \frac{\log 9}{\log 2} \doteqdot 3.17$$

b $$\log_2 9 = \frac{\log_{10} 9}{\log_{10} 2} \doteqdot 3.17$$

EXERCISE 4G

1 Use the rule $\log_b A = \dfrac{\log_{10} A}{\log_{10} b}$ to find correct to 3 significant figures:

a $\log_3 12$ **b** $\log_{\frac{1}{2}} 1250$ **c** $\log_3(0.067)$ **d** $\log_{0.4}(0.006\,984)$

2 Use the rule $\log_b A = \dfrac{\log_{10} A}{\log_{10} b}$ to solve, correct to 3 significant figures:

a $2^x = 0.051$ **b** $4^x = 213.8$ **c** $3^{2x+1} = 4.069$

Hint: In **a** $2^x = 0.051$ implies that $x = \log_2(0.051)$.

Example 20

Solve for x: $8^x - 5 \times 4^x = 0$

$$\begin{aligned} 8^x - 5 \times 4^x &= 0 \\ \therefore\ 2^{3x} - 5 \times 2^{2x} &= 0 \\ \therefore\ 2^{2x}(2^x - 5) &= 0 \\ \therefore\ 2^x &= 5 \quad \{\text{as } 2^{2x} > 0 \text{ for all } x\} \\ \therefore\ x &= \log_2 5 \\ \therefore\ x &= \frac{\log 5}{\log 2} \doteqdot 2.32 \end{aligned}$$

3 Solve for x:

a $25^x - 3 \times 5^x = 0$ **b** $8 \times 9^x - 3^x = 0$

4 Solve for x:

a $\log_4 x^3 + \log_2 \sqrt{x} = 8$ **b** $\log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x}$

5 Find the exact value of x for which $4^x \times 5^{4x+3} = 10^{2x+3}$

REVIEW SET 4A

1 Find the following *without* using a calculator. Show all working.

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_6 6$ **g** $\log_{81} 3$ **h** $\log_9(0.\overline{1})$

i $\log_{27} 3$ **j** $\log_k \sqrt{k}$ **k** $\log_m \sqrt{m^5}$ **l** $\log_n \left(\dfrac{1}{n^2}\right)$

2 Without using a calculator, find the logarithms of:

a $\sqrt{10}$ **b** $\dfrac{1}{\sqrt[3]{10}}$ **c** $10^a \times 10^{b+1}$

3 Write in the form 10^x:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

4 Find x if:

a $\log x = -3$ **b** $\log x \doteqdot 2.743$ **c** $\log x \doteqdot -3.145$

5 Write as a single logarithm:

a $\log 4 + 2\log 5$ **b** $\log 24 + \log 4 - \log 16$ **c** $2 - \log 25$

6 Write as logarithmic equations:

a $P = 3 \times b^x$ **b** $m = \dfrac{n^3}{p^2}$

7 Write the following equations without logarithms:

a $\log k \doteqdot 1.699 + x$ **b** $\log Q = 3\log P + \log R$ **c** $\log A \doteqdot 5\log B - 2.602$

8 Solve for x, giving your answer correct to 3 significant figures:

a $5^x = 7$ **b** $20 \times 2^{2x+1} = 500$

9 The weight of radioactive substance after t years is $W_t = 2500 \times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight.

b Find the time taken for the weight to reduce to 30% of its original value.

c Find the percentage weight loss after 1500 years.

d Sketch the graph of W_t against t.

REVIEW SET 4B

1 Without using a calculator, find the logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Use your calculator to write as powers of 10:

a 200 **b** $568\,000$ **c** 3.69×10^{-4}

3 Solve for x:

a $\log x = 3$ **b** $\log(x+2) = 1.732$ **c** $\log(\frac{x}{10}) = -0.671$

4 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log 16 - 2\log 3$ **c** $2 + \log 5$

5 Write as logarithmic equations:

a $M = ab^n$ **b** $T = \dfrac{5}{\sqrt{l}}$ **c** $G = \dfrac{a^2 b}{c}$

6 Write the following equations without logarithms:

a $\log T = 2\log x - \log y$ **b** $\log K = \log n + \frac{1}{2}\log t$

7 Solve for x: **a** $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

8 If $A = \log 2$ and $B = \log 3$, write the following in terms of A and B:

a $\log 36$ **b** $\log 54$ **c** $\log(8\sqrt{3})$ **d** $\log(20.25)$ **e** $\log(0.\dot{8})$

9 A population of seals after t years is given by $P_n = P_0\, 2^{\frac{t}{3}}$, $t \geqslant 0$. Find:

a the time taken for the population to double

b the percentage increase during the first 4 years

c the time taken for the population to increase by 200%.

Chapter 5

Natural logarithms

Contents:

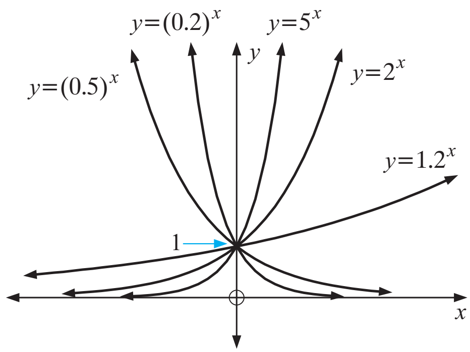
A Introduction
Investigation 1: e occurs naturally
Investigation 2: Continuous compound interest
B Natural logarithms
Investigation 3: The laws of natural logarithms
C Laws of natural logarithms
D Exponential equations involving e
E Growth and decay revisited
F Inverse functions revisited

Review set 5A
Review set 5B

A INTRODUCTION

The simplest exponential functions are of the form $f(x) = a^x$ where a is any positive number, $a \neq 1$.

Below are some examples of the graphs of simple exponential functions.



Note:

All members of the family $f(x) = a^x$ $(a > 0,\ a \neq 1)$ have graphs which

- pass through the point (0, 1)
- are above the x-axis for all values of x
- are asymptotic to the x-axis.

This means that as x gets large (positively or negatively) the graph gets closer and closer to the x-axis.

We can see from the graphs that a^x is positive for all x.

So, there are a vast number of possible choices for the base number.

However, in all branches of science, engineering, sociology, etc. where exponential data is being examined, the base e where $e \doteqdot 2.7183$ is commonly used.

We will examine the function $f(x) = e^x$ where $e \doteqdot 2.7183$ in detail.

The inverse function of $f(x) = e^x$ is $f^{-1}(x) = \log_e x$ also written $\ln x$.

"Where does e come from?" is a reasonable question to ask.

INVESTIGATION 1 — e OCCURS NATURALLY

Suppose $\$u_1$ is invested at a fixed rate of 10% p.a. for 10 years.

If *one* interest payment is made each year, then using $u_{n+1} = u_1 r^n$ where r is the rate per period and n is the number of periods, the investment will be worth $\$u_{11}$ after 10 years and $u_{11} = u_1(1.1)^{10} \doteqdot u_1 \times 2.593\,742$ {we are multiplying by 1.1 which equals 110%}

If 10 interest payments are made each year, then $u_{11} = u_1(1.01)^{100} \doteqdot u_1 \times 2.704\,814$

What to do:

1 Calculate u_{11} for
- **a** 100 interest payments per year $\{r = 1.001\}$
- **b** 1000 interest payments per year
- **c** 10 000 interest payments per year
- **d** 100 000 interest payments per year
- **e** 1 000 000 interest payments per year

2 If 1 000 000 interest payments are made each year, how frequently does this occur?

3 Comment on your results to **1** above

A more difficult but worthwhile investigation on 'where e comes from' follows:

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A formula for calculating the amount to which an investment grows is given by $u_n = u_0(1+i)^n$ where

u_n is the **final amount**
u_0 is the **initial amount**
i is the **interest rate per compounding period**
n is the **number of periods**
(i.e., the number of times the interest is compounded).

We are to investigate the final value of an investment for various values of n, and allow n to get extremely large.

What to do:

1 Suppose \$1000 is invested for 4 years at a fixed rate of 6% p.a. Use your calculator to find the final amount (sometimes called the *maturing value*) if the interest is paid:

a annually (once a year and so $n = 4$, $i = 6\% = 0.06$)

b quarterly (four times a year and so $n = 4 \times 4 = 16$ and $i = \frac{6\%}{4} = 0.015$)

c monthly **d** daily **e** by the second **f** by the millisecond.

2 Comment on your answers obtained in **1**.

3 If r is the percentage rate per year
t is the number of years
N is the number of interest payments per year, then $i = \dfrac{r}{N}$ and $n = Nt$.

This means that the growth formula becomes $u_n = u_0\left(1+\dfrac{r}{N}\right)^{Nt}$

a Show that $u_n = u_0\left(1+\dfrac{1}{\frac{N}{r}}\right)^{\frac{N}{r}\times rt}$.

b Now let $\dfrac{N}{r} = a$. Show that $u_n = u_0\left[\left(1+\dfrac{1}{a}\right)^a\right]^{rt}$.

4 For continuous compound growth, the number of interest payments per year, N, gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table:

Give answers as accurately as technology permits.

a	$\left(1+\dfrac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
⋮	

5 You should have discovered that for very large a values,

$$\left(1+\frac{1}{a}\right)^a \doteqdot 2.718\,281\,828\,235......$$

6 Now use the $\boxed{e^x}$ key of your calculator to find the value of e^1,

i.e., press 1 $\boxed{e^x}$ $\boxed{=}$ or $\boxed{e^x}$ 1 $\boxed{=}$. What do you notice?

7 For continuous growth, we have shown that:

$u_n = u_0 e^{rt}$ where u_0 is the initial amount
r is the annual percentage rate
t is the number of years

Use this formula to find the amount which an investment of \$1000 for 4 years at a fixed rate of 6% p.a., will reach if the interest is calculated continuously (instantaneously).

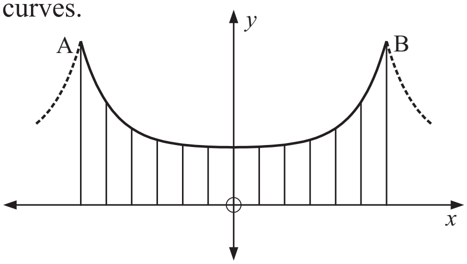
From **Investigation 2** we observed that:

"If interest is paid continuously (instantaneously) then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate p.a. and t is the number of years."

e is also found in the formulae for many important curves.

For example, a **catenary curve** is the shape of a wire or rope between two pole ends A and B.

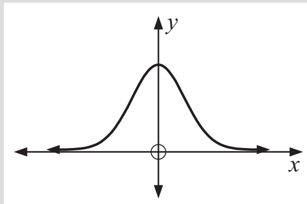
Its equation is of the form $y = k\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ where k and a are constants.



RESEARCH — RESEARCHING e

What to do:

1 The 'bell curve' which models statistical distributions is shown alongside. Research the equation of this curve.



2 $e^{i\pi}+1=0$ is called **Euler's equation** where $i=\sqrt{-1}$. Research the significance of this equation.

3 The series $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times 3}x^3 + \frac{1}{2\times 3\times 4}x^4 + ...$ has infinitely many terms.

It has been shown that $f(x) = e^x$.
Check this statement by finding an approximation for $f(1)$ using its first 20 terms.

EXERCISE 5A

1 Use the $\boxed{e^x}$ key of your calculator to find the approximate value of e to as many digits as are possible.

2 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$. Comment on any observations.

GRAPHING PACKAGE

3 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

4 For the general exponential function $y = ae^{kx}$, what is the y-intercept?

5 Consider $y = 2e^x$.

a Explain why y can never be negative.

b Find y if: **i** $x = -20$ **ii** $x = 20$.

6 Find to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

7 Write the following as powers of e:

a $\sqrt{e}$ **b** $e\sqrt{e}$ **c** $\dfrac{1}{\sqrt{e}}$ **d** $\dfrac{1}{e^2}$

8 Simplify:

a $(e^{0.36})^{\frac{t}{2}}$ **b** $(e^{0.064})^{\frac{t}{16}}$ **c** $(e^{-0.04})^{\frac{t}{8}}$ **d** $(e^{-0.836})^{\frac{t}{5}}$

9 Find, to five significant figures, the values of:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$

e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

10 On the same set of axes, sketch and clearly label the graphs of:

$f: x \longmapsto e^x$, $g: x \longmapsto e^{x-2}$, $h: x \longmapsto e^x + 3$

State the domain and range of each function.

11 On the same set of axes, sketch and clearly label the graphs of:

$f: x \longmapsto e^x$, $g: x \longmapsto -e^x$, $h: x \longmapsto 10 - e^x$

State the domain and range of each function.

12 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a What is the weight of the culture at:

i $t = 0$ **ii** $t = 30$ min **iii** $t = 1\frac{1}{2}$ hours **iv** $t = 6$ hours?

b Use **a** to sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

13 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.

a What current is still flowing in the circuit after: **i** $t = 1$ sec **ii** $t = 10$ sec?

b Use your graphics calculator to sketch $I(t) = 75e^{-0.15t}$ and $I = 1$.

c Find how long it would take for the current to fall to 1 amp.

14 Given $f: x \longmapsto e^x$

a find the defining equation of f^{-1}.

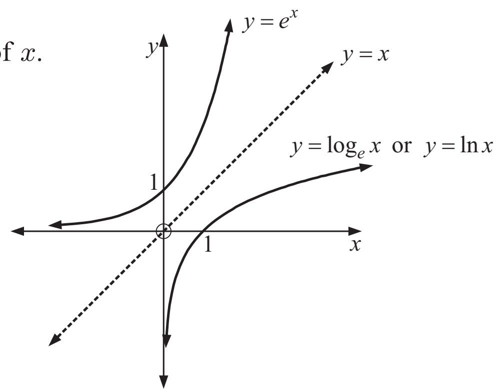
b Sketch the graphs of $y = e^x$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

B NATURAL LOGARITHMS

If f is the exponential function $x \mapsto e^x$ (i.e., $f(x) = e^x$ or $y = e^x$) then its inverse function, f^{-1} is $x = e^y$ or $y = \log_e x$.

So, $y = \log_e x$ is the reflection of $y = e^x$ in the mirror line $y = x$.

Notation: $\ln x$ is used to represent $\log_e x$
$\ln x$ is called the natural logarithm of x.



Note:

$\ln 1 = 0$ {as $1 = e^0$}

$\ln e = 1$ {as $e = e^1$}

$\ln e^2 = 2$

$\ln \sqrt{e} = \frac{1}{2}$ {as $\sqrt{e} = e^{\frac{1}{2}}$}

$\ln \left(\frac{1}{e}\right) = -1$ {as $\frac{1}{e} = e^{-1}$}

In general, $\boldsymbol{\ln e^x = x}$.

EXERCISE 5B

1 Without using a calculator find:

a $\ln e^3$ **b** $\ln 1$ **c** $\ln \sqrt[3]{e}$ **d** $\ln \left(\frac{1}{e^2}\right)$

2 Check your answers to question **1** using a calculator.

3 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

4 Use a calculator to find:

a $\ln 2$ **b** $\ln 10$ **c** $\ln 2158$ **d** $\ln 0.001\,677$

5 Simplify:

a $\ln e^a$ **b** $\ln(e \times e^a)$ **c** $\ln \left(\frac{e^n}{e}\right)$

d $\ln \left(e^a \times e^b\right)$ **e** $\ln(e^a)^b$ **f** $\ln \left(\frac{e^a}{e^b}\right)$

Example 1

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places: **a** 50 **b** 0.005

a 50
$= e^{\ln 50}$ {using $a = e^{\ln a}$}
$\doteqdot e^{3.9120}$

b 0.005
$= e^{\ln 0.005}$
$\doteqdot e^{-5.2983}$

6 Use your calculator to write the following in the form e^x where x is correct to 4 decimal places:

a	6	**b**	60	**c**	6000	**d**	0.6	**e**	0.006
f	15	**g**	1500	**h**	1.5	**i**	0.15	**j**	0.000 15

Example 2

Find x if: **a** $\ln x = 2.17$ **b** $\ln x = -0.384$

a Since $\ln x = 2.17$

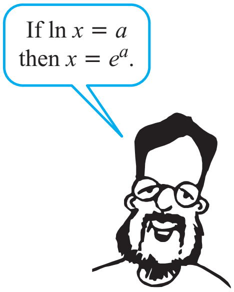
then $x = e^{2.17}$

$\therefore$ $x \doteqdot 8.76$ (to 3 s.f.)

b Since $\ln x = -0.384$

then $x = e^{-0.384}$

$\therefore$ $x \doteqdot 0.681$ (to 3 s.f.)



7 Find x if:

a	$\ln x = 3$	**b**	$\ln x = 1$	**c**	$\ln x = 0$	**d**	$\ln x = -1$
e	$\ln x = -5$	**f**	$\ln x \doteqdot 0.835$	**g**	$\ln x \doteqdot 2.145$	**h**	$\ln x \doteqdot -3.2971$

INVESTIGATION 3 THE LAWS OF NATURAL LOGARITHMS

What to do:

1 Use your calculator to find:

a	$\ln 2 + \ln 3$	**b**	$\ln 3 + \ln 7$	**c**	$\ln 4 + \ln 20$
d	$\ln 6$	**e**	$\ln 21$	**f**	$\ln 80$

From your answers, suggest a possible simplification for $\ln a + \ln b$.

2 Use your calculator to find:

a	$\ln 6 - \ln 2$	**b**	$\ln 12 - \ln 3$	**c**	$\ln 3 - \ln 5$
d	$\ln 3$	**e**	$\ln 4$	**f**	$\ln(0.6)$

From your answers, suggest a possible simplification for $\ln a - \ln b$.

3 Use your calculator to find:

a	$3 \ln 2$	**b**	$2 \ln 5$	**c**	$-4 \ln 3$
d	$\ln(2^3)$	**e**	$\ln(5^2)$	**f**	$\ln(3^{-4})$

From your answers, suggest a possible simplification for $n \ln a$.

C LAWS OF NATURAL LOGARITHMS

There are 3 important **laws of logarithms**. These are:

- $\ln A + \ln B = \ln(AB)$
- $\ln A - \ln B = \ln\left(\dfrac{A}{B}\right)$
- $n \ln A = \ln(A^n)$

These laws are easily established using index laws:

Since $A = e^{\ln A}$ and $B = e^{\ln B}$

- $AB = e^{\ln A} \times e^{\ln B} = e^{\ln A + \ln B}$.

  But, $AB = e^{\ln(AB)}$ $\quad \therefore \quad \ln A + \ln B = \ln(AB)$.

- $\dfrac{A}{B} = \dfrac{e^{\ln A}}{e^{\ln B}} = e^{\ln A - \ln B}$.

  But, $\dfrac{A}{B} = e^{\ln\left(\frac{A}{B}\right)}$ $\quad \therefore \quad \ln A - \ln B = \ln\left(\dfrac{A}{B}\right)$.

- $A^n = (e^{\ln A})^n = e^{n \ln A}$. But, $A^n = e^{\ln(A^n)}$ $\quad \therefore \quad n \ln A = \ln(A^n)$.

Example 3

Use the laws of logarithms to write the following as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$
$= \ln(5 \times 3)$
$= \ln 15$

b $\ln 24 - \ln 8$
$= \ln\left(\frac{24}{8}\right)$
$= \ln 3$

c $\ln 5 - 1$
$= \ln 5 - \ln e^1$
$= \ln\left(\frac{5}{e}\right)$

EXERCISE 5C

1 Write as a single logarithm:

a $\ln 8 + \ln 2$ **b** $\ln 8 - \ln 2$ **c** $\ln 40 - \ln 5$
d $\ln 4 + \ln 5$ **e** $\ln 5 + \ln(0.4)$ **f** $\ln 2 + \ln 3 + \ln 4$
g $1 + \ln 3$ **h** $\ln 4 - 1$ **i** $\ln 5 + \ln 4 - \ln 2$
j $2 + \ln 2$ **k** $\ln 40 - 2$ **l** $\ln 6 - \ln 2 - \ln 3$

Example 4

Use the laws of logarithms to simplify:

a $2 \ln 7 - 3 \ln 2$
b $2 \ln 3 - 1$

a $2 \ln 7 - 3 \ln 2$
$= \ln(7^2) - \ln(2^3)$
$= \ln 49 - \ln 8$
$= \ln\left(\frac{49}{8}\right)$

b $2 \ln 3 - 1$
$= \ln(3^2) - \ln e$
$= \ln 9 - \ln e$
$= \ln\left(\dfrac{9}{e}\right)$

2 Simplify:

a $5\ln 2 + \ln 3$ **b** $2\ln 3 + 3\ln 2$ **c** $3\ln 4 - \ln 8$

d $2\ln 5 - 3\ln 2$ **e** $\frac{1}{2}\ln 4 + \ln 3$ **f** $\frac{1}{3}\ln\left(\frac{1}{8}\right)$

g $-\ln 2$ **h** $-\ln\left(\frac{1}{3}\right)$ **i** $-2\ln\left(\frac{1}{4}\right)$

Example 5

Show that: **a** $\ln\left(\frac{1}{9}\right) = -2\ln 3$ **b** $\ln 500 \doteqdot 6.9078 - \ln 2$

a $$\begin{aligned}\ln\left(\tfrac{1}{9}\right) &= \ln(3^{-2})\\ &= -2\ln 3\end{aligned}$$

b $$\begin{aligned}\ln 500 &= \ln\left(\frac{1000}{2}\right)\\ &= \ln 1000 - \ln 2\\ &\doteqdot 6.9078 - \ln 2\end{aligned}$$

3 Show that:

a $\ln 9 = 2\ln 3$ **b** $\ln\sqrt{2} = \frac{1}{2}\ln 2$ **c** $\ln\left(\frac{1}{8}\right) = -3\ln 2$

d $\ln\left(\frac{1}{5}\right) = -\ln 5$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

g $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **h** $\ln\left(\frac{1}{32}\right) = -5\ln 2$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$

4 Show that $\ln\left(\dfrac{e^2}{8}\right) = 2 - 3\ln 2$.

Example 6

Write the following equations without logarithms:

a $\ln A = 2\ln c + 3$ **b** $\ln M = 3\ln a - 2$

a $$\begin{aligned}\ln A &= 2\ln c + 3\\ \therefore\ \ln A - 2\ln c &= 3\\ \therefore\ \ln A - \ln c^2 &= 3\\ \therefore\ \ln\left(\frac{A}{c^2}\right) &= 3\\ \therefore\ \frac{A}{c^2} &= e^3\\ \therefore\ A &= e^3c^2\end{aligned}$$

b $$\begin{aligned}\ln M &= 3\ln a - 2\\ \therefore\ \ln M - 3\ln a &= -2\\ \therefore\ \ln M - \ln a^3 &= -2\\ \therefore\ \ln\left(\frac{M}{a^3}\right) &= -2\\ \therefore\ \frac{M}{a^3} &= e^{-2}\\ \therefore\ M &= e^{-2}a^3 \quad\text{or}\quad M = \frac{a^3}{e^2}\end{aligned}$$

5 Write the following equations without logarithms:

a $\ln D = \ln x + 1$ **b** $\ln F = -\ln p + 2$ **c** $\ln P = \frac{1}{2}\ln x$

d $\ln M = 2\ln y + 3$ **e** $\ln B = 3\ln t - 1$ **f** $\ln N = -\frac{1}{3}\ln g$

g $\ln Q \doteqdot 3\ln x + 2.159$ **h** $\ln D \doteqdot 0.4\ln n - 0.6582$

D EXPONENTIAL EQUATIONS INVOLVING e

To solve exponential equations of the form $e^x = a$ we simply use the property:

If $e^x = a$ then $x = \ln a$.

This rule is clearly true as if $e^x = a$

then $\ln e^x = \ln a$ {finding ln of both sides}

and $x = \ln a$ $\{\ln e^x = x\}$

Example 7

Solve for x, giving your answer correct to 4 significant figures:

a $e^x = 30$ **b** $e^{\frac{x}{3}} = 21.879$ **c** $20e^{4x} = 0.0382$

a
$$e^x = 30$$
$$\therefore \quad x = \ln 30$$
$$\therefore \quad x \doteqdot 3.401$$

b
$$e^{\frac{x}{3}} = 21.879$$
$$\therefore \quad \frac{x}{3} = \ln 21.879$$
$$\therefore \quad \frac{x}{3} \doteqdot 3.0855.....$$
$$\therefore \quad x \doteqdot 9.257$$

c
$$20e^{4x} = 0.0382$$
$$\therefore \quad e^{4x} = 0.001\,91$$
$$\therefore \quad 4x = \ln 0.001\,91$$
$$\therefore \quad 4x \doteqdot -6.2607....$$
$$\therefore \quad x \doteqdot -1.565$$

EXERCISE 5D

1 Solve for x, giving answers correct to 4 significant figures:

a $e^x = 10$ **b** $e^x = 1000$ **c** $e^x = 0.008\,62$

d $e^{\frac{x}{2}} = 5$ **e** $e^{\frac{x}{3}} = 157.8$ **f** $e^{\frac{x}{10}} = 0.016\,82$

g $20 \times e^{0.06x} = 8.312$ **h** $50 \times e^{-0.03x} = 0.816$ **i** $41.83e^{0.652x} = 1000$

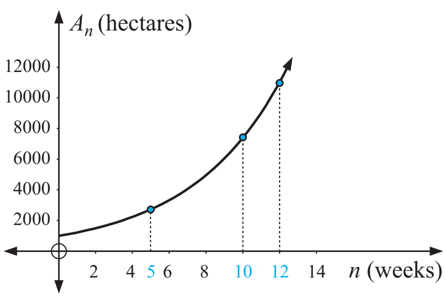
E GROWTH AND DECAY REVISITED

Earlier, in **Chapter 4**, we considered growth and decay problems in which we were required to find the value of the dependent variable for a given value of the independent variable.

For example:

Consider a locust plague for which the area of infestation is given by $A_n = 1000 \times e^{0.2n}$ hectares (n is the number of weeks after initial observation).

If we find A when $n = 0$, 5, 10 and 12 we can draw a graph of the growth in area of the infestation.



In this section we will consider the **reverse problem** of finding n (the independent variable) given values of A_n (the dependent variable). We can do this by:

- reading from an accurate **graph** to get approximate solutions
- using **logarithms** to solve the appropriate equation
- using **technology** (graphics calculator or computer graphing package).

Example 8

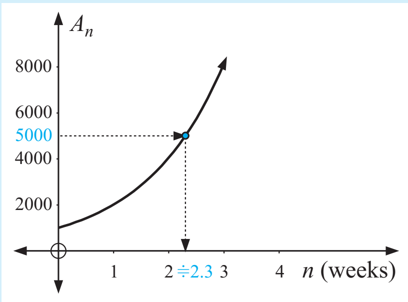
A biologist, monitoring a fire ant infestation, notices that the area affected by the ants is given by $A_n = 1000 \times e^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the infested area to reach 5000 ha.

b Find the answer to **a** using logarithms.

c Check your answer to **b** using suitable technology.

a



b When $A_n = 5000$,

$$1000 \times e^{0.7n} = 5000$$
$$\therefore \quad e^{0.7n} = 5$$
$$\therefore \quad 0.7n = \ln 5$$
$$\therefore \quad n = \frac{\ln 5}{0.7} \doteqdot 2.299$$

i.e., it takes about 2 weeks and 2 more days.

c Go to the graphing package or graphics calculator icon to find the intersection of $y = 1000 \times e^{0.7x}$ and $y = 5000$. $(x \doteqdot 2.299)$

As graphing by hand is rather tedious we will use logarithms and/or technology to solve problems of this kind.

EXERCISE 5E

1 The mass M_t grams, of bacteria in a culture t hours after establishment is given by $M_t = 20 \times e^{0.15t}$. Find the time for the mass of the culture to reach:

a 25 grams **b** 100 grams.

2 The temperature $T(^o\text{C})$, of a liquid which has been placed in a refrigerator is given by $T = 100 \times e^{-0.03t}$ where t is the time in minutes. Find the time required for the temperature to reach:

a 30^oC **b** 1^oC.

3 The mass M_t grams, of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

a halve **b** reach 25 grams **c** reach 1% of its original value.

4 A man jumps from an aeroplane and his speed of descent is given by $V = 50(1 - e^{-0.2t})$ m/s where t is the time in seconds. Find the time taken for his speed to reach 40 m/s.

5 Hot cooking oil is placed in a refrigerator and its temperature after m minutes is given by $T_m = (225 \times e^{-0.17m} - 6)$ ^{o}C. Find the time taken for the temperature to fall to 0^oC.

F INVERSE FUNCTIONS REVISITED

Recall that
- inverse functions are formed by interchanging x and y
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

Example 9

Given $f : x \longmapsto e^{x-3}$

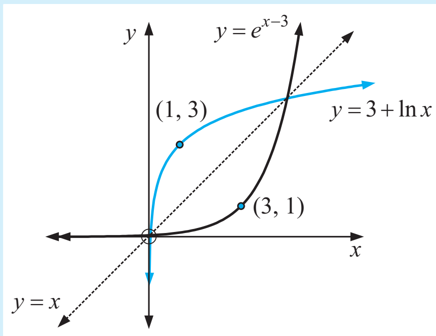
a find the defining equation of f^{-1}

b sketch the graphs of f and f^{-1} on the same set of axes

c state the domain and range of f and f^{-1}.

a
$$
\begin{aligned}
f(x) &= e^{x-3} \\
\text{i.e., } \quad y &= e^{x-3} \\
\therefore \quad f^{-1} \text{ is } x &= e^{y-3} \\
\text{i.e., } \quad y - 3 &= \ln x \\
\text{or } \quad y &= 3 + \ln x
\end{aligned}
$$

b



c

	f	f^{-1}
domain	$x \in \mathcal{R}$	$x > 0$
range	$y > 0$	$y \in \mathcal{R}$

$\in$ means 'is in'

EXERCISE 5F

GRAPHING PACKAGE

1 Consider $f : x \longmapsto e^x + 5$.

a Find the defining equation of f^{-1}.

b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

2 Consider $f : x \longmapsto e^{x+1} - 3$.

a Find the defining equation of f^{-1}.

b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

Example 10

Given $g: x \longmapsto \ln(x+2)$ where $x > -2$:

a find the defining equation of g^{-1}

b sketch the graphs of g and g^{-1} on the same set of axes

c state the domain and range of g and g^{-1}.

a g is $y = \ln(x+2)$ where $x > -2$

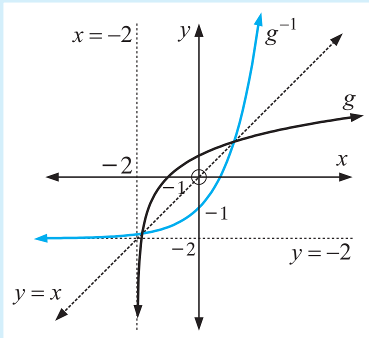
so, g^{-1} is $x = \ln(y+2)$ where $y > -2$

i.e., $y + 2 = e^x$

or $y = e^x - 2$

$\therefore g^{-1}(x) = e^x - 2$

b



c

	g	g^{-1}
domain	$x > -2$	$x \in \mathcal{R}$
range	$y \in \mathcal{R}$	$y > -2$

3 Consider $f: x \longmapsto \ln x - 4$ where $x > 0$.

a Find the defining equation of f^{-1}.

b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

4 Consider $g: x \longmapsto \ln(x-1) + 2$ where $x > 1$.

a Find the defining equation of g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

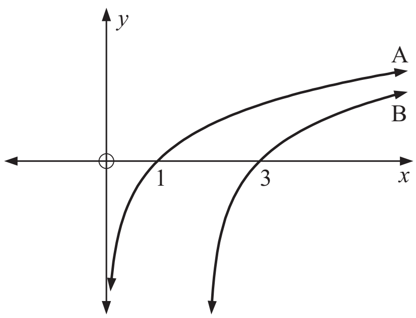
5 Given $f: x \longmapsto e^{2x}$ and $g: x \longmapsto 2x - 1$, find the defining equations of:

a $(f^{-1} \circ g)(x)$ **b** $(g \circ f)^{-1}(x)$

6 Consider the graphs A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x-2)$.

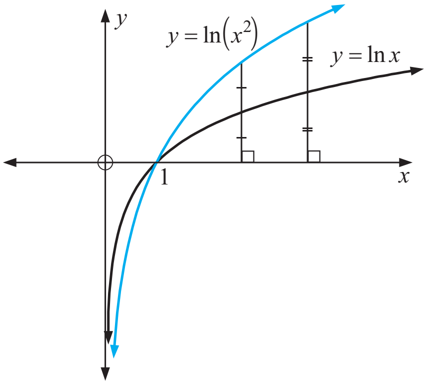
a Identify which is which. Give evidence for your anwer.

b Redraw the graphs on a new set of axes and add to them the graph of $y = \ln(x+2)$.



7 Lucia said that in order to graph $y = \ln(x^2)$, first graph $y = \ln x$ and double the distances away from the x-axis. Connecting these points will give the graph of $y = \ln x^2$.

Is she correct? Give evidence.



8 $y = \ln x$, $y = \ln(2x)$, $y = \ln(4x)$, $y = \ln(1.6x)$ are all members of the family of curves $y = \ln(kx)$ where $k > 0$, $x > 0$.

By graphing each of the above four functions using technology indicate how the last three may be obtained from the first. Give algebraic evidence.

REVIEW SET 5A

1 Find, to 3 significant figures, the value of:

a e^4 **b** $3e^2$ **c** $\dfrac{1}{6e}$ **d** $\dfrac{10}{\sqrt{e}}$

2 **a** On the same set of axes sketch and clearly label graphs of $f : x \longmapsto e^x$, $g : x \longmapsto e^{-x}$ and $h : x \longmapsto -e^{-x}$.

b What is the geometric connection between:
i f and g **ii** g and h?

3 Sketch on the same set of axes the graphs of $y = e^x$ and $y = 3e^x$.

4 A particle moves in a straight line such that its displacement from the origin O is given by $s(t) = 120t - 40e^{-\frac{t}{5}}$ metres, where t is the time in seconds, $t \geqslant 0$.

a Find the position of the particle at **i** $t = 0$ **ii** $t = 5$ **iii** $t = 20$.

b Hence sketch the graph of $s(t) = 120t - 40e^{-\frac{t}{5}}$ for $t \geqslant 0$.

5 Without using a calculator, find:

a $\ln(e^5)$ **b** $\ln(\sqrt{e})$ **c** $\ln\left(\dfrac{1}{e}\right)$

6 Simplify:

a $\ln(e^{2x})$ **b** $\ln(e^2e^x)$ **c** $\ln\left(\dfrac{e}{e^x}\right)$

7 Solve for x, giving your answers to 3 significant figures:

a $\ln x = 5$ **b** $3\ln x + 2 = 0$

8 Write as a single logarithm:

a $\ln 6 + \ln 4$ **b** $\ln 60 - \ln 20$
c $\ln 4 + \ln 1$ **d** $\ln 200 - \ln 8 + \ln 5$

9 Write in the form $a \ln k$ where a and k are positive whole numbers and k is as small as possible:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

10 Solve for x, giving answers correct to 3 significant figures:

a $e^x = 400$ **b** $e^{2x+1} = 11$ **c** $25e^{\frac{x}{2}} = 750$ **d** $e^{2x} = 7e^x - 12$

REVIEW SET 5B

1 Write as a single power of e:

a $\dfrac{1}{e^3}$ **b** $\dfrac{\sqrt{e}}{e^2}$ **c** $e^3\sqrt{e^3}$ **d** $\sqrt{10e}$

2 Evaluate correct to 6 significant figures:

a $4.8e^{1.725}$ **b** $0.66e^{-1.7968}$

3 On the same set of axes, sketch and clearly label the graphs of
$f: x \longmapsto e^x$, $g: x \longmapsto e^{-x}$, $h: x \longmapsto e^{-x} - 4$.
State the domain and range of each function.

4 Sketch on the same set of axes, the graphs of $y = e^x$ and $y = e^{3x}$.

5 Without using a calculator, find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln\left(\dfrac{e}{\sqrt{e^5}}\right)$

6 Write in the form e^x where x is correct to 3 significant figures:

a 20 **b** 3000 **c** 0.075

7 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

8 Write the following equations without logarithms:

a $\ln P = 1.5\ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5\ln N$

9 Consider $g: x \longmapsto 2e^x - 5$.

a Find the defining equation of g^{-1}.
b Sketch the graphs of g and g^{-1} on the same set of axes.
c State the domain and range of g and g^{-1}.

10 The weight W_t grams of radioactive substance remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.

Chapter 6

Graphing and transforming functions

Contents:

A Families of functions

Investigation: Function families

B Key features of functions

C Transformations of graphs

D Functional transformations

Review set 6

A FAMILIES OF FUNCTIONS

In this section we will consider these functions:

Name	General form	
Linear	$f(x) = ax + b, \quad a \neq 0$	$f : x \longmapsto ax + b, \quad a \neq 0$
Quadratic	$f(x) = ax^2 + bx + c, \quad a \neq 0$	$f : x \longmapsto ax^2 + bx + c, \quad a \neq 0$
Cubic	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$	$f : x \longmapsto ax^3 + bx^2 + cx + d, a \neq 0$
Absolute value	$f(x) = \|x\|$	$f : x \longmapsto \|x\|$
Exponential	$f(x) = a^x, \quad a > 0, \quad a \neq 1$	$f : x \longmapsto a^x, \quad a > 0, \quad a \neq 1$
Logarithmic	$f(x) = \log_e x$ or $f(x) = \ln x$	$f : x \longmapsto \ln x$
Reciprocal	$f(x) = \dfrac{k}{x}, \quad x \neq 0$	$f : x \longmapsto \dfrac{k}{x}, \quad x \neq 0$

Although the above functions have different graphs, they do have some similar features.

The main features we are interested in are:

- the axis intercepts (where the graph cuts the x and y-axes)
- slopes
- turning points (maxima and minima)
- values of x where the function does not exist
- the presence of asymptotes (lines or curves that the graph approaches).

INVESTIGATION FUNCTION FAMILIES

PORTFOLIO OPPORTUNITY

In this investigation you are encouraged to use the graphing package supplied. Click on the icon to access this package.

GRAPHING PACKAGE

What to do:

1 From the menu, graph on the same set of axes:
$y = 2x + 1, \quad y = 2x + 3, \quad y = 2x - 1$
Comment on all lines of the form $y = 2x + b$.

2 From the menu, graph on the same set of axes:
$y = x + 2, \quad y = 2x + 2, \quad y = 4x + 2, \quad y = -x + 2, \quad y = -\frac{1}{2}x + 2$
Comment on all lines of the form $y = ax + 2$.

3 On the same set of axes graph:
$y = x^2, \quad y = 2x^2, \quad y = \frac{1}{2}x^2, \quad y = -x^2, \quad y = -3x^2, \quad y = -\frac{1}{5}x^2$
Comment on all functions of the form $y = ax^2, \quad a \neq 0$.

4 On the same set of axes graph:
$y = x^2, \quad y = (x - 1)^2 + 2, \quad y = (x + 1)^2 - 3, \quad y = (x - 2)^2 - 1$
and other functions of the form $y = (x - h)^2 + k$ of your choice.
Comment on the functions of this form.

5 On the same set of axes, graph these absolute value functions:

a $y = |x|, \quad y = 2\,|x|, \quad y = |2x|$

b $y = |x|, \quad y = |x| + 2, \quad y = |x| - 3$

c $y = |x|, \quad y = |x - 2|, \quad y = |x + 3|, \quad y = |x - 1| + 2$

Write a brief report on your discoveries.

6 On the same set of axes, graph these functions:

a $y = \dfrac{1}{x}, \quad y = \dfrac{3}{x}, \quad y = \dfrac{10}{x}$

b $y = \dfrac{-1}{x}, \quad y = \dfrac{-2}{x}, \quad y = \dfrac{-5}{x}$

c $y = \dfrac{1}{x}, \quad y = \dfrac{1}{x-2}, \quad y = \dfrac{1}{x+3}$

d $y = \dfrac{1}{x}, \quad y = \dfrac{1}{x} + 2, \quad y = \dfrac{1}{x} - 2$

e $y = \dfrac{2}{x}, \quad y = \dfrac{2}{x-1} + 2, \quad y = \dfrac{2}{x+2} - 1$

Write a brief report on your discoveries.

Example 1

If $f(x) = x^2$, find in simplest form:

a $f(2x)$ **b** $f\left(\dfrac{x}{3}\right)$ **c** $2\,f(x) + 1$ **d** $f(x+3) - 4$

a
$$\begin{aligned} &f(2x) \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

b
$$\begin{aligned} &f\left(\frac{x}{3}\right) \\ &= \left(\frac{x}{3}\right)^2 \\ &= \frac{x^2}{9} \end{aligned}$$

c
$$\begin{aligned} &2\,f(x) + 1 \\ &= 2x^2 + 1 \end{aligned}$$

d
$$\begin{aligned} &f(x+3) - 4 \\ &= (x+3)^2 - 4 \\ &= x^2 + 6x + 9 - 4 \\ &= x^2 + 6x + 5 \end{aligned}$$

EXERCISE 6A

1 If $f(x) = x$, find in simplest form:

a $f(2x)$ **b** $f(x) + 2$ **c** $\frac{1}{2}\,f(x)$ **d** $2\,f(x) + 3$

2 If $f(x) = x^2$, find in simplest form:

a $f(3x)$ **b** $f\left(\dfrac{x}{2}\right)$ **c** $3\,f(x)$ **d** $2\,f(x-1) + 5$

3 If $f(x) = x^3$, find in simplest form:

a $f(4x)$ **b** $\frac{1}{2}\,f(2x)$ **c** $f(x+1)$ **d** $2\,f(x+1) - 3$

[**Note:** $(x+1)^3 = x^3 + 3x^2 + 3x + 1$]

4 If $f(x) = |x|$, find in simplest form:

a $f(2x) + 3$ **b** $f(-x)$ **c** $f(x-2)$ **d** $f(x+1) + 2$

5 If $f(x) = 2^x$, find in simplest form:

a $f(2x)$ **b** $f(-x)+1$ **c** $f(x-2)+3$ **d** $2\,f(x)+3$

6 If $f(x) = \dfrac{1}{x}$, find in simplest form:

a $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2\,f(x)+3$ **d** $3\,f(x-1)+2$

B KEY FEATURES OF FUNCTIONS

In this exercise you should use your graphics calculator to graph and find the key features of various kinds of functions.

EXERCISE 6B

1 Consider $f : x \longmapsto 2x+3$ or $y = 2x+3$.

a Graph this function using a graphics calculator.

b Find algebraically, the:

i x-axis intercept **ii** y-axis intercept **iii** slope.

c Use your graphics calculator to check that:

i the x-axis intercept is $-1\frac{1}{2}$ **ii** the y-axis intercept is 3.

2 Consider $f : x \longmapsto (x-2)^2 - 9$.

a Graph the function using a graphics calculator.

b Find algebraically the x and y-axis intercepts.

c Use your graphics calculator to check that:

i the x-axis intercepts are -1 and 5 **ii** the y-intercept is -5

iii the vertex is $(2, -9)$.

3 Consider $f : x \longmapsto 2x^3 - 9x^2 + 12x - 5$.

a Graph the function using your graphics calculator.

b Sketch the graph of the function.

c Check that:

i the x-intercepts are 1 and $2\frac{1}{2}$ **ii** the y-intercept is -5

iii the minimum turning point is at $(2, -1)$

iv the maximum turning point is at $(1, 0)$.

4 Use your graphics calculator to sketch the graph of $y = |x|$.

Note: $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$.

5 Consider $f : x \longmapsto 2^x$. After graphing on a graphics calculator, check these key features:

a as $x \to \infty$, $2^x \to \infty$ ($\to$ reads 'approaches')

b as $x \to -\infty$, $2^x \to 0$ i.e., the x-axis is an asymptote

c the y-intercept is 1

d 2^x is > 0 for all x.

6 Consider $f : x \longmapsto \log_e x$.

Graph on a graphics calculator and then check that:

- **a** as $x \to \infty$, $\ln x \to \infty$
- **b** as $x \to 0$ (from the right), $\ln x \to -\infty$
- **c** $\ln x$ only exists if $x > 0$
- **d** the x-intercept is 1
- **e** the y-axis is an asymptote.

7 Consider $f : x \longmapsto \dfrac{1}{x}$, $x \neq 0$.

Graph on a graphics calculator and then check that:

- **a** both axes are asymptotes
- **b** as $x \to \infty$, $y \to 0$ (from above)
- **c** as $x \to -\infty$, $y \to 0$ (from below)
- **d** as $x \to 0$ (from the right), $y \to \infty$
- **e** as $x \to 0$ (from the left), $y \to -\infty$.

C TRANSFORMATIONS OF GRAPHS

In the next exercise you should discover the graphical connection between $y = f(x)$ and the functions of the form

- $y = f(x) + b$, b is a constant
- $y = f(x - a)$, a is a constant
- $y = p\,f(x)$, p is a constant
- $y = f(kx)$, k is a constant
- $y = -f(x)$
- $y = f(-x)$

GRAPHING PACKAGE

Types $y = f(x) + b$ and $y = f(x - a)$

EXERCISE 6C.1

1 **a** Sketch the graph of $f(x) = x^2$.

b On the same set of axes sketch the graphs of:

- **i** $y = f(x) + 2$, i.e., $y = x^2 + 2$
- **ii** $y = f(x) - 3$, i.e., $y = x^2 - 3$

c What is the connection between the graphs of $y = f(x)$ and $y = f(x) + b$ if:

- **i** $b > 0$
- **ii** $b < 0$?

2 For each of the following functions f, sketch on the same set of axes $y = f(x)$, $y = f(x) + 1$ and $y = f(x) - 2$.

- **a** $f(x) = |x|$
- **b** $f(x) = 2^x$
- **c** $f(x) = x^3$
- **d** $f(x) = \dfrac{1}{x}$

Summarise your observations by describing the graphical transformation of $y = f(x)$ as it becomes $y = f(x) + b$.

3 **a** On the same set of axes, sketch the graphs of:

$f(x) = x^2$, $y = f(x - 3)$ and $y = f(x + 2)$.

b What is the connection between the graphs of $y = f(x)$ and $y = f(x - a)$ if:

- **i** $a > 0$
- **ii** $a < 0$?

4 For each of the following functions f, sketch on the same set of axes the graphs of $y = f(x)$, $y = f(x-1)$ and $y = f(x+2)$.

a $f(x) = |x|$ **b** $f(x) = x^3$ **c** $f(x) = \ln x$ **d** $f(x) = \dfrac{1}{x}$

Summarise your observations by describing the geometrical transformation of $y = f(x)$ as it becomes $y = f(x-a)$.

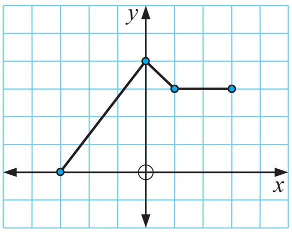
5 For each of the following functions sketch:

$y = f(x)$, $y = f(x-2)+3$ and $y = f(x+1)-4$ on the same set of axes.

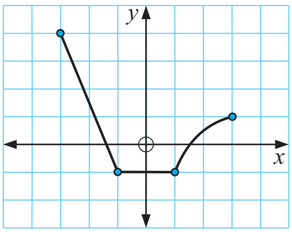
a $f(x) = x^2$ **b** $f(x) = e^x$ **c** $f(x) = \dfrac{1}{x}$

6 Copy these functions and then draw the graph of $y = f(x-2)-3$.

a



b



Types $y = p\,f(x)$, $p > 0$ and $y = f(kx)$, $k > 0$

EXERCISE 6C.2

1 Sketch on the same set of axes, the graphs of $y = f(x)$, $y = 2\,f(x)$ and $y = 3\,f(x)$ for each of:

a $f(x) = x^2$ **b** $f(x) = |x|$ **c** $f(x) = x^3$

d $f(x) = e^x$ **e** $f(x) = \ln x$ **f** $f(x) = \dfrac{1}{x}$

2 Sketch on the same set of axes, the graphs of $y = f(x)$, $y = \frac{1}{2}f(x)$ and $y = \frac{1}{4}f(x)$ for each of:

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = e^x$

3 Using **1** and **2**, summarise your observations by describing the graphical transformation of $y = f(x)$ to $y = p\,f(x)$ for $p > 0$.

4 Sketch on the same set of axes, the graphs of $y = f(x)$ and $y = f(2x)$ for each of:

a $y = x^2$ **b** $y = (x-1)^2$ **c** $y = (x+3)^2$

5 Sketch on the same set of axes, the graphs of $y = f(x)$ and $y = f(3x)$ for each of:

a $y = x$ **b** $y = x^2$ **c** $y = e^x$

6 Sketch on the same set of axes, the graphs of $y = f(x)$ and $y = f\left(\dfrac{x}{2}\right)$ for each of:

a $y = x^2$ **b** $y = 2x$ **c** $y = (x+2)^2$

7 Using **4**, **5** and **6**, summarise your observations by describing the graphical transformation of $y = f(x)$ to $y = f(kx)$ for $k > 0$.

8 Consider the function $f : x \longmapsto x^2$.

On the same set of axes sketch the graphs of:

- **a** $y = f(x)$, $y = 3f(x-2)+1$ and $y = 2f(x+1)-3$
- **b** $y = f(x)$, $y = f(x-3)$, $y = f\left(\frac{x}{2}-3\right)$, $y = 2f\left(\frac{x}{2}-3\right)$ and $y = 2f\left(\frac{x}{2}-3\right)+4$
- **c** $y = f(x)$ and $y = \frac{1}{4}f(2x+5)+1$.

Types $y = -f(x)$ and $y = f(-x)$

EXERCISE 6C.3

1 On the same set of axes, sketch the graphs of:

- **a** $y = 3x$ and $y = -3x$
- **b** $y = e^x$ and $y = -e^x$
- **c** $y = x^2$ and $y = -x^2$
- **d** $y = \ln x$ and $y = -\ln x$
- **e** $y = x^3 - 2$ and $y = -x^3 + 2$
- **f** $y = 2(x+1)^2$ and $y = -2(x+1)^2$

2 Based on question **1**, what transformation moves $y = f(x)$ to $y = -f(x)$?

3 **a** Find $f(-x)$ for:

- **i** $f(x) = 2x+1$
- **ii** $f(x) = x^2+2x+1$
- **iii** $f(x) = |x-3|$

b Graph $y = f(x)$ and $y = f(-x)$ for:

- **i** $f(x) = 2x+1$
- **ii** $f(x) = x^2+2x+1$
- **iii** $f(x) = |x-3|$

4 Based on question **3**, what transformation moves $y = f(x)$ to $y = f(-x)$?

Summary of graphical transformations on $y = f(x)$

- For $y = f(x) + b$, the effect of changes in b is to **translate** the graph of $y = f(x)$ **vertically** through b units.
  - If $b > 0$ it moves **upwards**.
  - If $b < 0$ it moves **downwards**.
- For $y = f(x-a)$, the effect of changes in a is to **translate** the graph of $y = f(x)$ **horizontally** through a units.
  - If $a > 0$ it moves to the **right**.
  - If $a < 0$ it moves to the **left**.
- For $y = p\,f(x)$, $p > 0$, the effect of changes in p is to **vertically stretch or compress** the graph of $y = f(x)$ by a factor of p.
  - If $p > 1$ it moves points of $y = f(x)$ **further away** from the x-axis.
  - If $0 < p < 1$ it moves points of $y = f(x)$ **closer** to the x-axis.

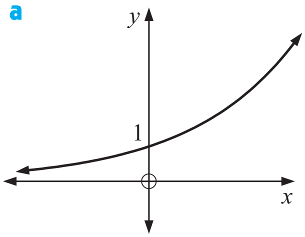
- For $y = f(kx)$, $k > 0$, the effect of changes in k is to **horizontally stretch or compress** the graph of $y = f(x)$ by a factor of $\frac{1}{k}$.
  - If $k > 1$ it moves points of $y = f(x)$ **closer** to the y-axis.
  - If $0 < k < 1$ it moves points of $y = f(x)$ **further away** from the y-axis.
- For $y = -f(x)$, the effect on the graph of $y = f(x)$ is to **reflect it in the x-axis**.
- For $y = f(-x)$, the effect on the graph of $y = f(x)$ is to **reflect it in the y-axis**.

D FUNCTIONAL TRANSFORMATIONS

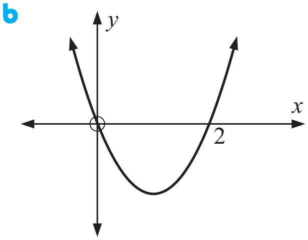
EXERCISE 6D

1 Copy the following sketch graphs for $y = f(x)$ and hence sketch the graph of $y = -f(x)$ on the same axes.

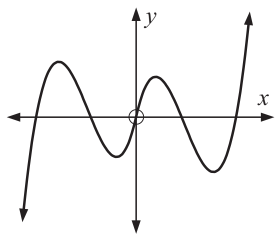
a



b

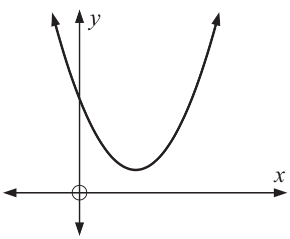


c

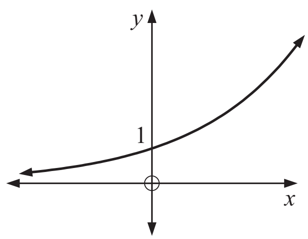


2 Given the following graphs of $y = f(x)$, sketch graphs of $y = f(-x)$:

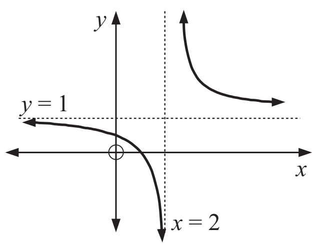
a



b



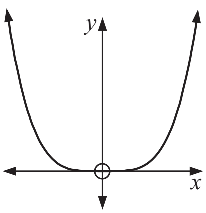
c



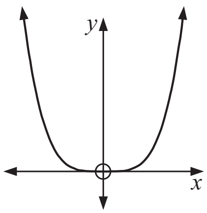
3 Match each equation to its graph drawn below:

A $y = x^4$ **B** $y = 2x^4$ **C** $y = \frac{1}{2}x^4$ **D** $y = 6x^4$

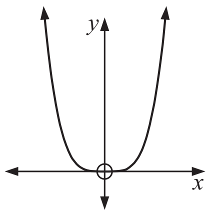
a



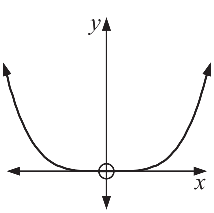
b



c



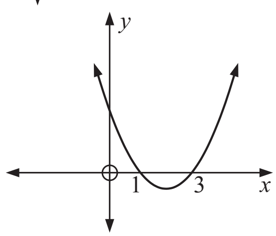
d



4 For the graph of $y = f(x)$ given, draw sketches of

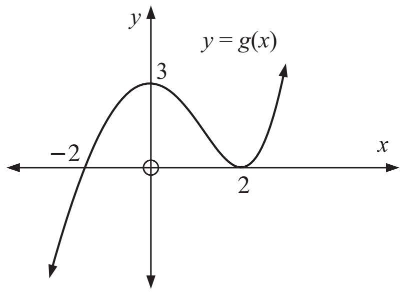
a $y = 2f(x)$ **b** $y = \frac{1}{2}f(x)$

c $y = f(x+2)$ **d** $y = f(2x)$

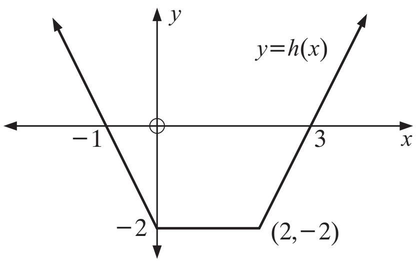


5 For the graph of $y = g(x)$ given, draw sketches of

- **a** $y = g(x) + 2$
- **b** $y = -g(x)$
- **c** $y = g(-x)$
- **d** $y = g(x + 1)$



6



For the graph of $y = h(x)$ given, draw sketches of

- **a** $y = h(x) + 1$
- **b** $y = \frac{1}{2}h(x)$
- **c** $y = h(-x)$
- **d** $y = h\left(\frac{x}{2}\right)$

REVIEW SET 6

1 Name the function type for:

- **a** $f : x \longmapsto 3x - 2$
- **b** $f : x \longmapsto 2^x$
- **c** $f : x \longmapsto \dfrac{5}{x}$
- **d** $f : x \longmapsto |\, x\, |$
- **e** $f : x \longmapsto x^2 - 3x + 1$
- **f** $f : x \longmapsto \ln x$

2 If $f(x) = x^2 - 2x$, find in simplest form:

- **a** $f(3)$
- **b** $f(-2)$
- **c** $f(2x)$
- **d** $f(-x)$
- **e** $3f(x) - 2$

3 If $f(x) = 5 - x - x^2$, find in simplest form:

- **a** $f(4)$
- **b** $f(-1)$
- **c** $f(x - 1)$
- **d** $f\left(\frac{x}{2}\right)$
- **e** $2f(x) - f(-x)$

4 If $f(x) = \dfrac{4}{x}$, find in simplest form:

- **a** $f(-4)$
- **b** $f(2x)$
- **c** $f\left(\frac{x}{2}\right)$
- **d** $4f(x + 2) - 3$

5 Consider $f(x) : x \longmapsto 3x - 2$

- **a** Sketch the function f.
- **b** Find algebraically the **i** x-intercept **ii** y-intercept **iii** slope.
- **c** **i** Find y when $x = 0.3$ **ii** Find x when $y = 0.7$

6 Consider $f(x) = (x + 1)^2 - 4$

- **a** Use your calculator to help graph the function.
- **b** Find algebraically **i** the x-intercepts **ii** the y-intercept
- **c** What are the coordinates of the vertex of the function?
- **d** Use your calculator to check your answers to **b** and **c**.

7 Consider $f : x \longmapsto 2^{-x}$.

a Use your calculator to help graph the function.

b True or false?

i as $x \to \infty$, $2^{-x} \to 0$ **ii** as $x \to -\infty$, $2^{-x} \to 0$

iii the y-intercept is $\frac{1}{2}$ **iv** $2^{-x} > 0$ for all x.

8 Sketch the graph of $f(x) = -x^2$, and on the same set of axes sketch the graph of:

a $y = f(-x)$ **b** $y = -f(x)$ **c** $y = f(2x)$ **d** $y = f(x-2)$

9 Sketch the graph of $g(x) = (x+1)^2$, and on the same set of axes sketch the graph of:

a $y = 2g(x)$ **b** $y = g(-x)$ **c** $y = g\left(\frac{x}{2}\right)$ **d** $y = g(x-1)$

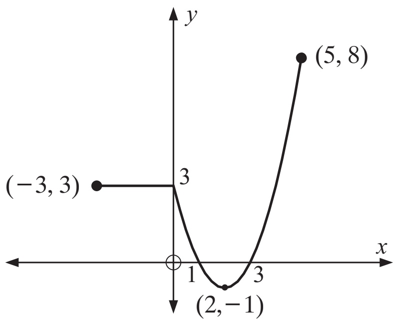
10 Consider the function $f : x \longmapsto x^2$.
On the same set of axes graph

a $y = f(x)$ **b** $y = f(x+2)$ **c** $y = 2f(x+2)$ **d** $y = 2f(x+2) - 3$

11 For the graph of $y = f(x)$, sketch graphs of:

a $y = f(-x)$ **b** $y = -f(x)$

c $y = f(x+2)$ **d** $y = f(x) + 2$



Chapter 7

Coordinate geometry

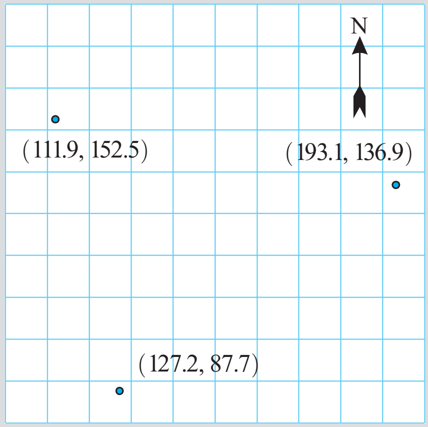
Contents:

A Assumed knowledge
Investigation: Finding where lines meet using technology
B Equations of lines
C Distance between two points
D Midpoints and perpendicular bisectors

Review set 7A
Review set 7B
Review set 7C

OPENING PROBLEM

Three small country towns decide to enter a combined team in the local football association. However, they cannot agree which town oval to use as each wants to use its own venue. This is a problem as there could be long distances for some players to travel to practice.



At a meeting they decide to place a new oval in such a position that it is equidistant from all three towns. The coordinates of each town are known from the local council map of the area, and are marked on the grid above.

Where exactly should the new venue be located? For you to think about:

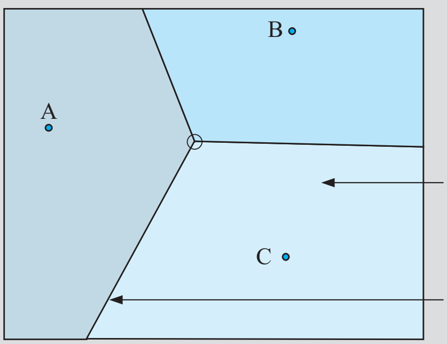
- What vital information given above is necessary to solve the problem?
- Can you see how to use coordinate geometry (midpoints, gradients, equations of lines, etc) to solve the problem?
- Can the problem be solved without using coordinate geometry?
- Jason Jolly says the answer is obvious. He says "Just average the x-coordinates and then the y-coordinates to give the correct answer". Is he correct?
- Can you see any problems with a mathematical solution to this problem?

HISTORICAL NOTE — VORONOI DIAGRAMS

Peter Dirichlet (1850) and later **Voronoi** (1908) considered the following problem:

"For a collection of points in the two dimensional plane, how can we partition the plane so that the partition lines are equidistant to two or more points?"

Below is a Voronoi diagram for three distinct, non collinear points A, B and C.



Notice that the plane has been partitioned into three regions and that each partition line is equidistant to any two of the points.

Points A, B and C are called **sites**.

This region is called the **Voronoi region** for site C. Every point in this region is closer to C than any other site.

This is a **Voronoi edge**, a boundary line between Voronoi regions. O is called a **Voronoi vertex**.

As Voronoi published solutions to the problem, the diagrams were named after him.

Voronoi diagrams have applications in zoology, archaeology, communications, crystallography and motion planning.

For example, Voronoi diagrams are useful in the solution to the problem of where to site mobile telephone towers. Given say three towers, how can they be sited so that any particular mobile call is carried by the closest tower?

Various methods (algorithms) are used to construct Voronoi diagrams. However, Steve Fortune (1985) and his *plane-sweep* method greatly reduced the time to draw these diagrams.

A ASSUMED KNOWLEDGE

THE NUMBER PLANE

The position or location of any point in the **number plane** can be specified in terms of an **ordered pair** of numbers (x, y), where:

x is the **horizontal step** from a fixed point O, and y is the **vertical step** from O.

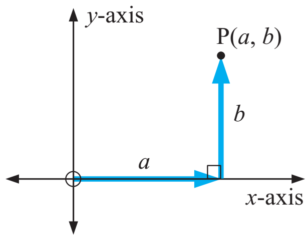
Once an **origin** O, has been given, two perpendicular axes are drawn.

The x**-axis** is horizontal and the y**-axis** is vertical.

DEMO

The **number plane** is also known as either:

- the **2-dimensional plane**, or
- the **Cartesian plane**, (named after **René Descartes**).



Note:

(a, b) is called an **ordered pair**, where a and b are often referred to as **the coordinates** of the point.

a is called the x**-coordinate** and

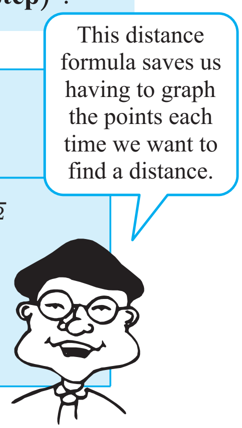
b is called the y**-coordinate**.

THE DISTANCE FORMULA

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a plane, then the distance d, between these points is given by

$$\mathbf{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $$\mathbf{AB} = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}.$$



Example 1

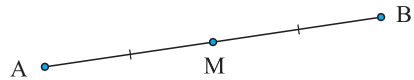
Find the distance between A$(-2, 1)$ and B$(3, 4)$.

$$\begin{array}{cccc} A(-2, & 1) & B(3, & 4) \\ \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & y_1 & x_2 & y_2 \end{array}$$

$$\begin{aligned} AB &= \sqrt{(3 - -2)^2 + (4 - 1)^2} \\ &= \sqrt{5^2 + 3^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

THE MIDPOINT FORMULA

If M is halfway between points A and B then M is the **midpoint** of AB.



If A(x_1, y_1) and B(x_2, y_2) are two points then the **midpoint** M of AB has coordinates $\left(\dfrac{\boldsymbol{x_1+x_2}}{\mathbf{2}}, \dfrac{\boldsymbol{y_1+y_2}}{\mathbf{2}}\right)$.

Example 2

Find the coordinates of the midpoint of AB for A(-1, 3) and B(4, 7).

x-coordinate of midpoint

$$= \frac{-1+4}{2}$$
$$= \tfrac{3}{2}$$
$$= 1\tfrac{1}{2}$$

y-coordinate of midpoint

$$= \frac{3+7}{2}$$
$$= 5$$

$\therefore$ the midpoint of AB is $(1\frac{1}{2}, 5)$

SLOPE (OR GRADIENT) OF A LINE

When looking at line segments drawn on a set of axes, it is clear that different line segments are inclined to the horizontal at different angles, i.e., some appear to be steeper than others.

The **slope** or **gradient** of a line is a measure of its steepness.

If A is (x_1, y_1) and B is (x_2, y_2) then the **gradient** of AB is $\dfrac{\boldsymbol{y_2-y_1}}{\boldsymbol{x_2-x_1}}$.

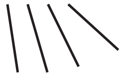
Example 3

Find the gradient of the line through (3, -2) and (6, 4).

$$\begin{array}{cccc} (3, & -2) & (6, & 4) \\ \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & y_1 & x_2 & y_2 \end{array}$$

$$\text{gradient} = \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{4--2}{6-3}$$
$$= \tfrac{6}{3}$$
$$= 2$$

Note:
- horizontal lines have a gradient of **0 (zero)**
- vertical lines have an **undefined** gradient

-  forward sloping lines have **positive** gradient
-  backward sloping lines have **negative** gradients
- **parallel lines** have equal gradients
- the gradients of **perpendicular lines** are *negative reciprocals*

  i.e., if the gradients are m_1 and m_2 then $\boldsymbol{m_2 = \dfrac{-1}{m_1}}$ or $\boldsymbol{m_1 m_2 = -1}$.

EQUATIONS OF LINES

The **equation of a line** is an equation which states the connection between the x and y values for every point on the line.

Equations of lines have various forms:

- All **vertical lines** have equations of the form $x = a$, (a is a constant).
- All **horizontal lines** have equations of the form $y = c$, (c is a constant).
- If a straight line has slope m and passes through (a, b) then it has equation

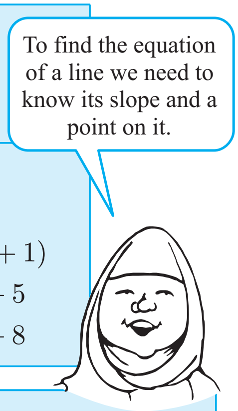
$$\boldsymbol{\frac{y-b}{x-a} = m}$$

which can be rearranged into $\boldsymbol{y = mx + c}$ {**slope-intercept** form}

or $\boldsymbol{Ax + By = C}$ {**general** form}

Example 4

Find, in *slope-intercept form*, the equation of the line through $(-1, 3)$ with a slope of 5.



The equation of the line is $\dfrac{y-3}{x--1} = 5$ i.e., $\dfrac{y-3}{x+1} = 5$

$\therefore\ y - 3 = 5(x+1)$

$\therefore\ y - 3 = 5x + 5$

$\therefore\ y = 5x + 8$

Example 5

Find, in *general form*, the equation of the line through $(1, -5)$ and $(5, -2)$.

The slope $= \dfrac{-2 - -5}{5-1}$

$= \frac{3}{4}$

So, the equation is $\dfrac{y - -2}{x-5} = \dfrac{3}{4}$

i.e., $\dfrac{y+2}{x-5} = \dfrac{3}{4}$

$\therefore\ 4(y+2) = 3(x-5)$

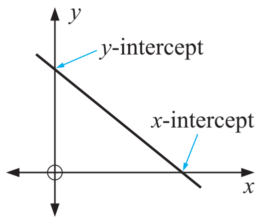
$\therefore\ 4y + 8 = 3x - 15$

$\therefore\ 3x - 4y = 23$

INTERCEPTS

Axis **intercepts** are the x- and y-values where a graph cuts the coordinate axes.

The x-intercept is found by letting $y = 0$.
The y-intercept is found by letting $x = 0$.



Example 6

For the line with equation $2x - 3y = 12$, find axis intercepts.

For $2x - 3y = 12$,

when $x = 0$, $-3y = 12$ $\therefore$ $y = -4$

when $y = 0$, $2x = 12$ $\therefore$ $x = 6$

DOES A POINT LIE ON A LINE?

A point lies on a line if its coordinates satisfy the equation of the line.

Example 7

Does $(3, -2)$ lie on the line with equation $5x - 2y = 20$?

Substituting $(3, -2)$ into $5x - 2y = 20$ gives
$5(3) - 2(-2) = 20$
i.e., $19 = 20$ which is false

$\therefore$ $(3, -2)$ does not lie on the line.

WHERE GRAPHS MEET

Example 8

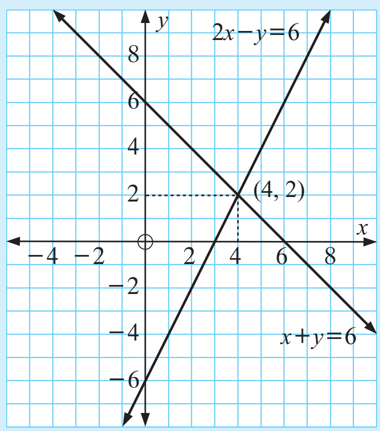
Use graphical methods to find where the lines $x + y = 6$ and $2x - y = 6$ meet.

For $x + y = 6$
when $x = 0$, $y = 6$
when $y = 0$, $x = 6$

x	0	6
y	6	0

For $2x - y = 6$
when $x = 0$, $-y = 6$,
$\therefore$ $y = -6$
when $y = 0$, $2x = 6$,
$\therefore$ $x = 3$

x	0	3
y	-6	0

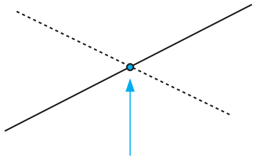


The graphs meet at $(4, 2)$.

Check: $4 + 2 = 6$ ✓ and $2 \times 4 - 2 = 6$ ✓

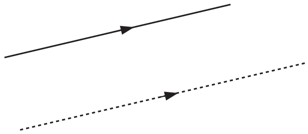
There are three possible situations which may occur. These are:

Case 1:



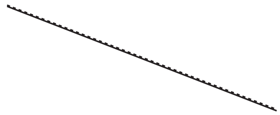
The lines meet in a single **point of intersection**.

Case 2:



The lines are **parallel** and **never meet**. So, there is no point of intersection.

Case 3:



The lines are **coincident** (the same line) and so there are infinitely many points of intersection.

INVESTIGATION FINDING WHERE LINES MEET USING TECHNOLOGY

PORTFOLIO OPPORTUNITY

Graphing packages and **graphics calculators** can be used to plot straight line graphs and hence find the point of intersection of the straight lines. This can be useful if the solutions are not integer values, although an algebraic method can also be used. However, most graphing packages and graphics calculators require the equation to be entered in the form $y = mx + c$.

Consequently, if an equation is given in **general form**, it must be rearranged into **slope-intercept form**.

For example, if we wish to use technology to find the point of intersection of $4x + 3y = 10$ and $x - 2y = -3$:

Step 1: We **rearrange** each equation into the form $y = mx + c$, i.e.,

$$4x + 3y = 10$$
$$\therefore \quad 3y = -4x + 10$$
$$\therefore \quad y = -\tfrac{4}{3}x + \tfrac{10}{3}$$

and

$$x - 2y = -3$$
$$\therefore \quad -2y = -x - 3$$
$$\therefore \quad y = \frac{x}{2} + \frac{3}{2}$$

Step 2: If you are using the **graphing package**, click on the icon to open the package and enter the two equations.

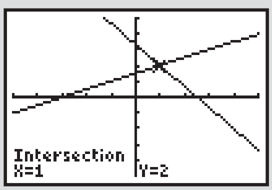
GRAPHING PACKAGE

If you are using a **graphics calculator**, enter the functions $Y_1 = -4X/3 + 10/3$ and $Y_2 = X/2 + 3/2$.

Step 3: Draw the **graphs** of the functions on the same set of axes. (You may have to change the viewing **window** if using a graphics calculator.)

Step 4: Use the built in functions to calculate the point of **intersection**.

Thus, the point of intersection is (1, 2).



TI

C

What to do:

1 Use technology to find the point of intersection of:

a $y = x + 4$
$5x - 3y = 0$

b $x + 2y = 8$
$y = 7 - 2x$

c $x - y = 5$
$2x + 3y = 4$

d $2x + y = 7$
$3x - 2y = 1$

e $y = 3x - 1$
$3x - y = 6$

f $y = -\dfrac{2x}{3} + 2$
$2x + 3y = 6$

2 Comment on the use of technology to find the point(s) of intersection in **1 e** and **1 f**.

EXERCISE 7A

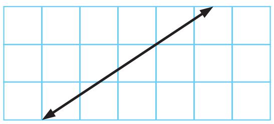
1 Use the distance formula to find the distance between the following pairs of points:

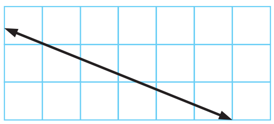
a A(1, 3) and B(4, 5)
b O(0, 0) and C(3, −5)
c P(5, 2) and Q(1, 4)
d S(0, −3) and T(−1, 0)

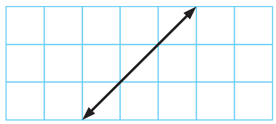
2 Find the midpoint of AB for:

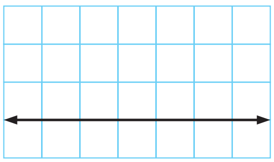
a A(3, 6) and B(1, 0)
b A(5, 2) and B(−1, −4)
c A(7, 0) and B(0, 3)
d A(5, −2) and B(−1, −3)

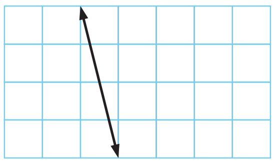
3 By finding an appropriate y-step and x-step, determine the slope of each of the following lines:

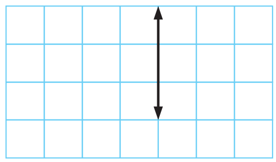
a


b


c


d


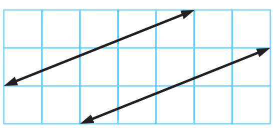
e


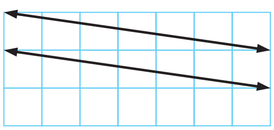
f


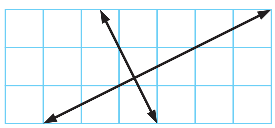
4 Find the gradient of the line passing through:

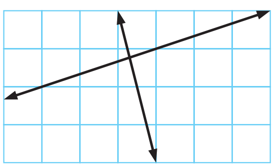
a (2, 3) and (4, 7)
b (3, 2) and (5, 8)
c (−1, 2) and (−1, 5)
d (4, −3) and (−1, −3)
e (0, 0) and (−1, 4)
f (3, −1) and (−1, −2)

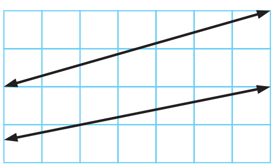
5 Classify the following pairs of lines as parallel, perpendicular or neither. Give a reason.

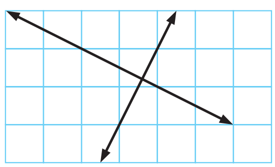
a


b


c


d


e


f


6 State the slope of the line which is perpendicular to the line with gradient (slope):

a $\frac{3}{4}$ **b** $\frac{11}{3}$ **c** 4 **d** $-\frac{1}{3}$ **e** -5 **f** 0

7 Find the equation of the line in slope-intercept form, through:

a $(4, 1)$ with slope 2 **b** $(1, 2)$ with slope -2 **c** $(5, 0)$ with slope 3
d $(-1, 7)$ with slope -3 **e** $(1, 5)$ with slope -4 **f** $(2, 7)$ with slope 1

8 Find the equation of the line, in general form, through:

a $(2, 1)$ with slope $\frac{3}{2}$ **b** $(1, 4)$ with slope $-\frac{3}{2}$ **c** $(4, 0)$ with slope $\frac{1}{3}$
d $(0, 6)$ with slope -4 **e** $(-1, -3)$ with slope 3 **f** $(4, -2)$ with slope $-\frac{4}{9}$

9 Find the equations of the lines through:

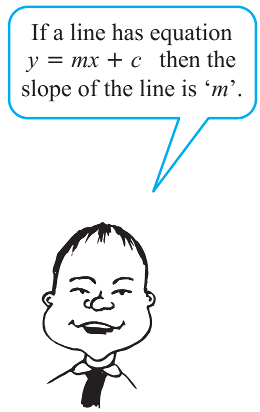
a $(0, 1)$ and $(3, 2)$ **b** $(1, 4)$ and $(0, -1)$
c $(2, -1)$ and $(-1, -4)$ **d** $(0, -2)$ and $(5, 2)$
e $(3, 2)$ and $(-1, 0)$ **f** $(-1, -1)$ and $(2, -3)$

10 Find the equations of the lines through:

a $(3, -2)$ and $(5, -2)$ **b** $(6, 7)$ and $(6, -11)$ **c** $(-3, 1)$ and $(-3, -3)$

11 Copy and complete:

	Equation of line	*Slope*	*x-intercept*	*y-intercept*
a	$2x - 3y = 6$			
b	$4x + 5y = 20$			
c	$y = -2x + 5$			
d	$x = 8$			
e	$y = 5$			
f	$x + y = 11$			
g	$4x + y = 8$			
h	$x - 3y = 12$			



12 **a** Does $(3, 4)$ lie on the line with equation $3x - 2y = 1$?
b Does $(-2, 5)$ lie on the line with equation $5x + 3y = -5$?
c Does $(6, -\frac{1}{2})$ lie on the line $3x - 8y = 22$?

13 Use graphical methods to find where the following lines meet:

a $x + 2y = 8$, $y = 2x - 6$
b $y = -3x - 3$, $3x - 2y = -12$
c $3x + y = -3$, $2x - 3y = -24$
d $2x - 3y = 8$, $3x + 2y = 12$
e $x + 3y = 10$, $2x + 6y = 11$
f $5x + 3y = 10$, $10x + 6y = 20$

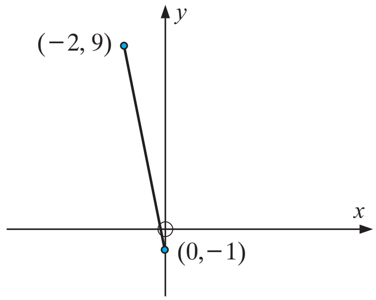
B EQUATIONS OF LINES

The equation of a line can be determined if we know, or have sufficient information to determine, the **gradient** and a **point** on the line.

RESTRICTED DOMAINS

The line segment connecting $(0,\ -1)$ to $(-2,\ 9)$ has equation $y = -5x - 1$. So to describe the line segment exactly we say that $y = -5x - 1$ where $-2 \leqslant x \leqslant 0$.

We say that $-2 \leqslant x \leqslant 0$ is the restricted domain in this example.



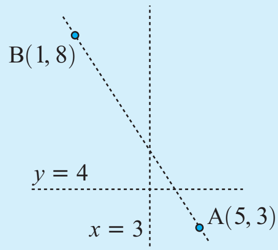
Example 9

A straight road is to pass through points A(5, 3) and B(1, 8).

a Find where this road meets the road given by:
i $x = 3$ **ii** $y = 4$

b If we wish to refer to points on road AB, but between A and B, how can we indicate this?

c Does C(23, −20) lie on the road?



First we must find the equation of the line representing the road.

Its slope is $m = \dfrac{3-8}{5-1} = -\frac{5}{4}$

So, its equation is $\dfrac{y-3}{x-5} = -\dfrac{5}{4}$

i.e., $4(y-3) = -5(x-5)$

i.e., $4y - 12 = -5x + 25$

i.e., $5x + 4y = 37$

a **i** when $x = 3$, $5(3) + 4y = 37$
$\therefore\ 15 + 4y = 37$
$\therefore\ 4y = 22$
$\therefore\ y = 5\frac{1}{2}$

i.e., meets at $(3,\ 5\frac{1}{2})$

ii when $y = 4$, $5x + 4(4) = 37$
$\therefore\ 5x + 16 = 37$
$\therefore\ 5x = 21$
$\therefore\ x = 4.2$

$\therefore$ meets at $(4.2,\ 4)$

b We state the possible x-value restriction i.e., $1 \leqslant x \leqslant 5$.

c If C(23, −20) lies on the line, its coordinates must satisfy the line's equation.

$$\begin{aligned}\text{Now LHS } &= 5(23) + 4(-20)\\ &= 115 - 80\\ &= 35\\ &\neq 37 \qquad \therefore\ \text{C does not lie on the road.}\end{aligned}$$

EXERCISE 7B.1

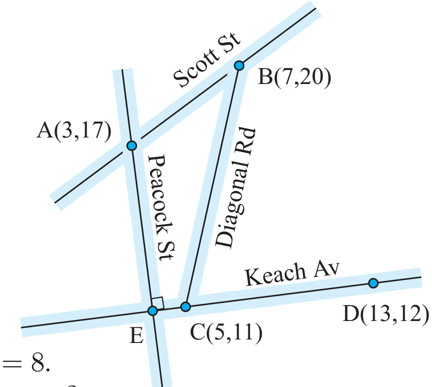
1 Find the equation of the:

- **a** horizontal line through $(3,\ -4)$
- **b** vertical line with x-intercept 5
- **c** vertical line through $(-1,\ -3)$
- **d** horizontal line with y-intercept 2
- **e** x-axis
- **f** y-axis

2 Find the equation of the line:

- **a** through A$(-1,\ 4)$ and with slope $\frac{3}{4}$
- **b** through P$(2,\ -5)$ and Q$(7,\ 0)$
- **c** which is parallel to the line with equation $y = 3x - 2$, but passes through $(0,\ 0)$
- **d** which is parallel to the line with equation $2x + 3y = 8$, but passes through $(-1,\ 7)$
- **e** which is perpendicular to the line with equation $y = -2x + 5$ and passes through $(3,\ -1)$
- **f** which is perpendicular to the line with equation $3x - y = 11$ and passes through $(-2,\ 5)$.

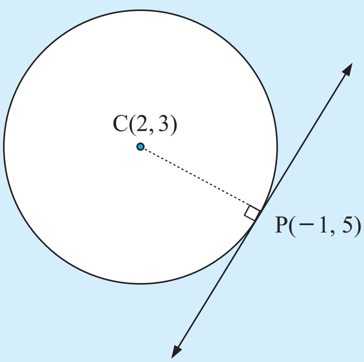
3 A is the town hall on Scott Street and D is a Post Office on Keach Avenue. Diagonal Road intersects Scott Street at B and Keach Avenue at C.



- **a** Find the equation of Keach Avenue.
- **b** Find the equation of Peacock Street.
- **c** Find the equation of Diagonal Road. (Be careful!)
- **d** Plunkit Street lies on map reference line $x = 8$. Where does Plunkit Street intersect Keach Avenue?

Example 10

Find the equation of the tangent to the circle with centre $(2,\ 3)$ at the point $(-1,\ 5)$.



slope of CP is $\dfrac{3-5}{2-(-1)} = \dfrac{-2}{3} = -\frac{2}{3}$

$\therefore$ the slope of the tangent at P is $\frac{3}{2}$,

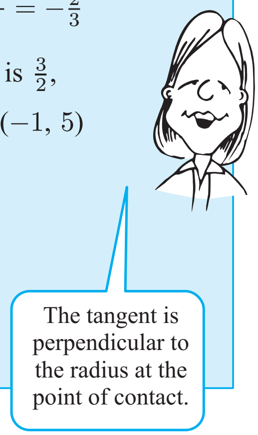
and since the tangent is through $(-1,\ 5)$ the equation is

$$\frac{y-5}{x--1} = \frac{3}{2}$$

$\therefore \quad 2(y-5) = 3(x+1)$

$\therefore \quad 2y - 10 = 3x + 3$

i.e., $\quad 3x - 2y = -13$



4 Find the equation of the tangent to the circle with centre:

a (0, 2) at (−1, 5) **b** (3, −1) at (−1, 1) **c** (2, −2) at (5, −2)

FINDING THE EQUATION OF A LINE QUICKLY

If a line has slope $\frac{3}{4}$, it must have form $y = \frac{3}{4}x + c$

i.e., $4y = 3x + 4c$ or $3x - 4y =$ a constant.

If a line has slope $-\frac{3}{4}$, using the same working we would obtain $3x + 4y =$ a constant.

This suggests that

> for slope $\dfrac{A}{B}$ the **form** of the line is $Ax - By = ...$
>
> for slope $-\dfrac{A}{B}$ the **form** of the line is $Ax + By = ...$

The constant term on the RHS is obtained by substituting a point which lies on the line into this form.

Example 11

Find the equation of the line:

a with slope $\frac{3}{4}$, passing through (5, −2)

b with slope $-\frac{3}{4}$, passing through (1, 7)

a The equation is $3x - 4y = 3(5) - 4(-2)$

i.e., $3x - 4y = 23$

b The equation is $3x + 4y = 3(1) + 4(7)$

i.e., $3x + 4y = 31$

EXERCISE 7B.2

1 Find the equation of the line:

a through (4, 1) with slope $\frac{1}{2}$

b through (−2, 5) with slope $\frac{2}{3}$

c through (5, 0) with slope $\frac{3}{4}$

d through (3, −2) with slope 3

e through (1, 4) with slope $-\frac{1}{3}$

f through (0, 4) with slope −3

2 We can use the reverse process to question **1** to write down the slope of a line given in general form. Find the slope of the line with equation:

a $2x + 3y = 8$ **b** $3x - 7y = 11$ **c** $6x - 11y = 4$

d $5x + 6y = -1$ **e** $3x + 6y = -1$ **f** $15x - 5y = 17$

3 Explain why:

a a line parallel to $3x + 5y = ...$ has form $3x + 5y = ...$

b a line perpendicular to $3x + 5y = ...$ has form $5x - 3y = ...$

4 Find the equation of the line which is:

a parallel to the line $3x+4y=6$ and passes through $(2,\ 1)$
b perpendicular to the line $5x+2y=10$ and passes through $(-1,\ -1)$
c perpendicular to the line $x-3y+6=0$ and passes through $(-4,\ 0)$
d parallel to the line $x-3y=11$ and passes through $(0,\ 0)$.

5 $2x-3y=6$ and $6x+ky=4$ are two straight lines.

a Write down the slope of each line. **b** Find k if the lines are parallel.
c Find k if the lines are perpendicular.

C DISTANCE BETWEEN TWO POINTS

A very common question is "What is the distance from point A to point B?" When travelling, time required and costs are directly affected by distance. Likewise, the costs of supplying services such as power lines, water mains, gas and sewerage pipes etc. depend on distance.

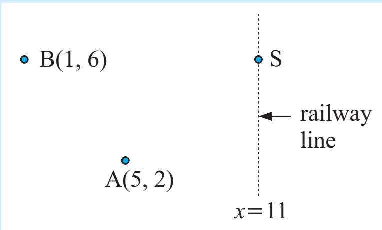
In the Cartesian coordinate system,

the **distance** between $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is given by $\boldsymbol{d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$.

The distance formula is a useful tool in coordinate geometry problem solving.

Example 12

Mining towns are situated at B(1, 6) and A(5, 2). Where should a railway siding S be located so that ore trucks from either A or B would travel equal distances to a railway line with equation $x=11$?



We let a general point on the line $x=11$ have coordinates $(11,\ a)$ say.

$$
\begin{aligned}
\text{Now}\quad \text{BS} &= \text{AS}\\
\therefore\quad \sqrt{(11-1)^2+(a-6)^2} &= \sqrt{(11-5)^2+(a-2)^2}\\
\therefore\quad 10^2+(a-6)^2 &= 6^2+(a-2)^2 \qquad \{\text{squaring both sides}\}\\
\therefore\quad 100+a^2-12a+36 &= 36+a^2-4a+4\\
\therefore\quad -12a+4a &= 4-100\\
\therefore\quad -8a &= -96\\
\therefore\quad a &= 12
\end{aligned}
$$

So, the railway siding should be located at (11, 12).

EXERCISE 7C

1 Show using *distances only* that:

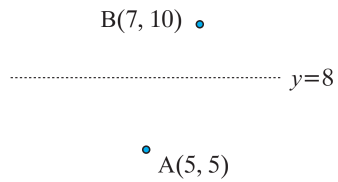
a the triangle with vertices A(3, 8), B(−11, 3) and C(−8, −2) is isosceles
b the triangle with vertices P(−1, 0), Q(0, $\sqrt{3}$) and R(1, 0) is equilateral

c the triangle with vertices K(6, -5), L(-2, -3) and M(-1, 1) is right angled

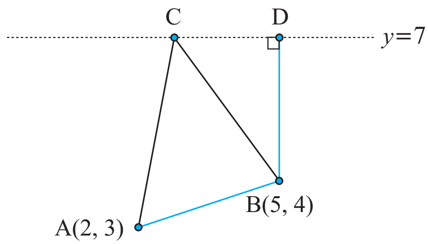
d the points A(1, -1), B(3, 5) and C(-4, -16) are collinear

e the triangle with vertices A(7, 5), B(2, 3) and C(6, -7) is right angled.

2 A(5, 5) and B(7, 10) are houses and $y = 8$ is a gas pipeline. Where should the one outlet from the pipeline be placed so that it is the same distance from both houses so they pay equal service costs?



3 CD is a water pipeline. A and B are two towns. A pumping station is to be located on the pipeline to pump water to A and B. Each town is to pay for their own service pipes and insist on equality of costs.



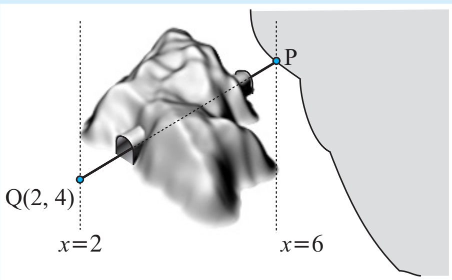
Scale: 1 unit ≡ 1 km

a Where should C be located for equality of costs to occur?

b What is the total length of service pipe required?

c If the towns agree to pay equal amounts, would it be cheaper to install the service pipeline from D to B to A?

Example 13



Scale: each unit is 1 km.

A tunnel through the mountains connects town Q(2, 4) to the port at P.

P is on grid reference $x = 6$ and the distance between the town and the port is 5 km.

Assuming the diagram is reasonably accurate, what is the horizontal grid reference of the port?

Let P be (6, a) say.

$$\begin{aligned}
\text{Now} \quad \text{PQ} &= 5 \\
\therefore \quad \sqrt{(6-2)^2 + (a-4)^2} &= 5 \\
\therefore \quad \sqrt{16 + (a-4)^2} &= 5 \\
\therefore \quad 16 + (a-4)^2 &= 25 \\
\therefore \quad (a-4)^2 &= 9 \\
\therefore \quad a - 4 &= \pm 3 \\
\therefore \quad a &= 4 \pm 3 = 7 \text{ or } 1
\end{aligned}$$

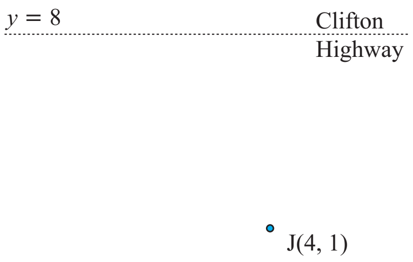
But P is further North than Q $\quad \therefore \quad a > 4$

So P is at (6, 7) and the horizontal grid reference is $y = 7$.

4 Find a if:

- **a** the distance from P$(3,\ a)$ to Q$(-3,\ 6)$ is 10 units
- **b** the distance from A$(a,\ 2)$ to B$(-1,\ 5)$ is $\sqrt{34}$ units
- **c** Q$(a,\ 2a)$ is $\sqrt{5}$ units from R$(-1,\ 1)$.

5 Jason's girlfriend lives in a house on Clifton Highway which has equation $y = 8$. The distance 'as the crow flies' from Jason's house to his girlfriend's house is 11.73 km. If Jason lives at $(4,\ 1)$, what are the coordinates of his girlfriend's house?



Scale: 1 unit $\equiv$ 1 km.

Example 14

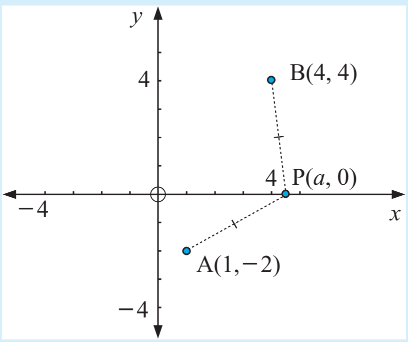
Find any points on the x-axis which are equidistant from A$(1,\ -2)$ and B$(4,\ 4)$. Illustrate.

Any point on the x-axis can be represented by P$(a,\ 0)$.

So, given that AP = BP,

$$\sqrt{(a-1)^2+(0--2)^2} = \sqrt{(a-4)^2+(0-4)^2}$$
$$\therefore \quad \sqrt{a^2-2a+1+4} = \sqrt{a^2-8a+16+16}$$
$$\therefore \quad a^2-2a+5 = a^2-8a+32 \qquad \{\text{squaring both sides}\}$$
$$\therefore \quad 6a = 27$$
$$\therefore \quad a = \tfrac{27}{6} = \tfrac{9}{2}$$

$\therefore$ $(\frac{9}{2},\ 0)$ is equidistant from A and B.



6 Find the coordinates of any point:

- **a** on the x-axis which is equidistant from A$(-3,\ 2)$ and B$(5,\ -1)$
- **b** on the y-axis which is equidistant from C$(5,\ 0)$ and D$(-1,\ -3)$
- **c** on the line with equation $y = 3x$ and 5 units from $(2,\ 1)$.
  (**Hint:** Let $(a,\ 3a)$ lie on the line $y = 3x$.)

D MIDPOINTS AND PERPENDICULAR BISECTORS

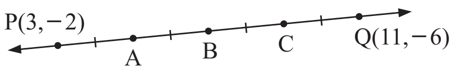
Recall that the **midpoint** of a line segment connecting (x_1, y_1) to (x_2, y_2) is

$$\underbrace{\frac{x_1+x_2}{2}}_{\text{average of } x\text{-coordinates}}, \quad \underbrace{\frac{y_1+y_2}{2}}_{\text{average of } y\text{-coordinates}}$$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

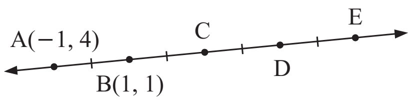
EXERCISE 7D.1

Note: Diagrams are often drawn to display facts or to help us solve a problem. Often no attempt is made to place points in correct positions relative to each other.

1 Given 

find the coordinates of A, B and C.

2 Find the coordinates of C, D and E.



3 Recall that "*the diagonals of a parallelogram bisect each other.*"

a Using midpoints check that ABCD is a parallelogram for A(−1, −2), B(0, 1), C(−3, 2) and D(−4, −1).

b Check that ABCD is a parallelogram by finding slopes of opposite sides.

4 Triangle ABC has vertices A(3, 6), B(−1, −2) and C(7, 4).

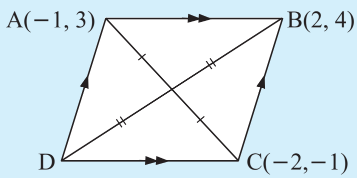
a Use slopes to prove that angle BAC is a right angle.

b Find the midpoint of BC.

c Find the equation of the tangent to the circle through A, B and C at point A.

Example 15

Use midpoints to find the fourth vertex of the given parallelogram:



Since ABCD is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of DB is the same as the midpoint of AC, and if D is (a, b),

$$\frac{a+2}{2} = \frac{-1+-2}{2} \quad \text{and} \quad \frac{b+4}{2} = \frac{3+-1}{2}$$

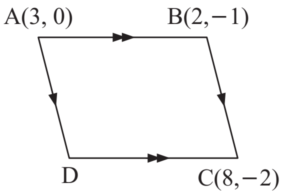
$$\therefore\ a+2=-3 \quad \text{and} \quad b+4=2$$

$$\therefore\ a=-5 \quad \text{and} \quad b=-2$$

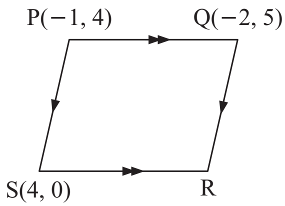
$\therefore$ D is $(-5, -2)$

5 Use midpoints to find the fourth vertex of the given parallelograms:

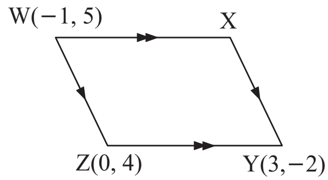
a



b

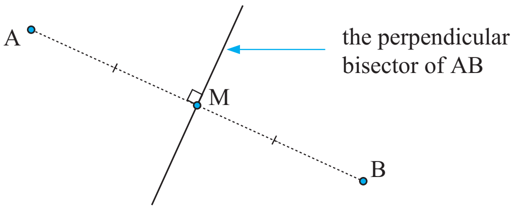


c



PERPENDICULAR BISECTORS

The perpendicular bisector of two points A and B divides the plane into two regions. On one side of the line points are closer to B than to A, and vice versa on the other side.

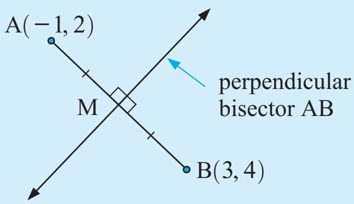


We observe that the midpoint of line segment AB must lie on the perpendicular bisector of AB.

Points on the perpendicular bisector of AB are **equidistant** to A and B.

Example 16

Find the equation of the perpendicular bisector of AB for A(−1, 2) and B(3, 4).



M is $\left(\dfrac{-1+3}{2}, \dfrac{2+4}{2}\right)$ i.e., M(1, 3)

slope of AB is $\dfrac{4-2}{3--1} = \dfrac{2}{4} = \frac{1}{2}$

$\therefore$ slope of perpendicular is $-\frac{2}{1}$

$\therefore$ equation of perpendicular bisector is $\dfrac{y-3}{x-1} = -2$ {using M(1, 3)}

$$\therefore \quad y - 3 = -2(x-1)$$
$$\therefore \quad y - 3 = -2x + 2$$
$$\therefore \quad y = -2x + 5$$

Note: We could have let P on the perpendicular bisector be (x, y) and as AP = PB then

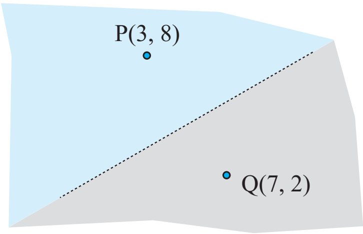
$$\sqrt{(x--1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y-4)^2}$$
$$\therefore \quad (x+1)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2$$
$$\therefore \quad x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 - 8y + 16$$
$$\therefore \quad 4y = -8x + 20$$
$$\text{or} \quad y = -2x + 5$$

EXERCISE 7D.2

1 Find the equation of the perpendicular bisector of AB for:

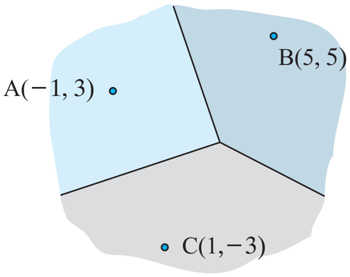
- **a** A(3, −3) and B(1, −1)
- **b** A(1, 3) and B(−3, 5)
- **c** A(3, 1) and B(−3, 6)
- **d** A(4, −2) and B(4, 4)

2 Two Post Offices are located at P(3, 8) and Q(7, 2) on a Council map. What is the equation of the line which should form the boundary between the two regions being serviced by the Post Offices?



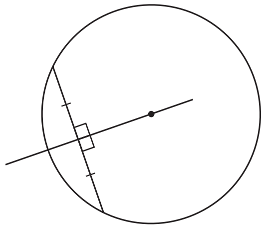
3 The diagram alongside shows the location of three Post Offices and the Voronoi diagram of regions of closest proximity. The Voronoi edges are the perpendicular bisectors of AB, BC and CA respectively. Find:

- **a** the equations of the Voronoi edges
- **b** the coordinates of the Voronoi vertex.



4 Recall that the perpendicular bisector of a chord of a circle passes through the centre of the circle.

A circle passes through points P(5, 7), Q(7, 1) and R(−1, 5). Find the perpendicular bisectors of PQ and QR and solve them simultaneously to find the centre of the circle.

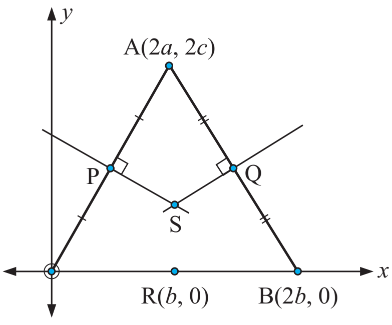


5 Revisit the **Opening Problem** on page **130** to find where the oval should be located.

6 By considering the figure alongside:

- **a** Find the equations of the perpendicular bisectors of OA and AB.
- **b** Find, using **a**, the x-coordinate of S.
- **c** Show that RS is perpendicular to OB.
- **d** Copy and complete:

  The perpendicular bisectors of the sides of a triangle ....



REVIEW SET 7A

1 Find the distance between P(−4, 7) and Q(−1, 3).

2 Determine the midpoint of the line segment joining K(3, 5) to L(7, −2).

3 Find the equation of the line in slope-intercept form, through:

- **a** (2, −1) with slope −3
- **b** (3, −2) and (−1, 4)

4 Find the equation of the line in general form, through:

  a $(1,\ -5)$ with slope $\frac{2}{3}$ **b** $(2,\ -3)$ and $(-4,\ -5)$

5 Find where the following lines cut the axes: **a** $y = -\frac{3}{2}x + 7$ **b** $5x - 3y = 12$

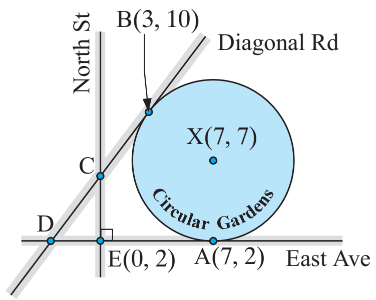
6 Does $(2,\ -5)$ lie on the line with equation $3x + 4y = -14$?

7 Find the equation of the line:

  a through P$(1,\ -5)$ which is parallel to the line with equation $4x - 3y = 6$

  b through Q$(-2,\ 1)$ which is perpendicular to the line with equation $y = -4x + 7$.

8 The Circular Gardens are bounded by East Avenue and Diagonal Road. Diagonal Road intersects North Street at C and East Avenue at D. Diagonal Rd is tangential to the Circular Gardens at B.



  a Find the equation of:

   i East Avenue **ii** North Street

   iii Diagonal Road.

  b Where does Diagonal Road intersect:

   i East Avenue **ii** North Street?

9 If $3x + ky = 7$ and $y = 3 - 4x$ are the equations of two lines, find k if:

  a the lines are parallel **b** the lines are perpendicular.

10 Find the equation connecting x and y if the distance of P$(x,\ y)$ from $(0,\ 4)$ is three times the distance of P from $(4,\ 0)$.

REVIEW SET 7B

1 State the equation of the line:

  a parallel to $3x + 5y = 7$ through $(-1,\ 3)$

  b perpendicular to $2x - 7y = 5$ through $(4,\ 2)$.

2 If $5x - 7y = 8$ and $3x + ky = -11$ are the equations of two lines, find k if:

  a the lines are parallel **b** the lines are perpendicular.

3 Find the coordinates of the point where the line through A$(-3,\ 2)$ and B$(1,\ 7)$ meets the line with equation $3x + 2y = 6$.

4 Determine the nature of the triangle KLM for K$(-5,\ -2)$, L$(0,\ 1)$ and M$(3,\ -4)$.

5 A$(3,\ 2)$ and B$(5,\ 7)$ are beach shacks and $y = 5$ is a power line. Where should the one power outlet be placed so that it is an equal distance to both shacks and they pay equal service costs?

6 Find k given that $(-3,\ k)$ is 7 units away from $(2,\ 4)$.

7 A point T on the y-axis, is 3 units from the point A$(-1,\ 2)$. Find:

  a the coordinates of T (there are two points T_1, T_2 say)

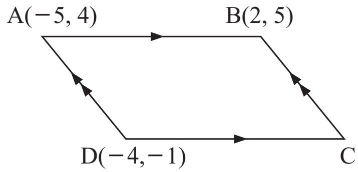
  b the equation of the line AT_1, given that T_1 is above T_2.

8 If P(x, y) is equidistant from A(-1, 4) and B(3, -2):

a draw a sketch of the possible positions of P

b find the equation connecting x and y.

9 Use midpoints to find the fourth vertex of the given parallelogram:



10 (4, 1) is one end of the diameter of a circle and the tangent through the point at the other end of the diameter has equation $3x - y = 1$. Determine the coordinates of the centre of the circle.

REVIEW SET 7C

1 Use midpoints to find the fourth vertex, K, of parallelogram HIJK for H(3, 4), I(-3, -1), and J(4, 10).

2 Find the equation of the perpendicular bisector of the line segment joining P(7, -1) to Q(-3, 5).

3 If A(-3, 2), B(6, 8) and C(2, k) are collinear, find the value of k.

4 AB is a diameter of a circle with centre (-3, 4). Find the coordinates of A if B has coordinates (6, 1).

5 For the points A(-1, 3), B(2, 1) and C(-3, 0), find:

a the equation of the line through C and parallel to AB

b the coordinates of D, the fourth vertex of parallelogram ABCD.

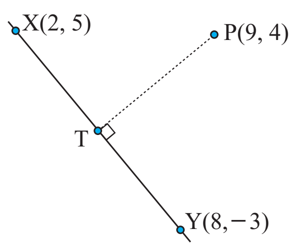
6 If A is a point on the line with equation $y = x - 1$ and $2\sqrt{2}$ units from B(4, -1), find the coordinates of A.

7 Two primary schools are located at P(5, 12) and Q(9, 4) on a council map. If the Local Education Authority wishes to zone the region so that children must attend that school which is closer to their place of residence, what is the equation of the line which should form this boundary?

8 A straight rail line passes through two stations with map references X(2, 5) and Y(8, -3) respectively. A spur line is to be built from a smelting works P(9, 4) to the line at T. To minimise costs the spur line PT is to be as short as possible. Find:

a the coordinates of T

b the length of the spur line from P to T given that the grid reference scale is 1 unit $\equiv$ 10 km.



9 A circle has centre A(4, -2) and a tangent at B(-1, 2). Find:

a the radius **b** the slope of AB **c** the equation of the tangent at B

d the equation of the other tangent parallel to the tangent at B.

10 A ship's radar system has a range of 100 km. If it is located at grid reference (4, 3), will its radar system detect another ship located at grid reference (7, 6) given that the scale is 1 unit $\equiv$ 20 km?

Chapter 8

Quadratic equations and functions

Contents:

A Function notation $f: x \mapsto ax^2 + bx + c$

B Graphs of quadratic functions

Investigation 1: Graphing $y = a(x-\alpha)(x-\beta)$

Investigation 2: Graphing $y = a(x-h)^2 + k$

C Completing the square

D Quadratic equations

E The quadratic formula

F Solving quadratic equations with technology

G Problem solving with quadratics

H Quadratic graphs (review)

I The discriminant, Δ

J Determining the quadratic from a graph

K Where functions meet

L Quadratic modelling

Review set 8A

Review set 8B

Review set 8C

Review set 8D

Review set 8E

INTRODUCTION

Consider the functions:

$f: x \longmapsto ax + b, \quad a \neq 0$	**Linear**
$f: x \longmapsto ax^2 + bx + c, \quad a \neq 0$	**Quadratic**
$f: x \longmapsto ax^3 + bx^2 + cx + d, \quad a \neq 0$	**Cubic**
$f: x \longmapsto ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	**Quartic**

These functions are the simplest members of the family of polynomials.

In this chapter we will examine quadratic functions in detail.

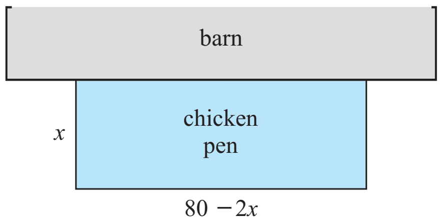
Quadratic functions arise in many situations.
Consider the following problem.

A MATHEMATICAL MODEL OF A CHICKEN PEN

You may need to review algebraic expansion and factorisation. To do this click on the 'Background Knowledge' icon on page 11.

Farmer Brown wishes to construct a rectangular chicken pen, using her barn for one of the sides. The other three sides will be constructed out of 80 m of chicken wire. (See the diagram.)

She can make the pen long and thin, or short and fat, or any rectangular shape in between, as long as she uses exactly 80 m of chicken wire. She is keen to know which rectangular shape will give her the maximum area.



We can create a mathematical model for this problem. Consider these steps involved in developing the model.

Step 1: **Identify the situation.**
Construct a well-defined question which would indicate exactly what Farmer Brown wants to know.

Step 2: **Simplify the situation.**
List the key features and relationships of this situation. Mark those we will ignore.

Step 3: **Build the model.**

Step 4: **Evaluate the model.**

Let x represent the width of the chicken pen. As there are two sides, each x metres long, and the total length of chicken wire is 80 metres, then the length of the pen, L, is given by $L = 80 - 2x$.

The area of a rectangle is given by $A = \text{width} \times \text{length}$.

For this problem $A = x(80 - 2x)$

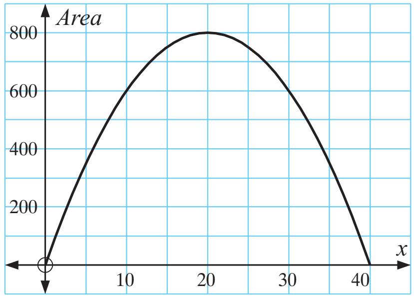
$\therefore \quad A = 80x - 2x^2 \quad$ {expanding}

This is an example of a quadratic function.

From the equation, we can make a table:

width (x)	0	5	10	15	20	25	30	35	40
area (A)	0	350	600	750	800	750	600	350	0

Draw a graph, by plotting the points and joining them with a smooth curve.



From either the graph or the table, note the following:

- The maximum area is 800 m^2, and is obtained by making the width of the pen, $x = 20$ m.
- Both the table and the graph are symmetric about this value.

Substituting into the formula for length:
$$\begin{aligned} L &= 80 - 2x \\ &= 80 - 2 \times 20 \\ &= 40 \end{aligned}$$

The pen should be constructed with a width of 20 m and a length of 40 m, to give a maximum area of 800 m^2.

This makes sense in terms of the original situation. The model could be revised to include other features, for example, adding a door, having a middle dividing fence, and so on. But the model is sufficient for our purposes for now.

HISTORICAL NOTE

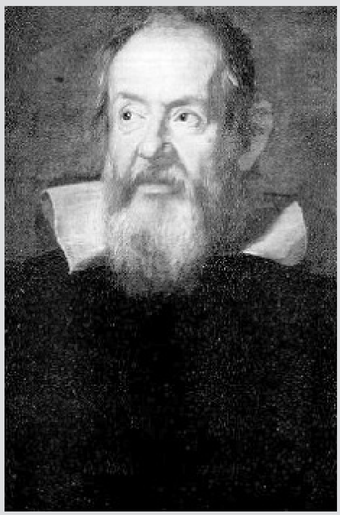
Over 400 years ago, **Galileo** conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor.

In a related experiment a horizontal shelf was placed at the end of the ramp, and the ball would travel along this shelf before descending to the floor.

In each experiment Galileo altered the release height (h) of the ball and measured the distance (d) the ball travelled before landing.

The units of measurement were called 'punti'.

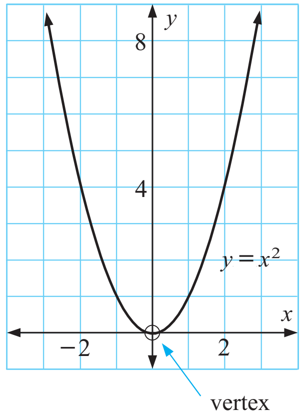


Galileo

THE SIMPLEST QUADRATIC FUNCTION

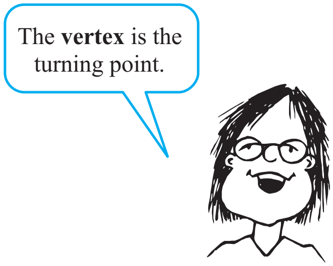
The simplest quadratic function is $y = x^2$ and its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



Note:

- The curve is a **parabola** and it opens upwards.
- There are no negative y values, i.e., the curve does not go below the x-axis.
- The curve is **symmetrical** about the y-axis because, for example, when $x = -3$, $y = (-3)^2$ and when $x = 3$, $y = 3^2$ have the same value.
- The curve has a **turning point** or **vertex** at (0, 0).



Special note:

It is essential that you can draw the graph of $y = x^2$ without having to refer to a table of values.

OPENING PROBLEM

A tennis ball is thrown vertically upwards and its height H, in m, above the ground is given at one second intervals as:

VIDEO CLIP

t	0	1	2	3	4	5
H	6.2	25.2	34.2	33.2	20.2	1.2

For you to consider:

- When the ball was released, was the thrower likely to be standing at ground level, standing on the roof of a single storey building or standing on the roof of a two storey building?
- What would the flight of the ball look like from a distance of 50 m away or from directly above the thrower?
- What is the function equation which gives the height H in terms of time t and what would its graph look like when H is plotted against t?
- What is the maximum height reached and when does this occur?
- When is the ball 30 m above the ground?

SIMULATION

A FUNCTION NOTATION $f: x \longmapsto ax^2+bx+c$

The function $f: x \longmapsto ax^2+bx+c, a \neq 0$ can be represented by $f(x) = ax^2+bx+c$.

As with linear functions, for any value of x a corresponding value of y can be found by substituting into the function equation.

For example, if $y = 2x^2 - 3x + 5$, and $x = 3$, then $y = 2 \times 3^2 - 3 \times 3 + 5$
$= 14$

Hence, the ordered pair (3, 14) satisfies the function $y = 2x^2 - 3x + 5$.

Similarly, using function notation we could write,

if $f(x) = 2x^2 - 3x + 5$ and $x = 3$, then $f(3) = 2 \times 3^2 - 3 \times 3 + 5$
$= 14$

Example 1

If $y = -2x^2 + 3x - 4$ find the value of y when:

a $x = 0$ **b** $x = 3$

a When $x = 0$

$$\begin{aligned} y &= -2(0)^2 + 3(0) - 4 \\ &= 0 + 0 - 4 \\ &= -4 \end{aligned}$$

b When $x = 3$

$$\begin{aligned} y &= -2(3)^2 + 3(3) - 4 \\ &= -2(9) + 9 - 4 \\ &= -18 + 9 - 4 \\ &= -13 \end{aligned}$$

EXERCISE 8A

1 Which of the following are quadratic functions?

a $y = 3x^2 - 4x + 1$ **b** $y = 5x - 7$ **c** $y = -x^2$

d $y = \frac{2}{3}x^2 + 4$ **e** $2y + 3x^2 - 5 = 0$ **f** $y = 5x^3 + x - 6$

2 For each of the following functions, find the value of y for the given value of x:

a $y = x^2 + 5x - 4$ $\{x = 3\}$ **b** $y = 2x^2 + 9$ $\{x = -3\}$

c $y = -2x^2 + 3x - 5$ $\{x = 1\}$ **d** $y = 4x^2 - 7x + 1$ $\{x = 4\}$

Example 2

If $f(x) = -2x^2 + 3x - 4$ find:

a $f(2)$ **b** $f(-4)$

a
$$\begin{aligned} f(2) &= -2(2)^2 + 3(2) - 4 \\ &= -2(4) + 6 - 4 \\ &= -8 + 6 - 4 \\ &= -6 \end{aligned}$$

b
$$\begin{aligned} f(-4) &= -2(-4)^2 + 3(-4) - 4 \\ &= -2(16) - 12 - 4 \\ &= -32 - 12 - 4 \\ &= -48 \end{aligned}$$

3 For each of the following functions find the value of $f(x)$ given in brackets:

a $f(x)=x^2-2x+3$ $\{f(2)\}$

b $f(x)=4-x^2$ $\{f(-3)\}$

c $f(x)=-\frac{1}{4}x^2+3x-4$ $\{f(0)\}$

d $f(x)=\frac{1}{2}x^2+3x$ $\{f(2)\}$

Example 3

State whether the following quadratic functions are satisfied by the given ordered pairs:

a $y=3x^2+2x$ $(2,\ 16)$

b $f(x)=-x^2-2x+1$ $(-3,\ 1)$

a $$\begin{aligned} y &= 3(2)^2+2(2)\\ &= 12+4\\ &= 16 \end{aligned}$$

i.e., when $x=2$, $y=16$

$\therefore$ $(2,\ 16)$ does satisfy $y=3x^2+2x$

b $$\begin{aligned} f(-3) &= -(-3)^2-2(-3)+1\\ &= -9+6+1\\ &= -2 \end{aligned}$$

i.e., $f(-3)\neq 1$

$\therefore$ $(-3,\ 1)$ does not satisfy $f(x)=-x^2-2x+1$

4 State whether the following quadratic functions are satisfied by the given ordered pairs:

a $f(x)=5x^2-10$ $(0,\ 5)$

b $y=2x^2+5x-3$ $(4,\ 9)$

c $y=-2x^2+3x$ $(-\frac{1}{2},\ 1)$

d $y=-7x^2+8x+15$ $(-1,\ 16)$

e $f(x)=3x^2-13x+4$ $(2,\ -10)$

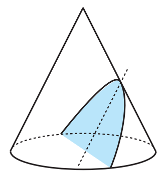
f $f(x)=-3x^2+x+2$ $(\frac{1}{3},\ 2)$

B GRAPHS OF QUADRATIC FUNCTIONS

The graphs of all quadratic functions are **parabolas**. The parabola is one of the conic sections.

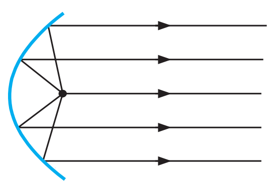
Conic sections are curves which can be obtained by cutting a cone with a plane. The Ancient Greek mathematicians were fascinated by conic sections.

You may like to find the conic sections for yourself by cutting an icecream cone. Cutting parallel to the side produces a parabola, i.e.,



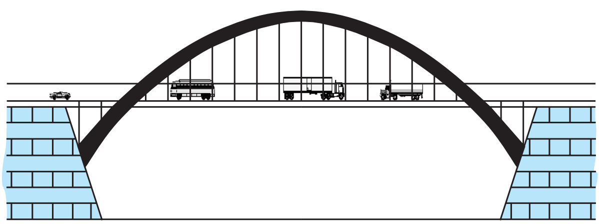
There are many examples of parabolas in every day life. The name parabola comes from the Greek word for **thrown** because when an object is thrown its path makes a parabolic shape.

Parabolic mirrors are used in car headlights, heaters, radar discs and radio telescopes because of their special geometric properties.



Alongside is a single span parabolic bridge.

Other suspension bridges, such as the Golden Gate bridge in San Francisco, also form parabolic curves.



INVESTIGATION 1 GRAPHING $y = a(x-\alpha)(x-\beta)$

PORTFOLIO OPPORTUNITY

This investigation is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

TI
C

What to do:

1 **a** Use technology to assist you to draw sketch graphs of:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$, $y = -3(x-1)(x-3)$ and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to assist you to draw sketch graphs of:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$, $y = 2x(x+5)$ and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of α and β in $y = 2(x-\alpha)(x-\beta)$?

3 **a** Use technology to assist you to draw sketch graphs of:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of α in $y = 2(x-\alpha)^2$?

4 Copy and complete:

- If a quadratic has factorisation $y = a(x-\alpha)(x-\beta)$ it ...... the x-axis at ......
- If a quadratic has factorisation $y = a(x-\alpha)^2$ it ...... the x-axis at ......

INVESTIGATION 2 GRAPHING $y = a(x-h)^2+k$

PORTFOLIO OPPORTUNITY

This investigation is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to assist you to draw sketch graphs of:
$y = (x-3)^2+2$, $y = 2(x-3)^2+2$, $y = -2(x-3)^2+2$, $y = -(x-3)^2+2$ and $y = -\frac{1}{3}(x-3)^2+2$

b Find the coordinates of the vertex for each function in **a**.

TI
C

c What is the geometrical significance of a in $y = a(x-3)^2+2$?

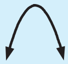
2 **a** Use technology to assist you to to draw sketch graphs of:
$y = 2(x-1)^2+3$, $y = 2(x-2)^2+4$, $y = 2(x-3)^2+1$, $y = 2(x+1)^2+4$, $y = 2(x+2)^2-5$ and $y = 2(x+3)^2-2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2+k$?

3 Copy and complete:
If a quadratic is in the form $y = a(x-h)^2+k$ then its vertex has coordinates ....

From **Investigations 1** and **2** you should have discovered that:

- The coefficient of x^2 (which is a) controls the degree of width of the graph and whether it opens upwards or downwards.
  - $a > 0$  whereas $a < 0$ produces 
  - If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
    If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.
- In the form $y = a(x-\alpha)(x-\beta)$ the graph **cuts** the x-axis at α and β.
- In the form $y = a(x-\alpha)^2$ the graph **touches** the x-axis at α.
- In the form $y = a(x-h)^2 + k$ the graph has vertex (h, k) and axis of symmetry $x = h$.

THE FORM $y = a(x-\alpha)(x-\beta)$

If we are given an equation of the form $y = a(x-\alpha)(x-\beta)$ we can easily graph it using

- the x-intercepts (α and β)
- the coordinates of its vertex $\left(\frac{\alpha+\beta}{2},\ f\left(\frac{\alpha+\beta}{2}\right)\right)$
- the axis of symmetry $\left(x = \frac{\alpha+\beta}{2}\right)$
- the y-intercept (let $x = 0$).

Example 4

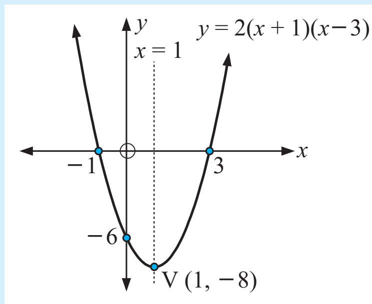
Using axis intercepts only, sketch the graphs of:

a $y = 2(x+1)(x-3)$ **b** $y = -2x(x+2)$

a $y = 2(x+1)(x-3)$
has x-intercepts -1 and 3, $\therefore$ the axis of symmetry is midway between the x-intercepts i.e., $x = 1$

when $x = 1$, $y = 2(2)(-2) = -8$
$\therefore$ the vertex is $(1, -8)$

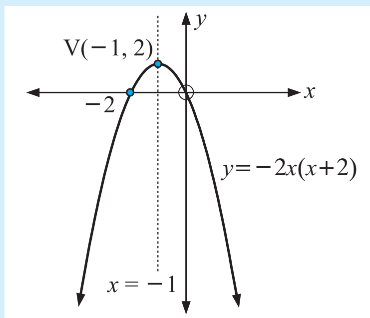
when $x = 0$, $y = 2(1)(-3) = -6$
$\therefore$ the y-intercept is -6



b $y = -2x(x+2)$
has x-intercepts 0 and -2
$\therefore$ the axis of symmetry is $x = -1$

when $x = -1$, $y = -2(-1)(1) = 2$
$\therefore$ the vertex is $(-1, 2)$

when $x = 0$, $y = 0$
$\therefore$ the y-intercept is 0



EXERCISE 8B.1

1 For each of the following functions:

- **i** state the x-intercepts
- **ii** state the equation of the axis of symmetry
- **iii** find the coordinates of the vertex
- **iv** find the y-intercept
- **v** sketch the graph of the function
- **vi** use technology to check your answers.

a $y = (x+2)(x-2)$ **b** $y = 2(x-1)(x-3)$ **c** $y = 3(x-1)(x-2)$

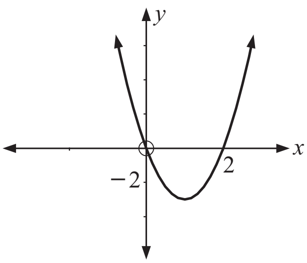
d $y = \frac{1}{2}x(x-4)$ **e** $y = -2x(x+3)$ **f** $y = -\frac{1}{2}(x+2)(x+3)$

2 Match each function with its corresponding graph:

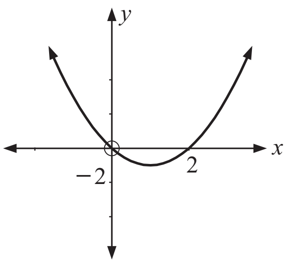
a $y = x(x-2)$ **b** $y = 3x(x-2)$ **c** $y = -x(x-2)$

d $y = (x+2)(x-1)$ **e** $y = 2(x+2)(x-1)$ **f** $y = -2(x+2)(x-1)$

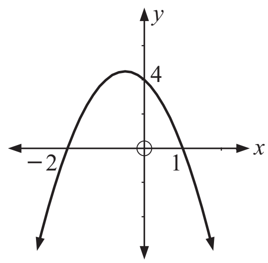
A



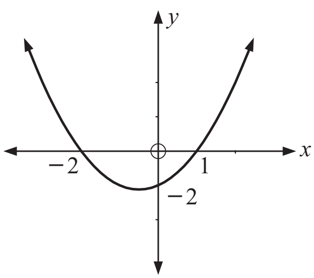
B



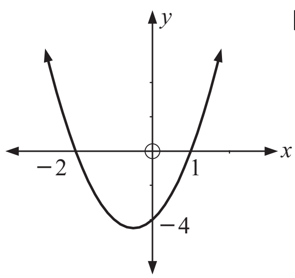
C



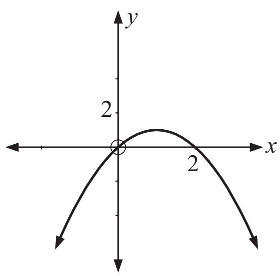
D



E



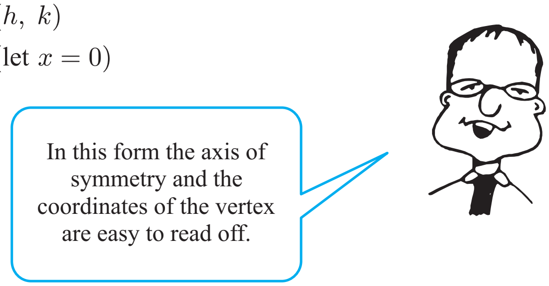
F



THE FORM $y = a(x-h)^2 + k$

If we are given an equation of the form $y = a(x-h)^2 + k$ we can easily graph it using:

- the axis of symmetry $(x = h)$
- the coordinates of the vertex $(h,\ k)$
- the y-intercept. (let $x = 0$)



Example 5

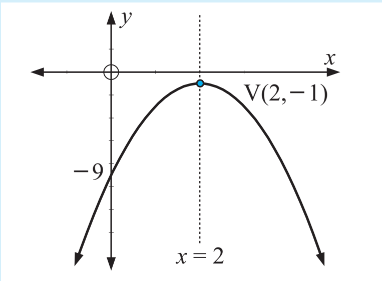
Use the vertex, axis of symmetry and y-intercept to sketch the graph of:

a $y = -2(x-2)^2 - 1$ **b** $y = \frac{1}{2}(x+3)^2$

a $y = -2(x-2)^2 - 1$
has axis of symmetry $x = 2$
and vertex $(2, -1)$

when $x = 0$, $y = -2(-2)^2 - 1$
$= -9$

$\therefore$ y-intercept is -9

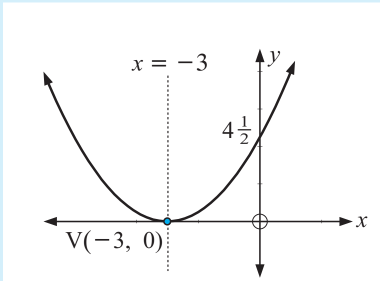


$a < 0$, $\therefore$ the shape is 

b $y = \frac{1}{2}(x+3)^2$
has axis of symmetry $x = -3$
and vertex $(-3, 0)$

when $x = 0$, $y = \frac{1}{2}(3)^2$
$= 4\frac{1}{2}$

$\therefore$ y-intercept is $4\frac{1}{2}$



$a > 0$, $\therefore$ the shape is 

EXERCISE 8B.2

1 For each of the following functions:

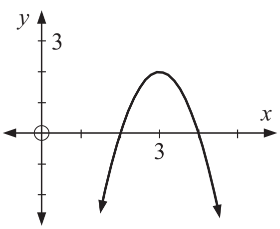
- **i** state the equation of the axis of symmetry
- **ii** find the coordinates of the vertex
- **iii** find the y-intercept
- **iv** sketch the graph of the function
- **v** use technology to check your answers.

a $y = (x-4)^2 + 3$ **b** $y = 2(x+1)^2$ **c** $y = -(x+3)^2 + 2$
d $y = 3(x+2)^2 - 4$ **e** $y = \frac{1}{2}(x-2)^2$ **f** $y = -\frac{3}{2}(x+2)^2 - 4$

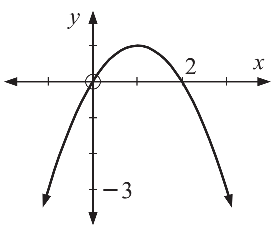
2 Match each quadratic function with its corresponding graph:

a $y = -(x+1)^2 + 3$ **b** $y = -2(x-3)^2 + 2$ **c** $y = x^2 + 2$
d $y = -(x-1)^2 + 1$ **e** $y = (x-2)^2 - 2$ **f** $y = \frac{1}{3}(x+3)^2 - 3$
g $y = -x^2$ **h** $y = -\frac{1}{2}(x-1)^2 + 1$ **i** $y = 2(x+2)^2 - 1$

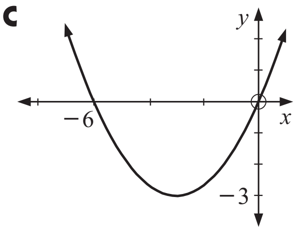
A



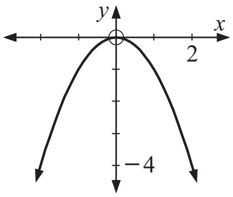
B



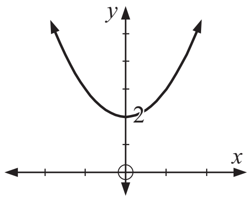
C



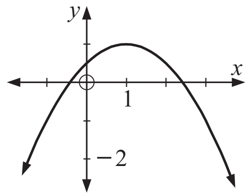
D



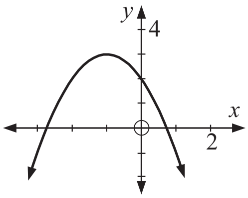
E



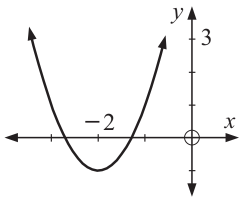
F



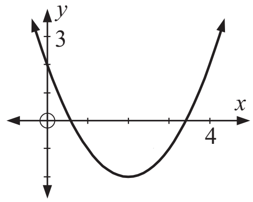
G



H

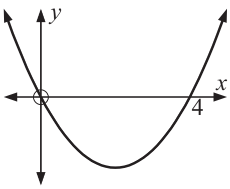


I

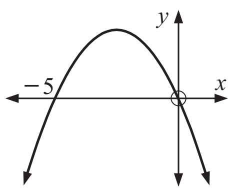


3 For each of the following find the equation of the axis of symmetry:

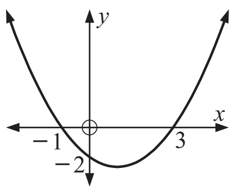
a



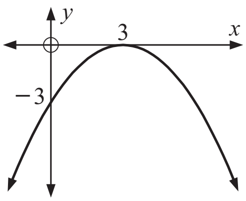
b



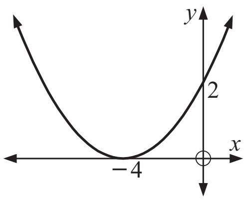
c



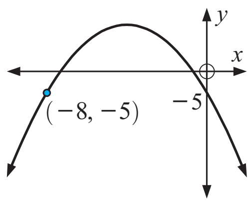
d



e



f



4 For each of the following quadratic functions:

- **i** sketch the graph using axes intercepts and hence find
- **ii** the equation of the axis of symmetry
- **iii** the coordinates of the vertex.

a $y = x^2 + 4x$ **b** $y = x(x-4)$ **c** $y = 3(x-2)^2$

d $y = 2(x-1)(x+3)$ **e** $y = -2(x-1)^2$ **f** $y = -3(x+2)(x-2)$

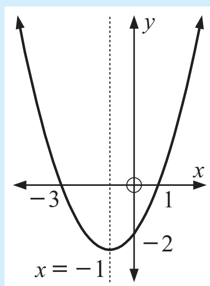
Example 6

Sketch the parabola which has x-intercepts -3 and 1, and y-intercept -2. Find the equation of the axis of symmetry.

The axis of symmetry lies halfway between the x-intercepts $\therefore$ axis of symmetry is $x = -1$.

$\{\dfrac{-3+1}{2} = -1\}$

Note: The graph must open upwards. Can you see why?



5 For each of the following:

   i sketch the parabola
   ii find the equation of the axis of symmetry.

   a x-intercepts 3 and -1, y-intercept -4
   b x-intercepts 2 and -2, y-intercept 4
   c x-intercept -3 (touching), y-intercept 6
   d x-intercept 1 (touching), y-intercept -4

6 Find all x-intercepts of the following graphs of quadratic functions:

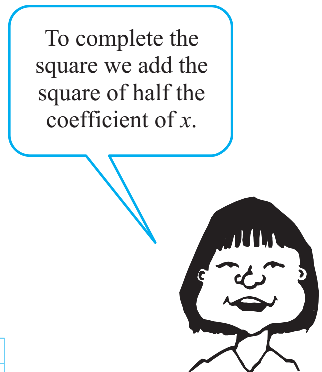
   a cuts the x-axis at 2, axis of symmetry $x = 4$
   b cuts the x-axis at -1, axis of symmetry $x = -3$
   c touches the x-axis at 3.

C COMPLETING THE SQUARE

If we wish to find the vertex of a quadratic given in general form $y = ax^2 + bx + c$ then one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the vertex (h, k). To do this we may choose to '**complete the square**'.

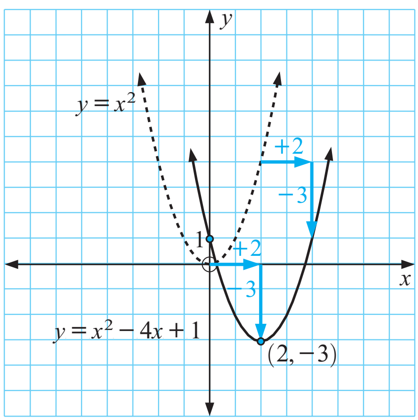
Consider a simple case where $a = 1$; $y = x^2 - 4x + 1$.

$$y = x^2 - 4x + 1$$
$$\therefore\quad y = x^2 - 4x + 2^2 + 1 - 2^2 \qquad \{\text{keeping the equation balanced}\}$$
$$\therefore\quad y = x^2 - 4x + 2^2 - 3$$
$$\therefore\quad y = (x-2)^2 - 3$$



So, $y = x^2 - 4x + 1$ is really $y = (x-2)^2 - 3$

and therefore the graph of $y = x^2 - 4x + 1$ can be considered as the graph of $y = x^2$ after it has been translated 2 units to the right and 3 units down, i.e., $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.



Example 7

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ using completing the square and hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

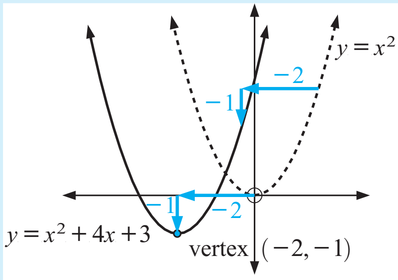
$y = x^2 + 4x + 3$

$\therefore\quad y = x^2 + 4x + 2^2 + 3 - 2^2$

$\therefore\quad y = (x+2)^2 - 1$

shift 2 units left; shift 1 unit down

Vertex is $(-2, -1)$ and the y-intercept is 3



EXERCISE 8C

1 Write the following quadratics in the form $y = (x-h)^2 + k$ using 'completing the square' and hence sketch each function, stating the vertex:

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$

d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

Example 8

Convert $y = 3x^2 - 4x + 1$ into the form $y = a(x-h)^2 + k$ by 'completing the square'. Hence, write down the coordinates of its vertex and sketch the graph of the function.

$y = 3x^2 - 4x + 1$

$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$ {take out a factor of 3}

$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 - (\frac{2}{3})^2 + \frac{1}{3}]$ {complete the square}

$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{1}{3}]$ {write as a perfect square}

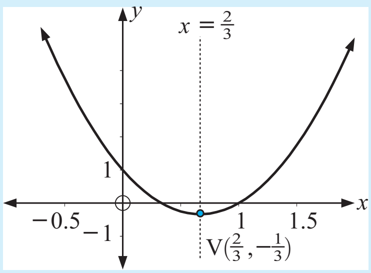
$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{3}{9}]$ {get common denominator}

$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$ {add fractions}

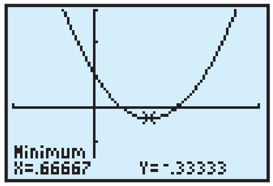
$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$ {expand to put into desired form}

So the vertex is $(\frac{2}{3}, -\frac{1}{3})$

The y-intercept is 1.



We can use technology to confirm this. For example:



2 For each of the following quadratics:

- **i** convert into the form $y = a(x-h)^2 + k$ by 'completing the square'
- **ii** state the coordinates of the vertex
- **iii** find the y-intercept.
- **iv** Hence, sketch the graph of the quadratic.
- **v** Use technology to check your answer.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$

3 By using your **graphing package** or **graphics calculator**, graph each of the following functions, and hence write each function in the form $y = a(x-h)^2 + k$:

a $y = x^2 - 4x + 7$ **b** $y = x^2 + 6x + 3$ **c** $y = -x^2 + 4x + 5$

d $y = 2x^2 + 6x - 4$ **e** $y = -2x^2 - 10x + 1$ **f** $y = 3x^2 - 9x - 5$

D QUADRATIC EQUATIONS

Acme Leather Jacket Co. makes and sells x leather jackets each day and their revenue function is given by $R = 12.5x^2 - 550x + 8125$ dollars.

How many jackets must be made and sold each week in order to obtain income of \$3000 each week?

Clearly we need to solve the equation:

$$12.5x^2 - 550x + 8125 = 3000$$

$$\text{i.e.,} \quad 12.5x^2 - 550x + 5125 = 0$$

This equation, which is of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

> A **quadratic equation**, with variable x, is an equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

To solve quadratic equations we can:

- **factorise** the quadratic and use the **Null Factor law**: "if $ab = 0$ then $a = 0$ or $b = 0$"
- **complete the square**
- use the **quadratic formula**
- use **technology**.

Definition:

The **roots** (or **solutions**) of $ax^2 + bx + c = 0$ are the values of x which satisfy the equation (i.e., make it true).

For example, $x = 2$ is a root of $x^2 - 3x + 2 = 0$ since, when $x = 2$

$$\begin{aligned} x^2 - 3x + 2 &= (2)^2 - 3(2) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \quad \checkmark \end{aligned}$$

SOLVING USING FACTORISATION

Step 1: Make one side of the equation 0 by transferring all terms to one side.
Step 2: Fully factorise the other side.
Step 3: Use the 'Null Factor law': "if $ab = 0$ then $a = 0$ or $b = 0$".
Step 4: Solve the resulting elementary equations.

Example 9

Solve for x: **a** $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

a
$$\begin{aligned} 3x^2 + 5x &= 0 \\ \therefore\ x(3x + 5) &= 0 && \{\text{factorise the LHS}\} \\ \therefore\ x = 0 \text{ or } 3x + 5 &= 0 && \{\text{use Null Factor law}\} \\ \therefore\ x = 0 \text{ or } x &= -\tfrac{5}{3} \end{aligned}$$

b
$$\begin{aligned} x^2 &= 5x + 6 \\ \therefore\ x^2 - 5x - 6 &= 0 && \{\text{equate to } 0\} \\ \therefore\ (x - 6)(x + 1) &= 0 && \{\text{fully factorise LHS}\} \\ \therefore\ x - 6 = 0 \text{ or } x + 1 &= 0 && \{\text{use Null Factor law}\} \\ \therefore\ x &= 6 \text{ or } -1 \end{aligned}$$

EXERCISE 8D.1

1 Solve the following using 'factorisation':

a $4x^2 + 7x = 0$	**b** $6x^2 + 2x = 0$	**c** $3x^2 - 7x = 0$
d $2x^2 - 11x = 0$	**e** $3x^2 = 8x$	**f** $9x = 6x^2$
g $x^2 - 5x + 6 = 0$	**h** $x^2 = 2x + 8$	**i** $x^2 + 21 = 10x$
j $9 + x^2 = 6x$	**k** $x^2 + x = 12$	**l** $x^2 + 8x = 33$

Example 10

Solve for x: **a** $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a
$$\begin{aligned} 4x^2 + 1 &= 4x \\ \therefore\ 4x^2 - 4x + 1 &= 0 \\ \therefore\ (2x-1)^2 &= 0 \\ \therefore\ x &= \tfrac{1}{2} \end{aligned}$$

b
$$\begin{aligned} 6x^2 &= 11x + 10 \\ \therefore\ 6x^2 - 11x - 10 &= 0 \\ \therefore\ (2x-5)(3x+2) &= 0 \end{aligned}$$
{using a factorisation technique}
$$\therefore\ x = \tfrac{5}{2} \text{ or } -\tfrac{2}{3}$$

2 Solve the following using factorisation:

a $9x^2 - 12x + 4 = 0$ **b** $2x^2 - 13x - 7 = 0$ **c** $3x^2 = 16x + 12$
d $3x^2 + 5x = 2$ **e** $2x^2 + 3 = 5x$ **f** $3x^2 = 4x + 4$
g $3x^2 = 10x + 8$ **h** $4x^2 + 4x = 3$ **i** $4x^2 = 11x + 3$
j $12x^2 = 11x + 15$ **k** $7x^2 + 6x = 1$ **l** $15x^2 + 2x = 56$

Example 11

Solve for x: $3x + \dfrac{2}{x} = -7$

$$3x + \frac{2}{x} = -7$$
$$\therefore\ x\left(3x + \frac{2}{x}\right) = -7x \quad \text{\{multiply both sides by } x \text{ to eliminate the fraction\}}$$
$$\therefore\ 3x^2 + 2 = -7x \quad \text{\{clear the bracket\}}$$
$$\therefore\ 3x^2 + 7x + 2 = 0 \quad \text{\{equate to 0\}}$$
$$\therefore\ (x+2)(3x+1) = 0 \quad \text{\{on factorising\}}$$
$$\therefore\ x = -2 \text{ or } -\tfrac{1}{3}$$

3 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$ **b** $(x+2)(1-x) = -4$

c $5 - 4x^2 = 3(2x+1) + 2$ **d** $x + \dfrac{2}{x} = 3$

e $2x - \dfrac{1}{x} = -1$ **f** $\dfrac{x+3}{1-x} = -\dfrac{9}{x}$

FINDING x GIVEN y IN $y = ax^2 + bx + c$

It is also possible to substitute a value for y to find a corresponding value for x. However, unlike linear functions, with quadratic functions there may be 0, 1 or 2 possible values for x for any one value of y.

Example 12

If $y = x^2 - 6x + 8$ find the value(s) of x when:

a $y = 15$ **b** $y = -1$

a If $y = 15$,
$$\begin{aligned} x^2 - 6x + 8 &= 15 \\ \therefore\ x^2 - 6x - 7 &= 0 \\ \therefore\ (x+1)(x-7) &= 0 \quad \{\text{factorising}\} \\ \therefore\ x = -1 \quad &\text{or} \quad x = 7 \qquad \text{i.e., 2 solutions.} \end{aligned}$$

b If $y = -1$,
$$\begin{aligned} x^2 - 6x + 8 &= -1 \\ \therefore\ x^2 - 6x + 9 &= 0 \\ \therefore\ (x-3)^2 &= 0 \quad \{\text{factorising}\} \\ \therefore\ x &= 3 \end{aligned}$$
i.e., only one solution

4 Find the value(s) of x for the given value of y for each of the following quadratic functions:

a $y = x^2 + 6x + 10$ $\{y = 1\}$
b $y = x^2 + 5x + 8$ $\{y = 2\}$
c $y = x^2 - 5x + 1$ $\{y = -3\}$
d $y = 3x^2$ $\{y = -3\}$

Example 13

If $f(x) = x^2 + 4x + 11$ find x when **a** $f(x) = 23$ **b** $f(x) = 7$

a If $f(x) = 23$
$$\begin{aligned} \therefore\ x^2 + 4x + 11 &= 23 \\ \therefore\ x^2 + 4x - 12 &= 0 \\ \therefore\ (x+6)(x-2) &= 0 \quad \{\text{factorising}\} \\ \therefore\ x &= -6 \text{ or } 2 \end{aligned}$$
i.e., 2 solutions.

b If $f(x) = 7$
$$\begin{aligned} \therefore\ x^2 + 4x + 11 &= 7 \\ \therefore\ x^2 + 4x + 4 &= 0 \\ \therefore\ (x+2)^2 &= 0 \quad \{\text{factorising}\} \\ \therefore\ x &= -2 \end{aligned}$$
i.e., one solution only.

5 Find the value(s) of x given that:

a $f(x) = 3x^2 - 2x + 5$ and $f(x) = 5$
b $f(x) = x^2 - x - 5$ and $f(x) = 1$
c $f(x) = -2x^2 - 13x + 3$ and $f(x) = -4$
d $f(x) = 2x^2 - 12x + 1$ and $f(x) = -17$

Example 14

A stone is thrown into the air and its height in metres above the ground is given by the function $h(t) = -5t^2 + 30t + 2$ where t is the time (in seconds) from when the stone is thrown.

a How high above the ground is the stone at time $t = 3$ seconds?

b How high above the ground was the stone released?

c At what time was the stone's height above the ground 27 m?

a $h(3) = -5(3)^2 + 30(3) + 2$
$= -45 + 90 + 2$
$= 47$
i.e., 47 m above ground.

b The stone is released when $t = 0$ sec
$\therefore$ $h(0) = -5(0)^2 + 30(0) + 2 = 2$
$\therefore$ released 2 m above ground level.

c When $h(t) = 27$

$-5t^2 + 30t + 2 = 27$

$\therefore$ $-5t^2 + 30t - 25 = 0$

$\therefore$ $t^2 - 6t + 5 = 0$ {dividing each term by -5}

$\therefore$ $(t-1)(t-5) = 0$ {factorising}

$\therefore$ $t = 1$ or 5

i.e., after 1 sec and after 5 sec. Can you explain the two answers?

6 An object is projected into the air with a velocity of 30 m/s. Its height in metres, after t seconds, is given by the function $h(t) = 30t - 5t^2$.

a Calculate the height after: **i** 1 second **ii** 5 seconds **iii** 3 seconds.

b Calculate the time(s) at which the height is: **i** 40 m **ii** 0 m.

c Explain your answers in part **b**.

7 A cake manufacturer finds that the profit in dollars, from making x cakes per day, is given by the function $P(x) = -\frac{1}{4}x^2 + 16x - 30$.

a Calculate the profit if:

i 0 cakes **ii** 10 cakes are made per day.

b How many cakes per day are made if the profit is \$57?

SOLVING USING 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. In fact, $x^2 + 4x + 1$ cannot be factorised by using a simple factorisation approach.

This means that we need a different approach in order to solve $x^2 + 4x + 1 = 0$.

One way is to use the 'completing the square' technique.

So, equations of the form $ax^2 + bx + c = 0$ can be converted to the form $(x+p)^2 = q$ from which the solutions are easy to obtain.

Example 15

Solve for x: **a** $(x+2)^2 = 7$ **b** $(x-1)^2 = -5$

a $(x+2)^2 = 7$

$\therefore \quad x+2 = \pm\sqrt{7}$

$\therefore \quad x = -2 \pm \sqrt{7}$

b $(x-1)^2 = -5$

has no real solutions

{the perfect square, $(x-1)^2$ cannot be negative}

EXERCISE 8D.2

1 Solve for x:

a $(x+5)^2 = 2$ **b** $(x+6)^2 = 11$

c $(x-4)^2 = 8$ **d** $(x-8)^2 = 7$

e $2(x+3)^2 = 10$ **f** $3(x-2)^2 = 18$

g $(x+1)^2 + 1 = 11$ **h** $(2x+1)^2 = 3$

Notice that if $X^2 = a$, then $X = \pm\sqrt{a}$ is used.

The squared number we add to both sides is $\left(\dfrac{\text{coefficient of } x}{2}\right)^2$

Example 16

Solve for x: $x^2 + 4x + 1 = 0$

$x^2 + 4x + 1 = 0$

$\therefore \quad x^2 + 4x = -1$ {put the constant on the RHS}

$\therefore \quad x^2 + 4x + 2^2 = -1 + 2^2$ {completing the square}

$\therefore \quad (x+2)^2 = 3$ {factorising}

$\therefore \quad x+2 = \pm\sqrt{3}$ {solving}

$\therefore \quad x = -2 \pm \sqrt{3}$

2 Solve for x:

a $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$

d $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$

g $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

3 If the coefficient of x^2 is not 1, we first divide throughout to make it 1.

For example, $2x^2 + 10x + 3 = 0$ becomes $x^2 + 5x + \frac{3}{2} = 0$

$5x^2 + 30x - 7 = 0$ becomes $x^2 + 6x - \frac{7}{5} = 0$

$-3x^2 + 12x + 5 = 0$ becomes $x^2 - 4x - \frac{5}{3} = 0$

Solve for x:

a $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$

d $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

E THE QUADRATIC FORMULA

USING 'THE QUADRATIC FORMULA'

Many quadratic equations cannot be solved by factorising, and completing the square is rather tedious. Consequently, the **quadratic formula** has been developed. This formula is:

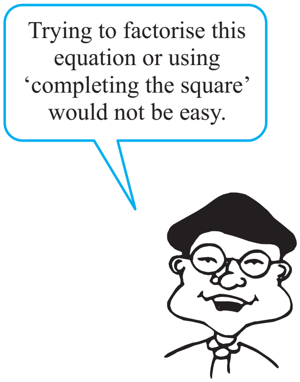
$$\text{If} \quad \boldsymbol{ax^2+bx+c=0}, \quad \text{then} \quad \boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$$

Consider the Acme Leather Jacket Co. equation from page **162**. We need to solve:

$$12.5x^2-550x+5125=0$$

Here we have $a=12.5$, $b=-550$, $c=5125$

$$\begin{aligned}\therefore \quad x &= \frac{550\pm\sqrt{(-550)^2-4(12.5)(5125)}}{2(12.5)}\\ &= \frac{550\pm\sqrt{46\,250}}{25}\\ &\doteq \frac{550+215.06}{25} \quad \text{or} \quad \frac{550-215.06}{25}\\ &\doteq 30.60 \quad \text{or} \quad 13.40\end{aligned}$$



But as x needs to be a whole number, $x=13$ or 31 would produce income of around \$3000 each week.

The following proof of the quadratic formula is worth careful examination.

Proof: If $ax^2+bx+c=0$,

then $x^2+\frac{b}{a}x+\frac{c}{a}=0$ {dividing each term by a, as $a\neq 0$}

$$\therefore \quad x^2+\frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$ {completing the square on LHS}

$$\begin{aligned}\therefore \quad \left(x+\frac{b}{2a}\right)^2 &= -\frac{c}{a}\left(\frac{4a}{4a}\right)+\frac{b^2}{4a^2}\\ \therefore \quad \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\ \therefore \quad x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}}\\ \therefore \quad x &= -\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}\\ \text{i.e.,} \quad x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\end{aligned}$$

Example 17

Solve for x: **a** $x^2 - 2x - 2 = 0$ **b** $2x^2 + 3x - 4 = 0$

a $x^2 - 2x - 2 = 0$ has $a = 1$, $b = -2$, $c = -2$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{4+8}}{2}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{12}}{2}$$

$$\therefore \quad x = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\therefore \quad x = 1 \pm \sqrt{3}$$

So, the solutions are $1 + \sqrt{3}$ and $1 - \sqrt{3}$.

b $2x^2 + 3x - 4 = 0$ has $a = 2$, $b = 3$, $c = -4$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9+32}}{4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{41}}{4}$$

So, the solutions are $\dfrac{-3+\sqrt{41}}{4}$ and $\dfrac{-3-\sqrt{41}}{4}$.

EXERCISE 8E

1 Use the quadratic formula to solve for x:

a $x^2 - 4x - 3 = 0$ **b** $x^2 + 6x + 7 = 0$ **c** $x^2 + 1 = 4x$

d $x^2 + 4x = 1$ **e** $x^2 - 4x + 2 = 0$ **f** $2x^2 - 2x - 3 = 0$

g $x^2 - 2\sqrt{2}x + 2 = 0$ **h** $(3x+1)^2 = -2x$ **i** $(x+3)(2x+1) = 9$

2 Use the quadratic formula to solve for x:

a $(x+2)(x-1) = 2-3x$ **b** $(2x+1)^2 = 3 - x$ **c** $(x-2)^2 = 1 + x$

d $\dfrac{x-1}{2-x} = 2x + 1$ **e** $x - \dfrac{1}{x} = 1$ **f** $2x - \dfrac{1}{x} = 3$

F SOLVING QUADRATIC EQUATIONS WITH TECHNOLOGY

A **graphics calculator** or **graphing package** could be used to solve quadratic equations.

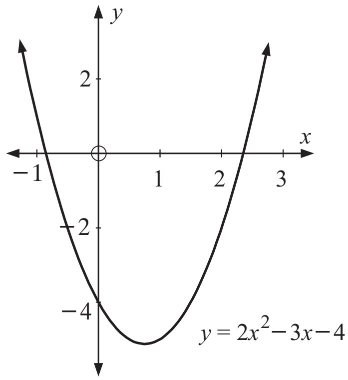
However, exact solutions in square root form would be lost in most cases. Approximate decimal solutions are usually generated. At this stage we will find solutions using **graphs** of quadratics and examine intersections with the x-axis (to get zeros) or the intersection of two functions to find the x-coordinates of the points where they meet.

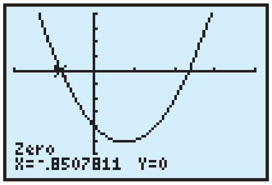
We have chosen to use this approach, even though it may not be the quickest, so that an understanding of the link between the algebra and the graphics is fully appreciated.

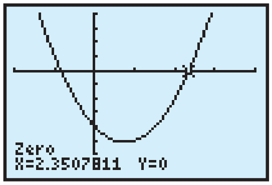
Consider the equation $2x^2 - 3x - 4 = 0$.

Our approach will be:

- draw the graph of $y = 2x^2 - 3x - 4$
- now $2x^2 - 3x - 4 = 0$ when $y = 0$ and this occurs at the x-intercepts of the graph.







The solutions are: $x \doteqdot -0.8508$ or 2.351

Click on the appropriate icon for helpful instructions if using a **graphics calculator** and/or **graphing package**.

GRAPHING PACKAGE

EXERCISE 8F

1 Use technology to solve:

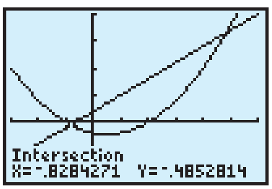
a $x^2 + 4x + 2 = 0$ **b** $x^2 + 6x - 2 = 0$ **c** $2x^2 - 3x - 7 = 0$

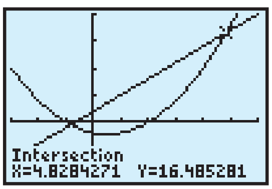
d $3x^2 - 7x - 11 = 0$ **e** $4x^2 - 11x - 13 = 0$ **f** $5x^2 + 6x - 17 = 0$

To solve a more complicated equation like $(x-2)(x+1) = 2 + 3x$ we could:

- make the RHS zero i.e., $(x-2)(x+1) - 2 - 3x = 0$.
  Plot $y = (x-2)(x+1) - 2 - 3x$ and find the x-intercepts.
- plot $y = (x-2)(x+1)$ and $y = 2 + 3x$ on the same axes and find the x-coordinates where the two graphs meet.

If using a graphics calculator with $Y_1 = (x-2)(x+1)$ and $Y_2 = 2 + 3x$ we get





GRAPHING PACKAGE

So, the solutions are $x \doteqdot -0.8284$ or 4.8284

2 Use technology to solve:

a $(x+2)(x-1) = 2-3x$ **b** $(2x+1)^2 = 3-x$ **c** $(x-2)^2 = 1+x$

d $\dfrac{x-1}{2-x} = 2x+1$ **e** $x - \dfrac{1}{x} = 1$ **f** $2x - \dfrac{1}{x} = 3$

G PROBLEM SOLVING WITH QUADRATICS

When solving some problems algebraically, a quadratic equation results. Consequently, we are only interested in any **real solutions** which result.

If the resulting quadratic equation has no real roots then the problem has no solution.

Also, any answer must be checked to see if it is reasonable. For example:

- if you are finding a length then it must be positive, so reject any negative solutions
- if you are finding 'how many people present' then clearly a fractional answer would be unacceptable.

General problem solving method:

Step 1: If the information is given in words, translate it into algebra using x for the unknown, say. An equation results.

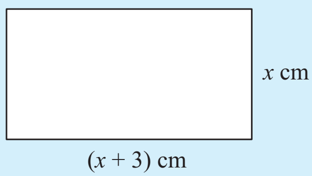
Step 2: Solve the equation by a suitable method.

Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Example 18

A rectangle has length 3 cm longer than its width and its area is 42 cm^2. Find its width.



If the width is x cm, then the length is $(x+3)$ cm.

Therefore $x(x+3) = 42$ {equating areas}

$\therefore\ x^2 + 3x - 42 = 0$

$$\therefore\ x = \frac{-3 \pm \sqrt{9 - 4(1)(-42)}}{2}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{177}}{2}$$

The solution $\dfrac{-3-\sqrt{177}}{2}$ is negative, so we reject it

$$\therefore\ x = \frac{-3+\sqrt{177}}{2} \doteq 5.152$$

$\therefore$ the width is approximately 5.152 cm.

EXERCISE 8G

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

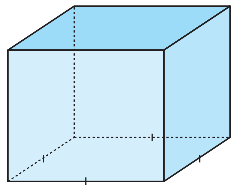
3 The sum of a natural number and its square is 210. Find the number.

4 The product of two consecutive even numbers is 360. Find the numbers.

5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$. A polygon has 90 diagonals. How many sides does it have?

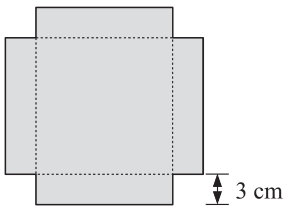
7 A rectangular box has a square base and its height is 1 cm longer than the length of one side of its base.



a If x cm is the length of one side of its base, show that its total surface area, A, is given by $A = 6x^2 + 4x$ cm^2.
b If the total surface area is 240 cm^2, find the dimensions of the box.

8 The length of a rectangle is 4 cm longer than its width. Find its width given that its area is 26 cm^2.

9

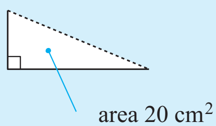


An open box contains 80 cm^3 and is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

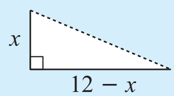
Example 19

Is it possible to bend a 12 cm length of wire to form the legs of a right angled triangle with area 20 cm^2?

i.e., becomes



$$\text{Area,} \quad A = \tfrac{1}{2}(12-x)x$$
$$\therefore \quad \tfrac{1}{2}x(12-x) = 20$$
$$\therefore \quad x(12-x) = 40$$
$$\therefore \quad 12x - x^2 - 40 = 0$$



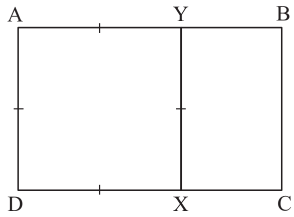
$$\therefore \quad x^2 - 12x + 40 = 0 \quad \text{which becomes} \quad x = \frac{12 \pm \sqrt{-16}}{2}$$

Thus there are no real solutions, indicating the **impossibility**.

10 Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of 30 cm^2?

11 The *golden rectangle* is the rectangle defined by the following statement:

The golden rectangle can be divided into a square and a smaller rectangle by a line which is parallel to its shorter sides, and the smaller rectangle is **similar** to the original rectangle.

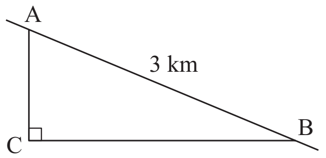


Thus, if ABCD is the golden rectangle, ADXY is a square and BCXY is similar to ABCD, (i.e., BCXY is a reduction of ABCD).

The ratio of $\dfrac{\text{AB}}{\text{AD}}$ for the golden rectangle is called the **golden ratio**.

Show that the golden ratio is $\dfrac{1+\sqrt{5}}{2}$. (**Hint:** Let AB $= x$ units and BC $= 1$ unit.)

12

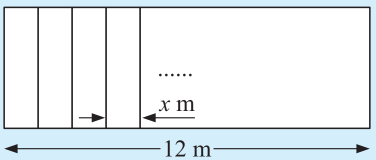


A triangular paddock has a road AB forming its hypotenuse. AB is 3 km long. The fences AC and CB are at right angles. If BC is 400 m longer than AC, find the area of the paddock in hectares.

13 Find the width of a uniform concrete path placed around a 30 m by 40 m rectangular lawn given that the concrete has area one quarter of the lawn.

Example 20

A wall is 12 m long and is timber panelled using vertical sheets of panelling of equal width. If the sheets had been 0.2 m wider, 2 less sheets would have been required. What is the width of the timber panelling used?



Let x m be the width of each panel used.

$\therefore$ $\dfrac{12}{x}$ is the number of sheets needed.

Now if the sheets are $\left(x+\frac{1}{5}\right)$ m in width

$\left(\dfrac{12}{x}-2\right)$ sheets are needed.

So, $\left(x+\frac{1}{5}\right)\left(\dfrac{12}{x}-2\right)=12$ {length of wall}

$\therefore\ 12-2x+\dfrac{12}{5x}-\frac{2}{5}=12$ {expanding LHS}

$\therefore\ -2x+\dfrac{12}{5x}-\frac{2}{5}=0$

$\therefore\ -10x^2+12-2x=0$ {$\times$ each term by $5x$}

$\therefore\ 5x^2+x-6=0$ {$\div$ each term by -2}

$\therefore\ (5x+6)(x-1)=0$

$\therefore\ x=-\frac{6}{5}$ or 1 where $x>0$

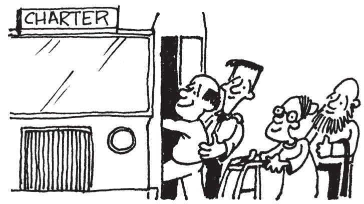
$\therefore$ each sheet is 1 m wide.

14 Chuong and Hassan both drive 40 km from home to work each day. One day, Chuong said to Hassan, "If you drive home at your usual speed, I will average 40 kmph faster than you and arrive home in 20 minutes less time." Find Hassan's speed.

15 If the average speed of an aeroplane had been 120 kmph less, it would have taken a half an hour longer to fly 1000 km. Find the speed of the plane.

16 Two trains travel a 105 km track each day. The express travels 10 kmph faster and takes 30 minutes less than the normal train. Find the speed of the express.

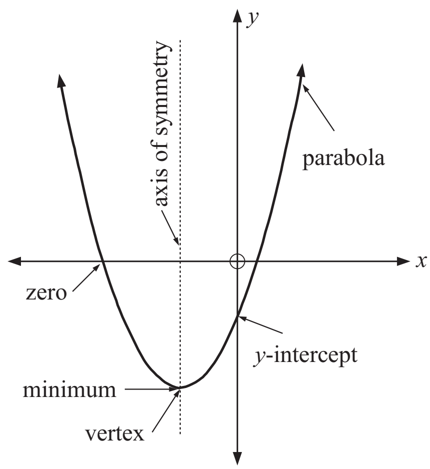
17 A group of elderly citizens chartered a bus for \$160. However, at the last minute, due to illness, 8 of them had to miss the trip. Consequently the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?



H QUADRATIC GRAPHS (REVIEW)

REVIEW OF TERMINOLOGY

The equation of a **quadratic function** is given by $y = ax^2 + bx + c$, where $a \neq 0$.



The graph of a quadratic function is called a **parabola**. The point where the graph 'turns' is called the **vertex**.

If the graph opens upward, the y-coordinate of the vertex is the **minimum**, while if the graph opens downward, the y-coordinate of the vertex is the **maximum**.

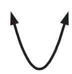
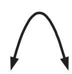
The vertical line that passes through the vertex is called the **axis of symmetry**. All parabolas are symmetrical about the axis of symmetry.

The point where the graph crosses the y-axis is the **y-intercept**.

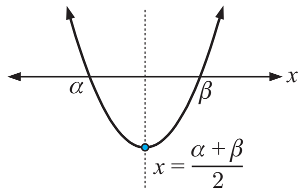
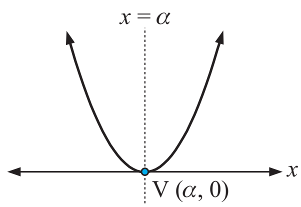
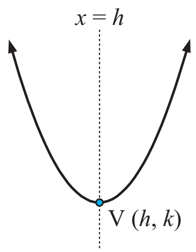
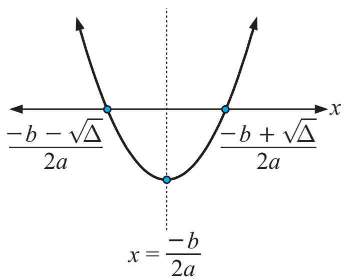
The points (if they exist) where the graph crosses the x-axis should be called the **x-intercepts**, but more commonly are called the **zeros** of the function.

OPENING UPWARDS OR DOWNWARDS?

If the coefficient of x^2:

- is positive, the graph opens upwards  $a > 0$
- is negative, the graph opens downwards  $a < 0$.

Summary of previously deduced facts

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-\alpha)(x-\beta)$ α, β are real		x-intercepts are α and β axis of symmetry is $x = \frac{\alpha+\beta}{2}$
• $y = a(x-\alpha)^2$ α is real		touches x-axis at α vertex is $(\alpha, 0)$ axis of symmetry is $x = \alpha$
• $y = a(x-h)^2 + k$		vertex is (h, k) axis of symmetry is $x = h$
• $y = ax^2 + bx + c$ (general quadratic form)		axis of symmetry is $x = \dfrac{-b}{2a}$ x-intercepts for $\Delta \geqslant 0$ are $\dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$

Notice that the axis of symmetry is always easily found.

Note: $-\dfrac{b}{2a}$ is the **average** of $\dfrac{-b-\sqrt{\Delta}}{2a}$ and $\dfrac{-b+\sqrt{\Delta}}{2a}$.

as the sum equals

$$\frac{-b-\sqrt{\Delta}}{2a} + \frac{-b+\sqrt{\Delta}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

and so the average is $\dfrac{\text{the sum}}{2} = \dfrac{-b}{2a}$

SKETCH GRAPHS USING KEY FACTS

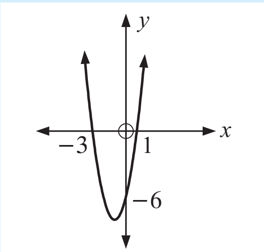
Example 21

Using axis intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

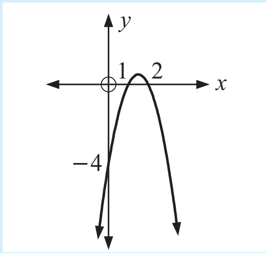
a $y = 2(x+3)(x-1)$ has x-intercepts -3, 1

when $x = 0$,
$y = 2(3)(-1)$
$= -6$
$\therefore$ y-intercept is -6



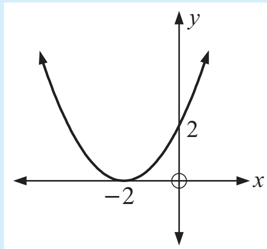
b $y = -2(x-1)(x-2)$ has x-intercepts 1, 2

when $x = 0$,
$y = -2(-1)(-2)$
$= -4$
$\therefore$ y-intercept -4



c $y = \frac{1}{2}(x+2)^2$ touches x-axis at -2

when $x = 0$,
$y = \frac{1}{2}(2)^2$
$= 2$
$\therefore$ has y-intercept 2



EXERCISE 8H

1 Using axis intercepts only, sketch the graphs of:

a $y = (x-4)(x+2)$ **b** $y = -(x-4)(x+2)$ **c** $y = 2(x+3)(x+5)$

d $y = -3x(x+4)$ **e** $y = 2(x+3)^2$ **f** $y = -\frac{1}{4}(x+2)^2$

2 What is the axis of symmetry of each graph in question **1**?

Example 22

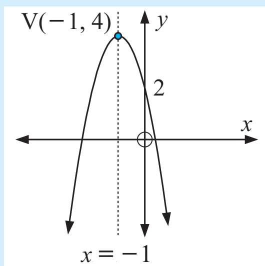
Use the vertex, axis of symmetry and y-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ $\therefore$ $\cap$ shape



3 Use the vertex, axis of symmetry and y-intercept to graph:

a $y = (x-1)^2 + 3$ **b** $y = 2(x+2)^2 + 1$ **c** $y = -2(x-1)^2 - 3$

d $y = \frac{1}{2}(x-3)^2 + 2$ **e** $y = -\frac{1}{3}(x-1)^2 + 4$ **f** $y = -\frac{1}{10}(x+2)^2 - 3$

Example 23

For the quadratic $y = 2x^2 + 6x - 3$, find:

a the equation of the axis of symmetry
b the coordinates of the vertex
c the axes intercepts.
d Hence, sketch the graph.

For $y = 2x^2 + 6x - 3$, $a = 2$, $b = 6$, $c = -3$ $\quad a > 0 \quad \therefore$

a $\dfrac{-b}{2a} = \frac{-6}{4} = -\frac{3}{2}$

$\therefore$ axis of symmetry is $x = -\frac{3}{2}$.

b When $x = -\frac{3}{2}$,

$y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 3$

$= -7.5$ {simplifying}

$\therefore$ vertex is $(-\frac{3}{2}, -7\frac{1}{2})$.

c When $x = 0$, $y = -3$

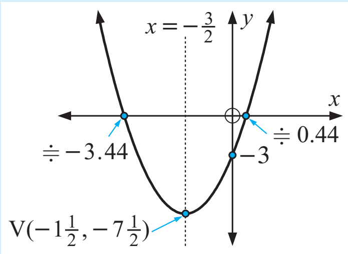
$\therefore$ y-intercept is -3.

When $y = 0$, $2x^2 + 6x - 3 = 0$

$\therefore \quad x = \dfrac{-6 \pm \sqrt{36 - 4(2)(-3)}}{4}$

$\therefore \quad x \doteqdot -3.44$ or 0.44

d



Example 24

Determine the coordinates of the vertex of $y = 2x^2 - 8x + 1$.

$y = 2x^2 - 8x + 1$ has $a = 2$, $b = -8$, $c = 1$

and so $\dfrac{-b}{2a} = \dfrac{-(-8)}{2 \times 2} = 2$

$\therefore$ equation of axis of symmetry is $x = 2$

and when $x = 2$, $y = 2(2)^2 - 8(2) + 1$

$= 8 - 16 + 1$

$= -7$

$\therefore$ the vertex has coordinates $(2, -7)$.

The vertex is sometimes called the maximum turning point or the minimum turning point depending on whether the graph is opening downwards or upwards.

4 Find the turning point (vertex) for the following quadratic functions:

a $y = x^2 - 4x + 2$
b $y = x^2 + 2x - 3$
c $y = 2x^2 + 4$
d $y = -3x^2 + 1$
e $y = 2x^2 + 8x - 7$
f $y = -x^2 - 4x - 9$
g $y = 2x^2 + 6x - 1$
h $y = 2x^2 - 10x + 3$
i $y = -\frac{1}{2}x^2 + x - 5$
j $y = -2x^2 + 8x - 2$

5 Find the x-intercepts for:

a $y = x^2 - 9$ **b** $y = 2x^2 - 6$ **c** $y = x^2 + 7x + 10$

d $y = x^2 + x - 12$ **e** $y = 4x - x^2$ **f** $y = -x^2 - 6x - 8$

g $y = -2x^2 - 4x - 2$ **h** $y = 4x^2 - 24x + 36$ **i** $y = x^2 - 4x + 1$

j $y = x^2 + 4x - 3$ **k** $y = x^2 - 6x - 2$ **l** $y = x^2 + 8x + 11$

6 For the following quadratics, find:

i the equation of the axis of symmetry **ii** the coordinates of the vertex

iii the axes intercepts, if they exist **iv** Hence, sketch the graph.

a $y = x^2 - 2x + 5$ **b** $y = x^2 + 4x - 1$ **c** $y = 2x^2 - 5x + 2$

d $y = -x^2 + 3x - 2$ **e** $y = -3x^2 + 4x - 1$ **f** $y = -2x^2 + x + 1$

g $y = 6x - x^2$ **h** $y = -x^2 - 6x - 8$ **i** $y = -\frac{1}{4}x^2 + 2x + 1$

I THE DISCRIMINANT, Δ

In the quadratic formula, $b^2 - 4ac$, which is under the square root sign, is called **the discriminant**.

The symbol **delta** Δ, is used to represent the discriminant, i.e., $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ if Δ replaces $b^2 - 4ac$.

Notice that

- if $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated root**) as the two roots are equal
- if $\Delta > 0$, $\sqrt{\Delta}$ is a real number and so there are **two distinct real roots**, $\dfrac{-b + \sqrt{\Delta}}{2a}$ and $\dfrac{-b - \sqrt{\Delta}}{2a}$
- if $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.

Note: If a, b and c are rational and Δ is a **perfect square** then the equation has two rational roots which can be found by factorisation.

Example 25

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 3x + 4 = 0$ **b** $4x^2 - 4x - 1 = 0$

a
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - 4(2)(4)\\ &= -23 \text{ which is } < 0\end{aligned}$$
$\therefore$ no real roots

b
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-4)^2 - 4(4)(-1)\\ &= 32 \text{ which is } > 0\end{aligned}$$
$\therefore$ has 2 distinct real roots

EXERCISE 8I.1

1 By using the discriminant only, state the nature of the solutions of:

a $x^2+7x-2=0$ **b** $x^2+4\sqrt{2}x+8=0$ **c** $2x^2+3x-1=0$

d $6x^2+5x-4=0$ **e** $x^2+x+6=0$ **f** $9x^2+6x+1=0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $2x^2+7x-4=0$ **b** $3x^2-7x-6=0$ **c** $2x^2+6x+1=0$

d $6x^2+19x+10=0$ **e** $4x^2-3x+3=0$ **f** $8x^2-10x-3=0$

Example 26

For $x^2-2x+m=0$, determine Δ in simplest form and hence find the values of m for which the equation has:

a a repeated root **b** two distinct real roots **c** no real roots.

$x^2-2x+m=0$ has $a=1$, $b=-2$ and $c=m$

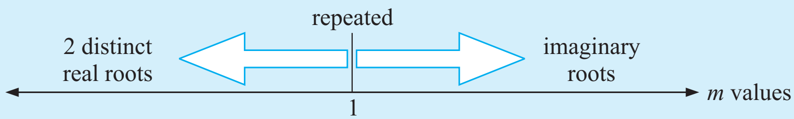
$\therefore\ \Delta=b^2-4ac=(-2)^2-4(1)(m)=4-4m$

a For a repeated root
$$\Delta=0$$
$$\therefore\ 4-4m=0$$
$$\therefore\ 4=4m$$
$$\therefore\ m=1$$

b For 2 distinct real roots
$$\Delta>0$$
$$\therefore\ 4-4m>0$$
$$\therefore\ -4m>-4$$
$$\therefore\ m<1$$

c For no real roots
$$\Delta<0$$
$$\therefore\ 4-4m<0$$
$$\therefore\ -4m<-4$$
$$\therefore\ m>1$$

Note:



3 For the following quadratic equations, determine the discriminant in simplest form and hence find the values of m for which the equation has:

i a repeated root **ii** two distinct roots **iii** no real roots.

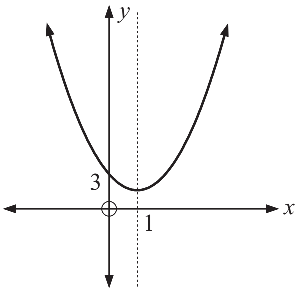
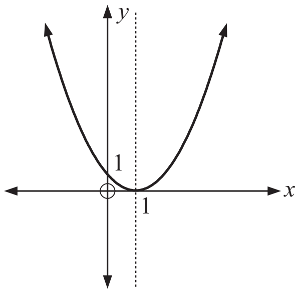
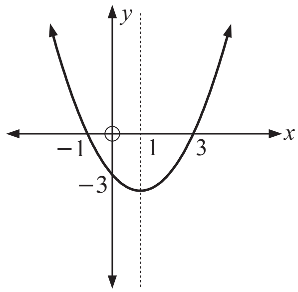
a $x^2+3x+m=0$ **b** $x^2-5x+m=0$ **c** $mx^2-x+1=0$

d $mx^2+2x+3=0$ **e** $2x^2+7x+m=0$ **f** $mx^2-5x+4=0$

THE DISCRIMINANT AND THE QUADRATIC GRAPH

Consider the graphs of:
$$y=x^2-2x+3$$
$$y=x^2-2x+1$$
$$y=x^2-2x-3$$

All of these curves have axis of symmetry with equation $x=1$.

<table>
<tr><th>$y = x^2 - 2x + 3$</th><th>$y = x^2 - 2x + 1$</th><th>$y = x^2 - 2x - 3$</th></tr>
<tr>
<td>
</td>
<td>
</td>
<td>
</td>
</tr>
<tr>
<td>$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(3) \\ &= 4 - 12 \\ &= -8\end{aligned}$</td>
<td>$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(1) \\ &= 4 - 4 \\ &= 0\end{aligned}$</td>
<td>$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(-3) \\ &= 4 + 12 \\ &= 16\end{aligned}$</td>
</tr>
<tr><td>$\Delta < 0$</td><td>$\Delta = 0$</td><td>$\Delta > 0$</td></tr>
<tr><td>does not cut the x-axis</td><td>touches the x-axis</td><td>cuts the x-axis twice</td></tr>
</table>

Thus, the **discriminant** Δ, helps us to decide between the possibilities of:

- not cutting the x-axis $(\Delta < 0)$
- touching the x-axis $(\Delta = 0)$
- cutting the x-axis twice $(\Delta > 0)$.

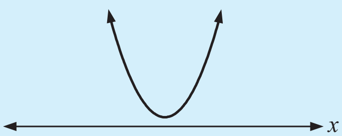
Example 27

Use the discriminant to determine the relationship between the graph and the x-axis of: **a** $y = x^2 + 3x + 4$ **b** $y = -2x^2 + 5x + 1$

a $a = 1, \quad b = 3, \quad c = 4$

$$\therefore \quad \begin{aligned}\Delta &= b^2 - 4ac \\ &= 9 - 4(1)(4) \\ &= 9 - 16 \\ &= -7 \\ &< 0\end{aligned}$$

$a > 0 \quad \therefore$ opens upwards

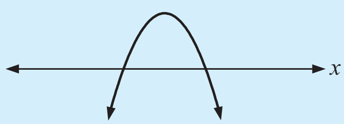


$\therefore$ graph does not cut the x-axis. i.e., lies entirely above the x-axis.

b $a = -2, \quad b = 5, \quad c = 1$

$$\therefore \quad \begin{aligned}\Delta &= b^2 - 4ac \\ &= 25 - 4(-2)(1) \\ &= 25 + 8 \\ &= 33 \\ &> 0\end{aligned}$$

$a < 0 \quad \therefore$ opens downwards

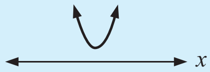


$\therefore$ graph cuts x-axis twice.

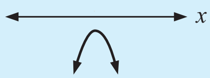
POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Definitions:

Positive definite quadratics are quadratics which are positive for all values of x, i.e., $ax^2 + bx + c > 0$ for all x.



Negative definite quadratics are quadratics which are negative for all values of x, i.e., $ax^2 + bx + c < 0$ for all x.



TESTS

- A quadratic is **positive definite** if $a > 0$ and $\Delta < 0$.
- A quadratic is **negative definite** if $a < 0$ and $\Delta < 0$.

EXERCISE 81.2

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y = x^2 + 7x - 2$ **b** $y = x^2 + 4\sqrt{2}x + 8$ **c** $y = -2x^2 + 3x + 1$

d $y = 6x^2 + 5x - 4$ **e** $y = -x^2 + x + 6$ **f** $y = 9x^2 + 6x + 1$

Example 28

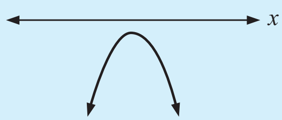
Show that $-x^2 + 3x - 3$ is negative definite.

$a = -1, \quad b = 3, \quad c = -3$

$$\begin{aligned} \text{So } \Delta &= b^2 - 4ac \\ &= 9 - 4(-1)(-3) \\ &= 9 - 12 \\ &= -3 \\ &< 0 \end{aligned}$$

Since $a < 0$ and $\Delta < 0$
$-x^2 + 3x - 3 < 0$ for all x
and is $\therefore$ negative definite

i.e.,



2 Show that:

a $x^2 - 3x + 6 > 0$ for all x **b** $4x - x^2 - 6 < 0$ for all x

c $2x^2 - 4x + 7$ is positive definite **d** $-2x^2 + 3x - 4$ is negative definite

3 Explain why $3x^2 + kx - 1$ is never always positive for any value of k.

4 Under what conditions is $2x^2 + kx + 2$ positive definite?

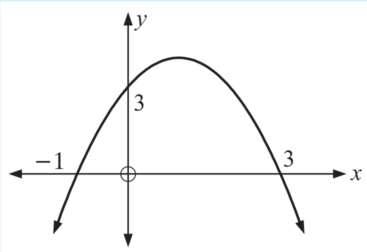
J DETERMINING THE QUADRATIC FROM A GRAPH

If we are given sufficient information on or about a graph we can determine the quadratic function in whatever form required.

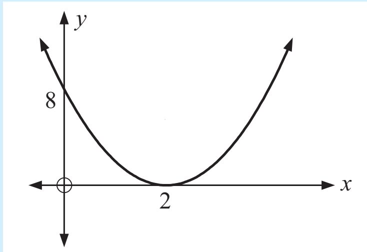
Example 29

Find the equation of the quadratic with graph:

a



b



a Since the x-intercepts are -1 and 3
then $y = a(x+1)(x-3)$, $a \neq 0$.

But when $x = 0$, $y = 3$

$\therefore$ $3 = a(1)(-3)$

$\therefore$ $a = -1$

So, $y = -(x+1)(x-3)$.

b Since it touches at 2, then
$y = a(x-2)^2$, $a \neq 0$.

But when $x = 0$, $y = 8$

$\therefore$ $8 = a(-2)^2$

$\therefore$ $8 = 4a$

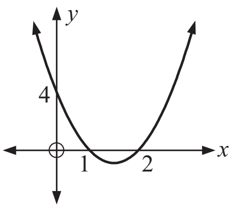
and $\therefore$ $a = 2$

So, $y = 2(x-2)^2$.

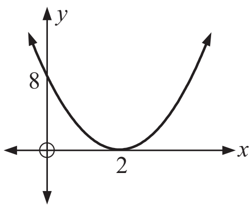
EXERCISE 8J

1 Find the equation of the quadratic with graph:

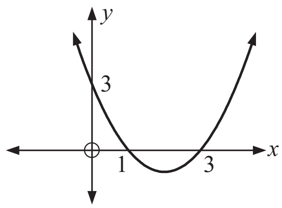
a



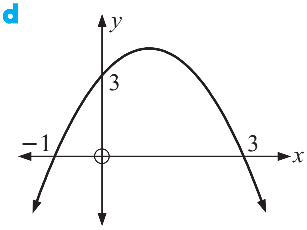
b



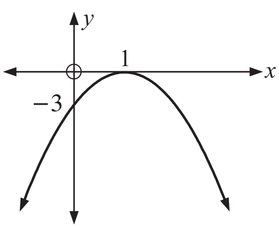
c



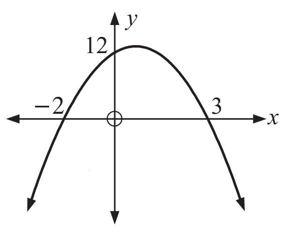
d



e

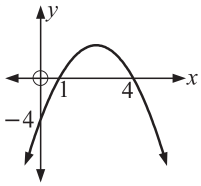


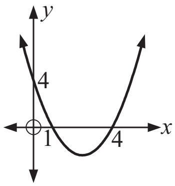
f

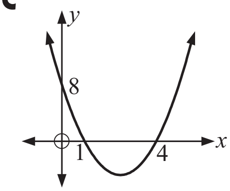


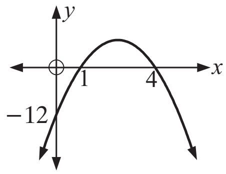
2 Match the given graphs to the possible formulae stated:

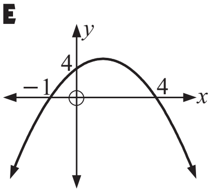
- **a** $y = 2(x-1)(x-4)$
- **b** $y = -(x+1)(x-4)$
- **c** $y = (x-1)(x-4)$
- **d** $y = (x+1)(x-4)$
- **e** $y = 2(x+4)(x-1)$
- **f** $y = -3(x+4)(x-1)$
- **g** $y = -(x-1)(x-4)$
- **h** $y = -3(x-1)(x-4)$

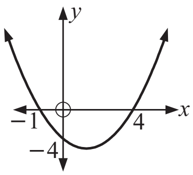
A


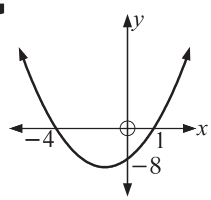
B


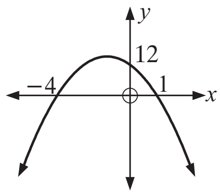
C


D


E


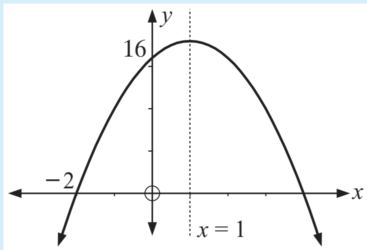
F


G


H


Example 30

Find the equation of the quadratic with graph:



As the axis of symmetry is $x = 1$, the other x-intercept is 4

$$\therefore \quad y = a(x+2)(x-4)$$

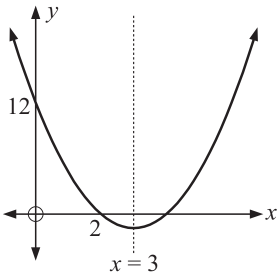
But when $x = 0$, $y = 16$

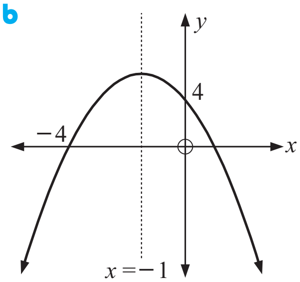
$$\therefore \quad 16 = a(2)(-4)$$

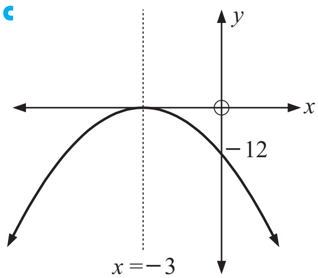
$$\therefore \quad a = -2$$

$\therefore$ quadratic is $y = -2(x+2)(x-4)$

3 Find the quadratic with graph:

a


b


c


Example 31

Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x-axis at 4 and -3 and passes through the point $(2, -20)$.

Since the x-intercepts are 4 and -3, the equation is
$y = a(x-4)(x+3)$ where $a \neq 0$.

But when $x = 2$, $y = -20$

$\therefore\ -20 = a(2-4)(2+3)$
$\therefore\ -20 = a(-2)(5)$
$\therefore\ a = 2$

$\therefore$ equation is $y = 2(x-4)(x+3)$
i.e., $y = 2(x^2 - x - 12)$
i.e., $y = 2x^2 - 2x - 24$

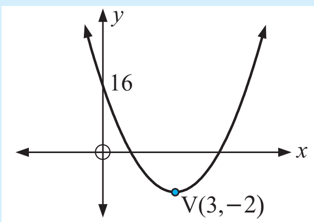
4 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at 5 and 1, and passes through $(2, -9)$
b cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
c touches the x-axis at 3 and passes through $(-2, -25)$
d touches the x-axis at -2 and passes through $(-1, 4)$
e cuts the x-axis at 3, passes through $(5, 12)$ and has axis of symmetry $x = 2$
f cuts the x-axis at 5, passes through $(2, 5)$ and has axis of symmetry $x = 1$.

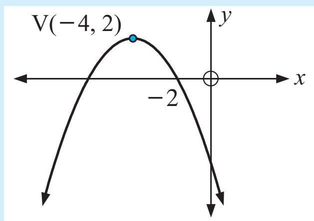
Example 32

Find the equation of the quadratic given its graph is:

a



b



a For vertex $(3, -2)$
the quadratic has form
$y = a(x-3)^2 - 2$

But when $x = 0$, $y = 16$

$\therefore\ 16 = a(-3)^2 - 2$
$\therefore\ 16 = 9a - 2$
$\therefore\ 9a = 18$
$\therefore\ a = 2$

i.e., $y = 2(x-3)^2 - 2$

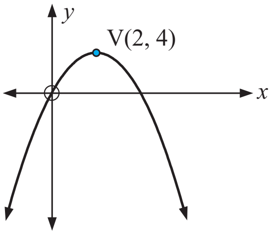
b For vertex $(-4, 2)$
the quadratic has form
$y = a(x+4)^2 + 2$

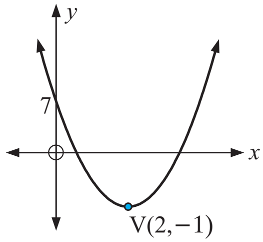
But when $x = -2$, $y = 0$

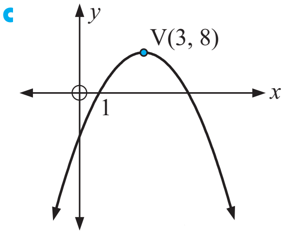
$\therefore\ 0 = a(2)^2 + 2$
$\therefore\ 4a = -2$
$\therefore\ a = -\frac{1}{2}$

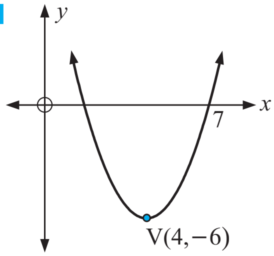
i.e., $y = -\frac{1}{2}(x+4)^2 + 2$

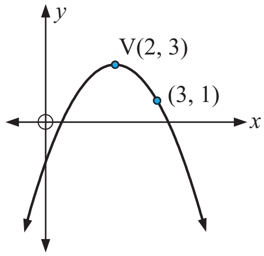
5 Find the equation of the quadratic given its graph is:

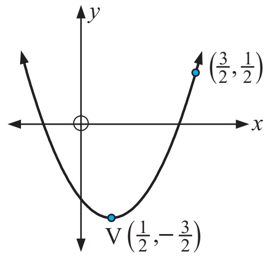
a


b


c


d


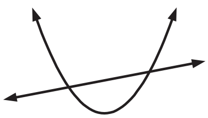
e


f


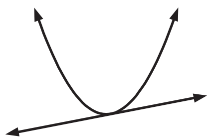
K WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

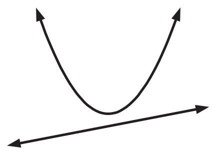
Notice that we could have:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

The graphs could meet and the coordinates of the points of intersection of the graphs of the two functions can be found by *solving the two equations simultaneously*.

Example 33

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$

$$\therefore \quad x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$

$$\therefore \quad (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$

$$\therefore \quad x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ graphs meet at (5, 2) and (−3, −6).

EXERCISE 8K

1 Find the coordinates of the point(s) of intersection of the graphs with equations:

a $y = x^2 - 2x + 8$ and $y = x + 6$
b $y = -x^2 + 3x + 9$ and $y = 2x - 3$
c $y = x^2 - 4x + 3$ and $y = 2x - 6$
d $y = -x^2 + 4x - 7$ and $y = 5x - 4$

2 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to two decimal places) of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$
b $y = x^2 - 5x + 2$ and $y = x - 7$
c $y = -x^2 - 2x + 4$ and $y = x + 8$
d $y = -x^2 + 4x - 2$ and $y = 5x - 6$

GRAPHING PACKAGE

TI

C

3 Find, by algebraic means, the points of intersection of the graphs with equations:

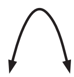
a $y = x^2$ and $y = x + 2$
b $y = x^2 + 2x - 3$ and $y = x - 1$
c $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$
d $xy = 4$ and $y = x + 3$

4 Use technology to check your solutions to the questions in **3**.

GRAPHING PACKAGE

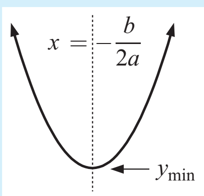
L QUADRATIC MODELLING

There are many situations in the real world where the relationship between two variables is a quadratic function.

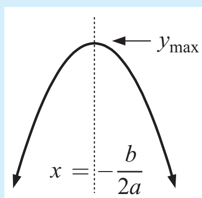
This means that the graph of such a relationship will be either  or  and the function will have a minimum or maximum value.

For $y = ax^2 + bx + c$:

- if $a > 0$, the **minimum** value of y occurs at $x = -\dfrac{b}{2a}$



- if $a < 0$, the **maximum** value of y occurs at $x = -\dfrac{b}{2a}$.



The process of finding the maximum or minimum value of a function is called **optimisation**.

Optimisation is a very useful tool when looking at such issues as:

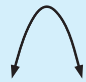
- maximising profits
- minimising costs
- maximising heights reached etc.

Example 34

The height H metres, of a rocket t seconds after it is fired vertically upwards is given by $H(t) = 80t - 5t^2,\ t \geqslant 0$.

a How long does it take for the rocket to reach its maximum height?
b What is the maximum height reached by the rocket?
c How long does it take for the rocket to fall back to earth?

a $$H(t) = 80t - 5t^2$$
$$\therefore\quad H(t) = -5t^2 + 80t \quad \text{where} \quad a = -5 \quad \therefore$$



The maximum height reached occurs when $t = \dfrac{-b}{2a}$

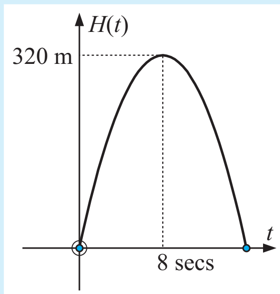
i.e., $t = \dfrac{-80}{2(-5)}$

$\therefore\quad t = 8$

i.e., the maximum height is reached after 8 seconds.

b $$\begin{aligned} H(8) &= 80 \times 8 - 5 \times 8^2 \\ &= 640 - 320 \\ &= 320 \end{aligned}$$

i.e., the maximum height reached is 320 m.



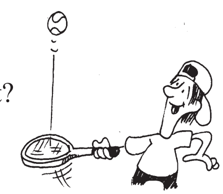
c The rocket falls back to earth when $H(t) = 0$

$$\therefore\quad 0 = 80t - 5t^2$$
$$\therefore\quad 5t^2 - 80t = 0$$
$$\therefore\quad 5t(t - 16) = 0 \qquad \{\text{factorising}\}$$
$$\therefore\quad t = 0 \text{ or } t = 16$$

i.e., the rocket falls back to earth after 16 seconds.

EXERCISE 8L

1 The height H metres, of a ball hit vertically upwards t seconds after it is hit is given by $H(t) = 36t - 2t^2$.



a How long does it take for the ball to reach its maximum height?
b What is the maximum height of the ball?
c How long does it take for the ball to hit the ground?

2 A skateboard manufacturer finds that the cost $\$C$ of making x skateboards per day is given by $C(x) = x^2 - 24x + 244$.

a How many skateboards should be made per day to minimise the cost of production?
b What is the minimum cost?
c What is the cost if no skateboards are made in a day?

3 The driver of a car travelling downhill on a road applied the brakes. The velocity, v, of the car in m/s, t seconds after the brakes were applied is given by $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15$.

- **a** How fast was the car travelling when the driver applied the brakes?
- **b** After how many seconds did the car reach its maximum velocity? Explain why this may have happened.
- **c** What was the maximum velocity reached?
- **d** How long does it take for the car to stop?

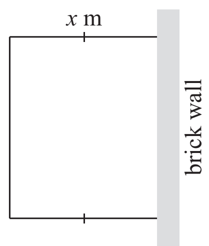
4 The hourly profit ($\$P$) obtained from operating a fleet of n taxis is given by $P(n) = 84n - 45 - 2n^2$.

- **a** What number of taxis gives the maximum hourly profit?
- **b** What is the maximum hourly profit?
- **c** How much money is lost per hour if no taxis are on the road?

5 The temperature T^o Celsius in a greenhouse t hours after dusk (7.00 pm) is given by $T(t) = \frac{1}{4}t^2 - 5t + 30, \quad (t \leqslant 20)$.

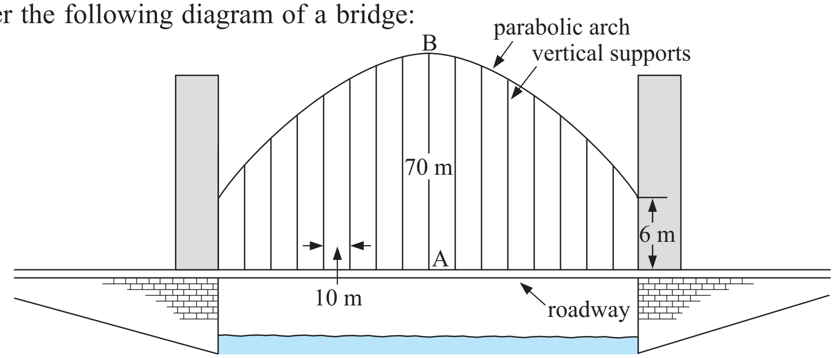
- **a** What was the temperature in the greenhouse at dusk?
- **b** At what time was the temperature at a minimum?
- **c** What was the minimum temperature?

6 A vegetable gardener has 40 m of fencing to enclose a rectangular garden plot where one side is an existing brick wall. If the width is x m as shown:



- **a** Show that the area (A) enclosed is given by $A = -2x^2 + 40x$ m^2.
- **b** Find x such that the vegetable garden has maximum area.
- **c** What is the maximum area?

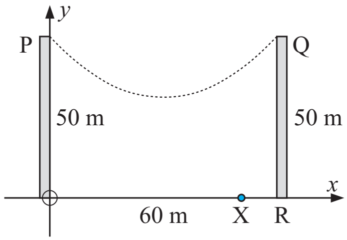
7 Consider the following diagram of a bridge:



AB is the longest vertical support of a bridge which contains a parabolic arch. The vertical supports are 10 m apart. The arch meets the vertical end supports 6 m above the road.

- **a** If axes are drawn on the diagram of the bridge above, with x-axis the road and y-axis on AB, find the equation of the parabolic arch in the form $y = ax^2 + c$.
- **b** Hence, determine the lengths of all other vertical supports.

8 Two towers OP and RQ of a suspension bridge are 50 m high and 60 m apart. A cable is suspended between P and Q and approximates the shape of a parabola under its own weight.



The maximum sag in the middle of the cable is 20 m.

a Find the coordinates of the vertex of the parabola.

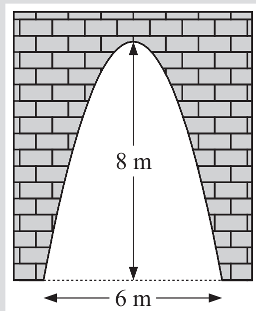
b Hence, find the equation of the parabola.

c How high is the cable directly above X, given that XR $= 10$ m?

GRAPHICS CALCULATOR INVESTIGATION TUNNELS AND TRUCKS

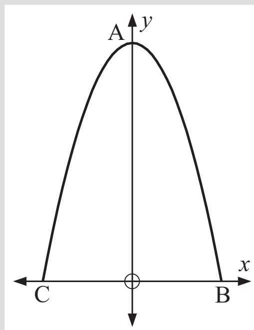
A tunnel is parabolic in shape with dimensions shown:

A truck carrying a wide load is 4.8 m high and 3.9 m wide and needs to pass through the tunnel. Your task is to determine if the truck will fit through the tunnel.



What to do:

1 If a set of axes is fitted to the parabolic tunnel as shown, state the coordinates of points A, B and C.



2 Using a **graphics calculator**:

a enter the x-coordinates of A, B and C into **List 1**

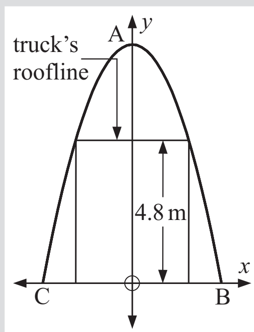
b enter the y-coordinates of A, B and C into **List 2**.

3 Draw a **scatterplot** of points A, B and C.

4 Set your calculator to display 4 decimal places and determine the equation of the parabolic boundary of the tunnel in the form $y = ax^2 + bx + c$, by fitting a **quadratic model** to the data.

5 Place the end view of the truck on the same set of axes as above.

What is the equation of the truck's roofline?



6 You should have found that the equation of the parabolic boundary of the tunnel is $y = -0.8889x^2 + 8$ and the equation of the truck's roofline is $y = 4.8$.

Graph these equations on the same set of axes. Calculate the **points of intersection** of the graphs of these functions.

7 Using the points of intersection found in **6**, will the truck pass through the tunnel? What is the maximum width of a truck that is 4.8 m high if it is to pass through the tunnel?

8 Investigate the maximum width of a truck that is 3.7 m high if it is to pass through the tunnel.

9 What is the maximum width of a 4.1 m high truck if it is to pass through a parabolic tunnel 6.5 m high and 5 m wide?

REVIEW SET 8A

1 For $y = -2(x+2)(x-1)$:

a state the x-intercepts
b state the equation of the axis of symmetry
c find the coordinates of the vertex
d find the y-intercept
e sketch the graph of the function
f use technology to check your answers.

2 For $y = \frac{1}{2}(x-2)^2 - 4$:

a state the equation of the axis of symmetry
b find the coordinates of the vertex
c find the y-intercept
d sketch the graph of the function
e use technology to check your answers.

3 For $y = x^2 - 4x - 1$:

a convert into the form $y = (x-h)^2 + k$ by 'completing the square'
b state the coordinates of the vertex
c find the y-intercept.
d Hence sketch the graph of the quadratic.
e Use technology to check your answer.

4 For $y = 2x^2 + 6x - 3$:

a convert into the form $y = a(x-h)^2 + k$ by 'completing the square'
b state the coordinates of the vertex
c find the y-intercept.
d Hence sketch the graph of the quadratic.
e Use technology to check your answer.

5 Solve the following equations:

a $x^2 - 11x = 60$
b $3x^2 - x - 10 = 0$
c $3x^2 - 12x = 0$

6 Solve the following equations:

a $x^2 + 10 = 7x$
b $x + \dfrac{12}{x} = 7$
c $2x^2 - 7x + 3 = 0$

7 Solve the following equation by completing the square: $x^2 + 7x - 4 = 0$

8 Solve the following equation by completing the square: $x^2 + 4x + 1 = 0$

9 Solve the following using the quadratic formula:

a $x^2 - 7x + 3 = 0$ **b** $2x^2 - 5x + 4 = 0$

REVIEW SET 8B

1 Draw the graph of $y = -x^2 + 2x$.

2 Determine the equation of the axis of symmetry and the vertex of the quadratic relation $y = -3x^2 + 8x + 7$.

3 Determine the equation of the axis of symmetry and the vertex of the quadratic relation $y = 2x^2 + 4x - 3$.

4 Use the discriminant only to determine the number of solutions to:

a $3x^2 - 5x + 7 = 0$ **b** $-2x^2 - 4x + 3 = 0$

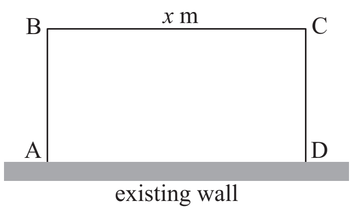
5 Show that $5 + 7x + 3x^2$ is positive definite.

6 Find the maximum or minimum value of the relation $y = -2x^2 + 4x + 3$ and the value of x for which the maximum or minimum occurs.

7 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

8 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x-axis?

9 60 m of chicken wire is available for constructing a chicken enclosure against an existing wall. The enclosure is to be rectangular.



a If $\text{BC} = x$ m, show that the area of rectangle ABCD is given by $A = (30x - \frac{1}{2}x^2)$ m^2.

b Find the dimensions of the enclosure which will maximise the area enclosed.

REVIEW SET 8C

1 Solve the following using the quadratic formula:

a $x^2 + 5x + 3 = 0$ **b** $3x^2 + 11x - 2 = 0$

2 Solve the following equations:

a $x^2 - 5x - 3 = 0$ **b** $2x^2 - 7x - 3 = 0$

3 Use technology to solve:

a $x^2 + 6x + 1 = 0$ **b** $3x^2 - x - 5 = 0$

4 Use technology to solve:

a $(x - 2)(x + 1) = 3x - 4$ **b** $2x - \dfrac{1}{x} = 5$

5 Using the discriminant only, determine the nature of the solutions of:

a $2x^2 - 5x - 7 = 0$ **b** $3x^2 - 24x + 48 = 0$

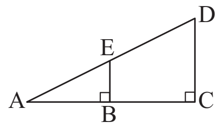
6 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root **b** two distinct real roots **c** no real roots

7 Find the value of t for which the quadratic $3x^2 + 4x + t = 0$ has:

a a repeated root **b** two distinct real roots **c** no real roots

8 If AB is the same length as CD, BC is 2 cm shorter than AB and BE is 7 cm in length, find the length of AB.



9 Find the length of the hypotenuse of a right angled triangle with one leg 7 cm longer than the other and the hypotenuse 2 cm longer than the longer leg.

REVIEW SET 8D

1 Use axis intercepts only to sketch the graph of $y = 3x(x-2)$.

2 Use the vertex, axis of symmetry and y-intercept to graph:

a $y = (x-2)^2 - 4$ **b** $y = -\frac{1}{2}(x+4)^2 + 6$

3 For the quadratic $y = 2x^2 + 4x - 1$, find:

a the equation of the axis of symmetry

b the coordinates of the vertex

c the axis intercepts.

d Hence sketch the graph of the quadratic.

4 Use the discriminant only to find the relationship between the graph and the x-axis for:

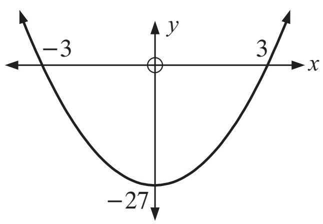
a $y = 2x^2 + 3x - 7$ **b** $y = -3x^2 - 7x + 4$

5 Determine if the quadratic functions are positive definite, negative definite or neither:

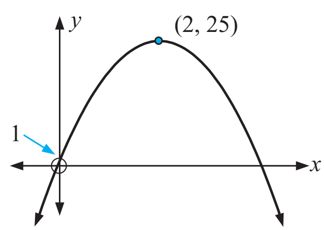
a $y = -2x^2 + 3x + 2$ **b** $y = 3x^2 + x + 11$

6 Find the equation of the quadratic relation with graph:

a



b



7 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

8 An open square container is made by cutting 4 cm square pieces out of a piece of tinplate. If the capacity is 120 cm^3, find the size of the original piece of tinplate.

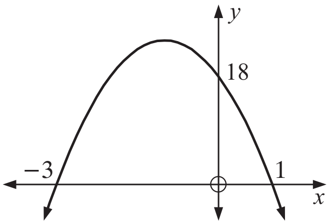
9 Find the point of intersection of the graphs with equations

$$y = -x^2 - 5x + 3 \quad \text{and} \quad y = x^2 + 3x + 11.$$

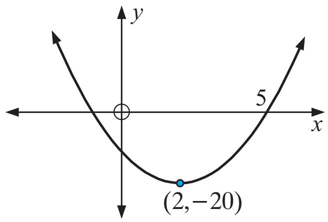
REVIEW SET 8E

1 Find the equation of the quadratic relation with graph:

a

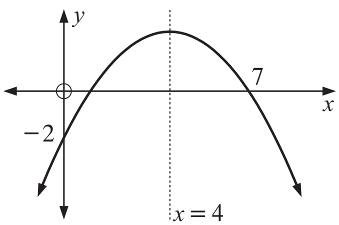


b

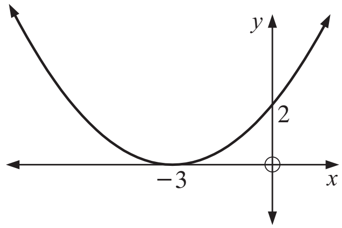


2 Find the equation of the quadratic with graph:

a



b



3 Find an expression for a quadratic which cuts the x-axis at 3 and -2 and has y-intercept 24. Give your answer in the form $y = ax^2 + bx + c$.

4 Find in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph touches the x-axis at 4 and passes through (2, 12).

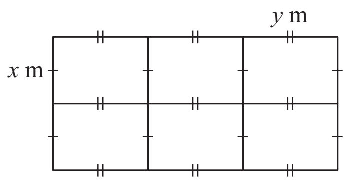
5 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph has vertex $(-4, 1)$ and passes through (1, 11).

6 Find the maximum or minimum value of the following quadratics, and the corresponding values of x:

a $y = 3x^2 + 4x + 7$ **b** $y = -2x^2 - 5x + 2$

7 For what values of k would the graph of $y = x^2 - 2x + k$ cut the x-axis twice? Check your answer(s) using technology.

8 600 m of fencing are used to construct 6 rectangular animal pens as shown.



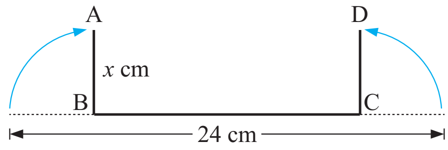
a Show that $y = \dfrac{600 - 8x}{9}$.

b Find the area A, of each pen, in terms of x.

c Find the dimensions of each pen when each pen has a maximum area.

d What is the maximum area of each pen?

9 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown in the illustration.



a If AB $= x$ cm, show that the area of cross-section of the gutter ABCD is given by $A = x(24 - 2x)$ cm^2.

b Where must the bends be made in order to maximise the volume of water carried by the gutter?

Chapter 9

The binomial theorem

Contents:

A Binomial expansions

Investigation 1: The binomial expansions of $(a+b)^n$, $n \geqslant 4$

Investigation 2: nCr or C_r^n values

B The general binomial expansion

Review set 9

A BINOMIAL EXPANSIONS

Consider the following algebraic expansions of the binomial $(a+b)^n$.

$$\begin{aligned}
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= (a+b)(a+b)^2 \\
&= (a+b)(a^2+2ab+b^2) \\
&= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 \\
&= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= (a+b)(a+b)^3 \\
&= (a+b)(a^3+3a^2b+3ab^2+b^3) \\
&= \quad \vdots
\end{aligned}$$

$a^2+2ab+b^2$ is the binomial expansion of $(a+b)^2$

$a^3+3a^2b+3ab^2+b^3$ is the binomial expansion of $(a+b)^3$

INVESTIGATION 1 THE BINOMIAL EXPANSIONS OF $(a+b)^n, n \geqslant 4$

What to do:

1 Complete the expansion of $(a+b)^4$ as outlined above.

2 Similarly, expand algebraically $(a+b)^5$ using your answer for the expansion of $(a+b)^4$ from **1**.

3 Likewise, expand $(a+b)^6$ using your expansion for $(a+b)^5$.

4 The $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ expansion contains 4 terms; a^3, $3a^2b$, $3ab^2$ and b^3. The coefficients of these terms are: 1 3 3 1

a What can be said about the powers of a and b in each term of the expansion of $(a+b)^n$ for $n = 1, 2, 3, 4, 5$ and 6?

b Write down the triangle of coefficients to row 6:

$$\begin{array}{lccccccc}
n=1 & & & 1 & & 1 & & \\
n=2 & & 1 & & 2 & & 1 & \\
n=3 & 1 & & 3 & & 3 & & 1 \quad \leftarrow \text{row } 3 \\
 & & & & \vdots & \text{etc.} & &
\end{array}$$

5 This triangle of coefficients is called **Pascal's triangle**. Investigate:

a the predictability of each row from the previous one

b a formula for finding the sum of the numbers in the nth row of Pascal's triangle.

6 Use your results from **5** to predict the elements of the 7th row of Pascal's triangle and hence write down the binomial expansion of $(a+b)^7$.

Check your result algebraically by using $(a+b)^7 = (a+b)(a+b)^6$ and your results from **3** above.

From the **Investigation** we obtained

$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4\end{aligned}$$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1 whilst the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4 + 1 = 5$.

In **general**, for the expansion of $(a+b)^n$ where $n = 1, 2, 3, 4, 5, ...$:

- As we look from left to right across the expansion, the powers of a decrease by 1 whilst the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n + 1$.

Notice also that:

- $\boldsymbol{a+b}$ is called a **binomial** as it contains two terms
- any expression of the form $\boldsymbol{(a+b)^n}$ is called a **power of a binomial**.

The expansion of $(a+b)^3$, which is $a^3+3a^2b+3ab^2+b^3$ can be used to expand other cubes.

Example 1

Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, find the binomial expansion of:

a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a = (2x)$, $b = (3)$

$$\begin{aligned}\therefore\quad (2x+3)^3 &= (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27 \quad \text{on simplifying}\end{aligned}$$

b This time, $a = (x)$ and $b = (-5)$

$$\begin{aligned}\therefore\quad (x-5)^3 &= (x)^3 + 3(x^2)(-5) + 3(x)(-5)^2 + (-5)^3 \\ &= x^3 - 15x^2 + 75x - 125\end{aligned}$$

EXERCISE 9A

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(x+1)^3$ **b** $(x+2)^3$ **c** $(x-4)^3$ **d** $(2x+1)^3$

e $(2x-1)^3$ **f** $(3x-1)^3$ **g** $(2x+5)^3$ **h** $\left(2x+\dfrac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(x+2)^4$ **b** $(x-2)^4$ **c** $(2x+3)^4$

d $(3x-1)^4$ **e** $\left(x+\frac{1}{x}\right)^4$ **f** $\left(2x-\frac{1}{x}\right)^4$

Example 2

a What is the 5th row of Pascal's triangle?

b Find the binomial expansion of $\left(x-\frac{2}{x}\right)^5$.

a

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} \quad \leftarrow \text{the 5th row}$$

b So, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

and we let $a = (x)$ and $b = \left(\frac{-2}{x}\right)$

$$\therefore \left(x-\frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$

$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

3 Expand and simplify:

a $(x+2)^5$ **b** $(x-2)^5$ **c** $(2x+1)^5$ **d** $\left(2x-\frac{1}{x}\right)^5$

4 **a** Write down the 6th row of Pascal's triangle.

b Find the binomial expansion of:

i $(x+2)^6$ **ii** $(2x-1)^6$ **iii** $\left(x+\frac{1}{x}\right)^6$

5 Expand and simplify:

a $(1+\sqrt{2})^3$ **b** $(1+\sqrt{5})^4$ **c** $(2-\sqrt{2})^5$

6 **a** Expand $(2+x)^6$.

b Use the expansion of **a** to find the value of $(2.01)^6$.

7 Expand and simplify $(2x+3)(x+1)^4$.

8 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$

b a^3b^3 in the expansion of $(2a+3b)^6$.

INVESTIGATION 2 nCr OR C_r^n VALUES

Calculators have a $\boxed{C_r^n}$ or $\boxed{{}_nC_r}$ key.

What to do:

1 Use a scientific calculator to find C_r^n for $n = 3$ and $r = 0$, 1, 2 and 3.

To find C_2^3 press: 3 $\boxed{C_r^n}$ 2 $\boxed{=}$ What do you notice?

or use, for example, the MATH menu of a graphics calculator, PRB, nCr with a TI-83.

e.g., 8 $\boxed{\text{MATH}}$ PRB 3 $\boxed{\text{ENTER}}$ 2 $\boxed{\text{ENTER}}$ is the value of C_2^8 (and is 28).

2 Use your calculator to find C_0^4, C_1^4, C_2^4, C_3^4, C_4^4.

What do you notice?

3 What does nCr or C_r^n represent?

Note:
- C_r^n is also written as $\binom{n}{r}$
- Values of C_r^n or $\binom{n}{r}$ can be found from **Pascal's triangle** or from your **calculator**.

B THE GENERAL BINOMIAL EXPANSION

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots.. + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, \ldots., n$.

The **general term**, or $(\boldsymbol{r+1})$th term is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

Note: $\binom{n}{r}$ or C_r^n also represents the number of combinations of n objects when r are taken at a time.

For example: If we want to *select* any two people, from Ann, Bob, Chloe and David we can do this in $C_2^4 = 6$ ways.

(These are: AB, AC, AD, BC, BD, CD)

Example 3

Write down the first 3 and last 2 terms of the expansion of $\left(2x + \dfrac{1}{x}\right)^{12}$.

$$\left(2x+\frac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right) + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^2 + \ldots..$$
$$+\binom{12}{11}(2x)\left(\frac{1}{x}\right)^{11} + \binom{12}{12}\left(\frac{1}{x}\right)^{12}$$

EXERCISE 9B

1 Write down the first three and last two terms of the binomial expansion of:

a $(1+2x)^{11}$ **b** $\left(3x+\dfrac{2}{x}\right)^{15}$ **c** $\left(2x-\dfrac{3}{x}\right)^{20}$

Example 4

Find the 7th term of $\left(3x-\dfrac{4}{x^2}\right)^{14}$. Do not simplify.

For $\left(3x-\dfrac{4}{x^2}\right)^{14}$, $a=(3x)$ and $b=\left(\dfrac{-4}{x^2}\right)$

So, as $T_{r+1}=\binom{n}{r}a^{n-r}b^r$, we let $r=6$

$\therefore \quad T_7=\binom{14}{6}(3x)^8\left(\dfrac{-4}{x^2}\right)^6$

2 Without simplifying, find:

a the 6th term of $(2x+5)^{15}$ **b** the 4th term of $\left(x^2+\dfrac{5}{x}\right)^9$

c the 10th term of $\left(x-\dfrac{2}{x}\right)^{17}$ **d** the 9th term of $\left(2x^2-\dfrac{1}{x}\right)^{21}$

Example 5

Find the coefficient of x^7 in the expansion of $\left(x^2+\dfrac{4}{x}\right)^{11}$.

$a=(x^2)$, $b=\left(\dfrac{4}{x}\right)$ and $n=11$

$$\begin{aligned}\therefore \quad T_{r+1} &= \binom{11}{r}(x^2)^{11-r}\left(\frac{4}{x}\right)^r\\ &= \binom{11}{r}x^{22-2r}\frac{4^r}{x^r}\\ &= \binom{11}{r}x^{22-2r-r}4^r\\ &= \binom{11}{r}4^r x^{22-3r}\end{aligned}$$

Letting $22-3r=7$ we have $3r=15$

$\therefore \quad r=5$

and so, $T_6=\binom{11}{5}4^5x^7$

$\therefore$ the coefficient of x^7 is $\binom{11}{5}4^5$ or $473\,088$.

3 Find the coefficient of:

a x^{10} in the expansion of $(3+2x^2)^{10}$ **b** x^3 in the expansion of $\left(2x^2-\dfrac{3}{x}\right)^6$

c x^{12} in the expansion of $\left(2x^2-\dfrac{1}{x}\right)^{12}$

4 Find the constant term in:

a the expansion of $\left(x + \dfrac{2}{x^2}\right)^{15}$ **b** the expansion of $\left(x - \dfrac{3}{x^2}\right)^{9}$

5 **a** Write down the first 5 rows of Pascal's triangle.

b What is the sum of the numbers in:

i row 1 **ii** row 2 **iii** row 3 **iv** row 4 **v** row 5?

c Copy and complete: It seems that the sum of the numbers in row n of Pascal's triangle is ......

d Show that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Hence deduce that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n$

Example 6

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned}
&(x+3)(2x-1)^6\\
&= (x+3)[(2x)^6 + \tbinom{6}{1}(2x)^5(-1) + \tbinom{6}{2}(2x)^4(-1)^2 + ...]\\
&= (x+3)(2^6x^6 - \tbinom{6}{1}2^5x^5 + \tbinom{6}{2}2^4x^4 - ....)
\end{aligned}$$

(2) (1)

So terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1) and

$-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5$

$= -336$

6 Find the coefficient of:

a x^5 in the expansion of $(x+2)(x^2+1)^8$

b x^6 in the expansion of $(2-x)(3x+1)^9$

Extension:

7 **a** $\binom{n}{1} = n$ and $\binom{n}{2} = \dfrac{n(n-1)}{2}$ are true statements.

Use your calculator to verify them for $n = 3$, 8 and 20.

b The third term of $(1+x)^n$ is $36x^2$. Find the fourth term.

c If $(1+kx)^n = 1 - 12x + 60x^2 - .....$, find the values of k and n.

REVIEW SET 9

1 Use Pascal's triangle to expand $(a+b)^6$.
Hence, find the binomial expansion of:

a $(x-3)^6$ **b** $\left(1+\dfrac{1}{x}\right)^6$

2 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

3 Expand and simplify $(\sqrt{3}+2)^5$ giving your answer in the form $a+b\sqrt{2}$ where a, b are in $\mathbf{Z}$.

4 Use the expansion of $(4+x)^3$ to find the exact value of $(4.02)^3$.

5 Find the constant term of the expansion of $\left(3x^2+\dfrac{1}{x}\right)^8$.

6 Expand and simplify $\left(3a^2-\dfrac{2}{b}\right)^5$.

7 Find the coefficient of x^{-6} in the expansion of $\left(2x-\dfrac{3}{x^2}\right)^{12}$.

8 Find the coefficient of x^{15} in the expansion of $\left(x^2+\dfrac{5}{x}\right)^{15}$.

9 Find the coefficient of x^5 in the expansion of $(2x+3)(x-2)^6$.

Chapter 10

Practical trigonometry with right angled triangles

Contents:

A Pythagoras' rule (review)
B Pythagoras' rule in 3-D problems
Investigation: Shortest distance
C Right angled triangle trigonometry
D Finding sides and angles
E Problem solving using trigonometry
F The slope of a straight line

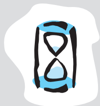
Review set 10A
Review set 10B
Review set 10C

INTRODUCTION

Trigonometry is the study of the relationship between lengths and angles of geometrical figures.

We can apply trigonometry in engineering, astronomy, architecture, navigation, surveying, the building industry and in many branches of applied science.

HISTORICAL NOTE



Astronomy leads to the development of trigonometry

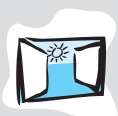
The Greek astronomer **Hipparchus** (140 BC) is credited with being the originator of trigonometry. To aid his calculations regarding astronomy, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest* which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

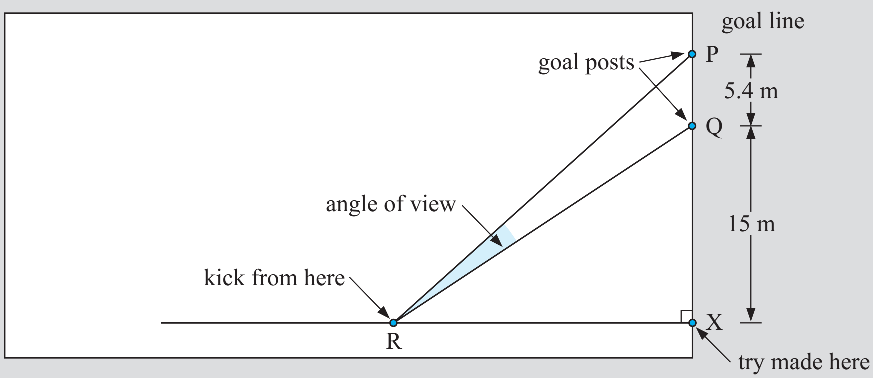
Aryabhata, a Hindu mathematician in the 6th Century AD, drew up a table of the lengths of half-chords of a circle with radius one unit. After completing this chapter you will see that the length of the half-chord is sine θ. So Aryabhata actually drew up the first table of sine values.

In the late 16th century, **Rhaeticus** produced comprehensive and remarkably accurate tables of all six trigonometric ratios (you will learn about three of these in this chapter). These involved a tremendous number of tedious calculations, all without the aid of calculators or computers!

OPENING PROBLEM



After a try is scored in a Rugby game Ray O'Farrell makes a place kick for goal to earn extra points. This kick must be taken from a point on an imaginary line which is perpendicular to the goal line out from the point where the try was made.



Later in this chapter you will be able to answer these questions:

1 If Ray is 30 m from the goal line, how far is he from the nearer goal post Q and what is the angle of view to the goal posts that Ray faces?

2 Can you use a scale diagram to check your answers to **1**, and to what degree of accuracy would your answer be?

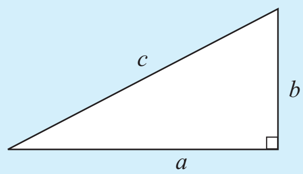
3 If Ray is x m from the goal line find an expression for the angle of view θ^o, using the tangent ratio.

4 Find how far Ray should place the ball from the goal line to maximise the angle of view which in theory would maximise his chance of kicking the goal.

A PYTHAGORAS' RULE (REVIEW)

The **Pythagoras' Rule** is:

In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. i.e., $\boldsymbol{c^2 = a^2 + b^2}$.



Reminder: The **hypotenuse** is always the *longest side* and is *opposite the right angle.*

This theorem, known to the ancient Greeks, is particularly valuable in that:

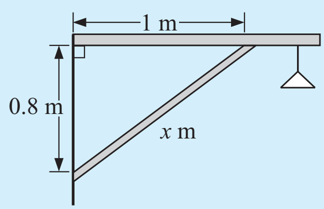
- if we know the lengths of any two sides of a right angled triangle then we can calculate the length of the third side
- if we know the lengths of the three sides then we can determine whether or not the triangle is right angled.

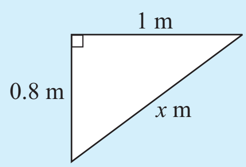
The second statement here relies on the **converse of Pythagoras' Rule**, which is:

If a triangle has sides of length a, b and c units and $a^2 + b^2 = c^2$ say, then the triangle is right angled and its hypotenuse is c units long.

Example 1

Find the unknown length in:





$x^2 = 0.8^2 + 1^2$

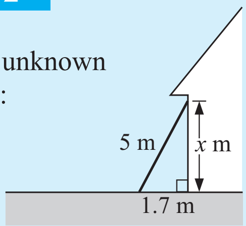
$\therefore \quad x = \sqrt{(0.8^2 + 1^2)}$

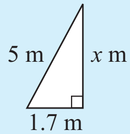
$\therefore \quad x = 1.2806......$

So, the length is 1.28 m.

Example 2

Find the unknown length in:





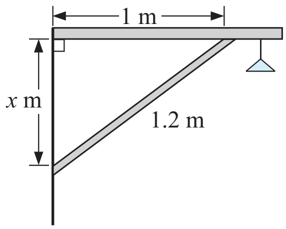
$$x^2 + 1.7^2 = 5^2$$
$$\therefore \quad x^2 = 5^2 - 1.7^2$$
$$\therefore \quad x = \sqrt{(5^2 - 1.7^2)}$$
$$\therefore \quad x = 4.7021....$$

So, the length is 4.70 m.

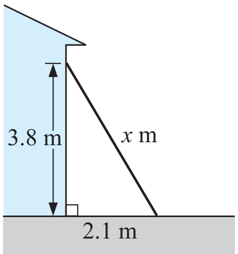
EXERCISE 10A

1 Find, correct to 3 significant figures, the value of x in:

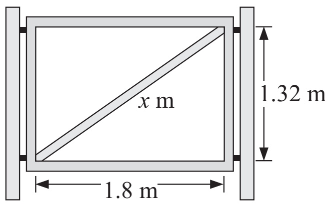
a



b

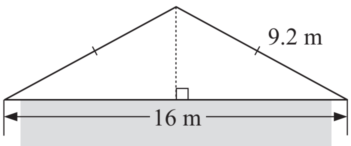


c

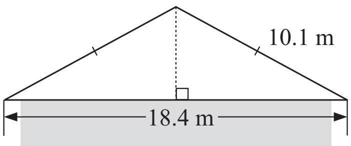


2 How high is the roof above the walls in the following roof structures?

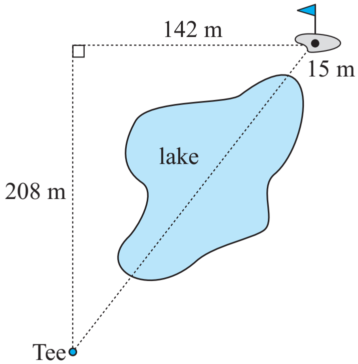
a



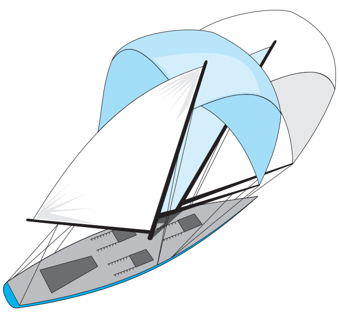
b



3 Bob is about to tee off on the sixth, a par 4 at the Royal Golf Club. If he chooses to hit over the lake, directly at the flag, how far must he hit the ball to clear the lake, given that the pin is 15 m from the water's edge?



4



A sailing ship sails 46 km North then 74 km East.

a Draw a fully labelled diagram of the ship's course.

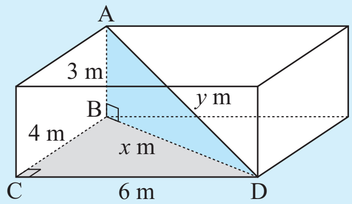
b How far is the ship from its starting point?

B PYTHAGORAS' RULE IN 3-D PROBLEMS

The rule of Pythagoras is often used twice in 3-D problem solving.

Example 3

A room is 6 m by 4 m at floor level and the floor to ceiling height is 3 m. Find the distance from a floor corner point to the opposite corner point on the ceiling.



The required distance is AD. We join BD.

In ΔBCD, $x^2 = 4^2 + 6^2$ {Pythagoras}

$\therefore\ x^2 = 16 + 36 = 52$

In ΔABD $y^2 = x^2 + 3^2$

$\therefore\ y^2 = 52 + 9 = 61$

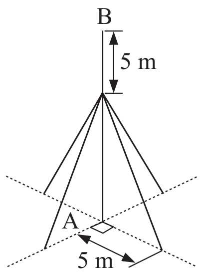
$\therefore\ y = \sqrt{61} \doteqdot 7.81$

i.e., the required distance is 7.81 m.

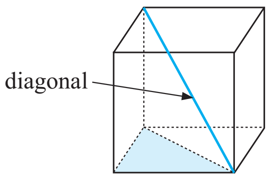
EXERCISE 10B

1 A pole AB is 16 m tall above the ground. At a point 5 m below B, four wires are connected from the pole to the ground.

Each wire is pegged to the ground 5 m from the base of the pole. What is the total length of the wire needed given that a total of 2 m extra is needed for tying?



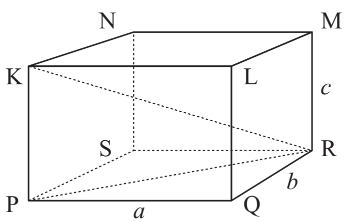
2 A cube has sides of length 10 cm.
Find the length of a diagonal of the cube.



3 A room is 7 m by 4 m and has a height of 3 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

4 A pyramid of height 40 m has a square base with edges 50 m. Determine the length of the slant edges.

5



The figure given represents a rectangular prism.

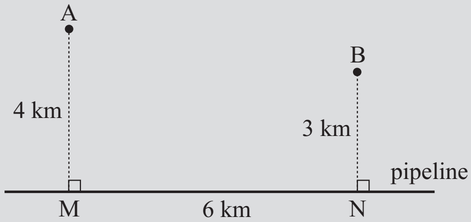
a What are the measures of angles PQR and KPR?

b Find PR^2.

c Explain why $KR = \sqrt{a^2 + b^2 + c^2}$.

INVESTIGATION SHORTEST DISTANCE

A and B are two farm houses which are 4 km and 3 km away from a water pipeline. M and N are the nearest points (on the pipeline) to A and B respectively, and MN = 6 km. The cost of running a spur pipeline across country from the main pipe line is \$3000 per km and the cost of a pump is \$8000. Your task is to determine the most economic way of pumping the water from the pipeline to A and B. Should you use two pumps (located at M and N) or use one pump located somewhere between M and N knowing that one pump would be satisfactory to pump sufficient water to meet the needs of both farm houses?



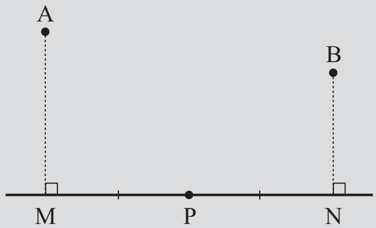
What to do:

1 Find the total cost of the pumps and pipelines if two pumps are used (one at M and the other at N).

2 Suppose one pump is used and it is located at P, the midpoint of MN.

  a Find AP and PB to the nearest metre.

  b Find the total cost of the pipeline and pump in this case.

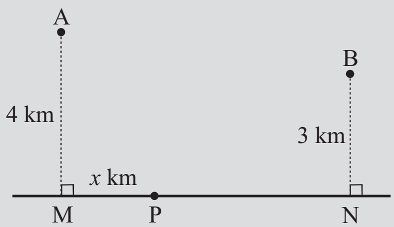


3 Now suppose P is x km from M.

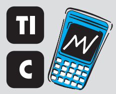
  a Find distance AP in terms of x.

  b Find distance BP in terms of x.

  c Show that AP + BP is given by $\sqrt{x^2+16}+\sqrt{x^2-12x+45}$ km.

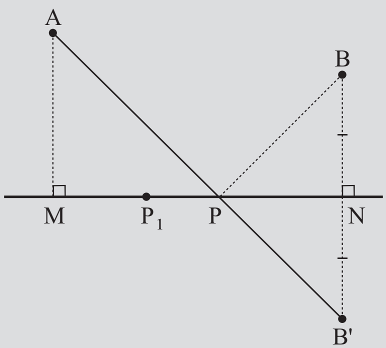


4 Use a **graphics calculator** to find the smallest value of AP + BP using the formula in **3**.



5 From your answer in **4** calculate the total cost of the pipeline and pump for the shortest distance AP + BP.

6



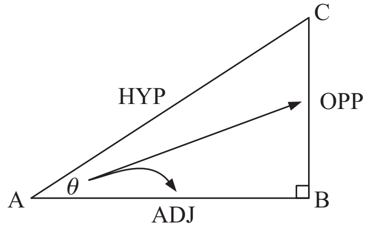
To locate P geometrically we reflect B in the mirror line MN. Its image is B′. We then join AB′. P is located where AB′ meets MN.

Show that the above statement is correct. (**Hint:** AP + PB = AP + PB′ and compare APB′ with AP$_1$B′ for any other point P$_1$ on MN.)

C RIGHT ANGLED TRIANGLE TRIGONOMETRY

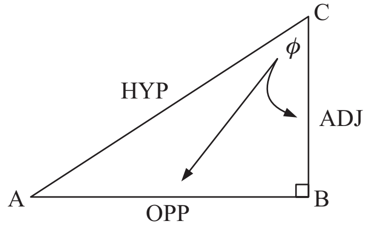
LABELLING RIGHT ANGLED TRIANGLES

For the given right angled triangle, the **hypotenuse (HYP)** is the side which is opposite the right angle and is the longest side of the triangle.



For the angle marked θ:

- BC is the side **opposite (OPP)** angle θ
- AB is the side **adjacent (ADJ)** angle θ.



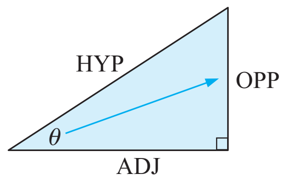
Notice that, for the angle marked ϕ:

- AB is the side **opposite (OPP)** angle ϕ
- BC is the side **adjacent (ADJ)** angle ϕ.

THE THREE BASIC TRIGONOMETRIC RATIOS

By definition, the three basic trigonometric ratios are sine, cosine and tangent where

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \qquad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \qquad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}.$$



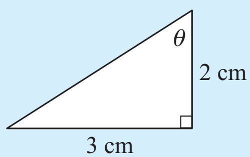
$\sin\theta$, $\cos\theta$ and $\tan\theta$ are abbreviations for sine θ, cosine θ and tangent θ.

The three formulae above are called the **trigonometric ratios** and are the tools we use for finding sides and angles of right angled triangles.

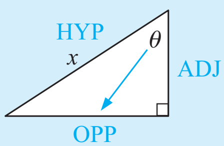
However, before doing this we will calculate the trigonometric ratios in right angled triangles where we know two of the sides.

Example 4

Given



find without using a calculator $\sin\theta$, $\cos\theta$ and $\tan\theta$.



If the hypotenuse is x cm long

$x^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore\ x^2 = 13$

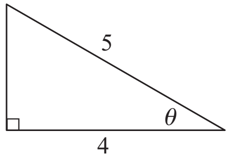
$\therefore\ x = \pm\sqrt{13}$

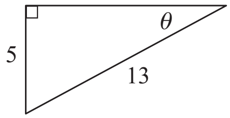
$\therefore\ x = \sqrt{13}$ {as $x > 0$}

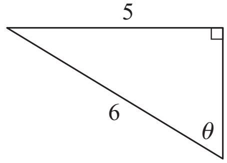
So, $\sin\theta = \frac{3}{\sqrt{13}}$ $\qquad \cos\theta = \frac{2}{\sqrt{13}}$ $\qquad \tan\theta = \frac{3}{2}$.

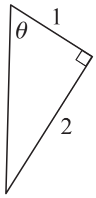
EXERCISE 10C

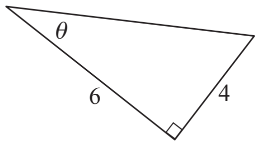
1 For the following triangles, find the length of the third side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:

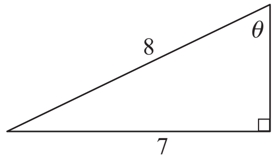
a


b


c


d


e


f


Example 5

If θ is an acute angle and $\sin\theta = \frac{1}{3}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.

We draw a right angled triangle with OPP $= 1$ unit
HYP $= 3$ units

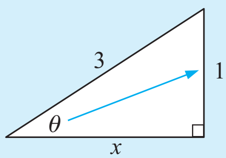
Now $x^2 + 1^2 = 3^2$ {Pythagoras}

$\therefore\ x^2 + 1 = 9$

$\therefore\ x^2 = 8$

$\therefore\ x = \pm\sqrt{8}$

so, $x = \sqrt{8}$ as $x > 0$.



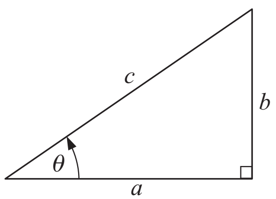
$\therefore\ \cos\theta = \frac{\sqrt{8}}{3}$ and $\tan\theta = \frac{1}{\sqrt{8}}$.

Note: Generally, if $\sin\theta = \frac{1}{3}$ we cannot say that OPP $= 1$ and HYP $= 3$ as it could be that OPP $= 2$ and HYP $= 6$. However, using the simplest ratio produces the required result.

2 **a** If θ is an acute angle and $\cos\theta = \frac{1}{2}$, find $\sin\theta$ and $\tan\theta$.

b If α is an acute angle and $\sin\alpha = \frac{2}{3}$, find $\cos\alpha$ and $\tan\alpha$.

c If β is an acute angle and $\tan\beta = \frac{4}{3}$, find $\sin\beta$ and $\cos\beta$.

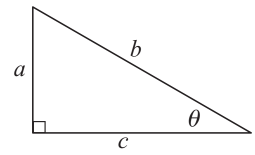
3


a For the triangle given, write down expressions for sin θ, cos θ and tan θ.

b Write $\dfrac{\sin\theta}{\cos\theta}$ in terms of a, b and c

and hence show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

4 The remaining angle of the illustrated triangle is $90 - \theta$, which is the complement of θ.

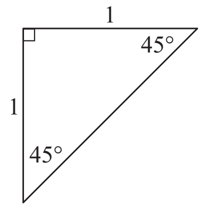


Recall: Two angles are *complementary* if their sum is 90^o.

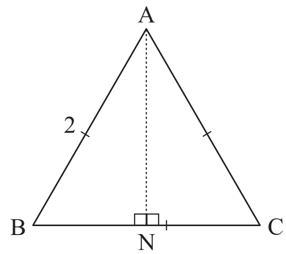
a Find:
i $\sin\theta$ **ii** $\cos\theta$ **iii** $\sin(90-\theta)$ **iv** $\cos(90-\theta)$

b Use your results of **a** to complete the following statements:
i *The sine of an angle is the cosine of its ......*
ii *The cosine of an angle is the sine of its ......*

5 **a** Find the length of the remaining side.
b Find $\sin 45^o$, $\cos 45^o$, $\tan 45^o$ using the figure.
c Use your calculator to check your answers.



6 Triangle ABC is equilateral. AN is the altitude to BC.
a State the measures of angles ABN and BAN.
b Find the length of BN and AN.
c Hence, without using a calculator, find:
i $\sin 60^o$, $\cos 60^o$ and $\tan 60^o$
ii $\sin 30^o$, $\cos 30^o$ and $\tan 30^o$.



We can summarise the ratios for special angles in table form.

Try to learn them.

Summary

θ (degrees)	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

D FINDING SIDES AND ANGLES

NOTE ON CALCULATOR USE

Before commencing calculations make sure that you check that the $\boxed{\text{MODE}}$ is set on **degrees**. In this chapter all problem solving is in degrees.

SIDE FINDING

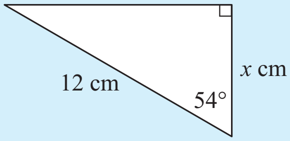
In a right angled triangle, if we wish to find the **length of a side** we first need to know:

- one angle and
- one other side.

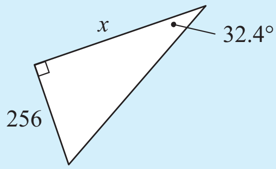
Example 6

Find, correct to 3 significant figures, the value of x in:

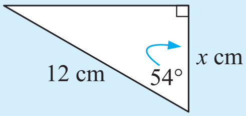
a



b



a



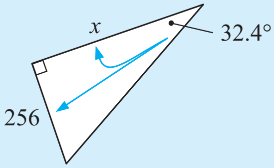
For the 54^o angle, HYP $= 12$, ADJ $= x$.

So, $\cos 54^o = \dfrac{x}{12}$

$\therefore \quad 12\cos 54^o = x$

$\therefore \quad x \doteqdot 7.05$

b For the 32.4^o angle, OPP $= 12$, ADJ $= x$



So, $\tan 32.4^o = \dfrac{256}{x}$

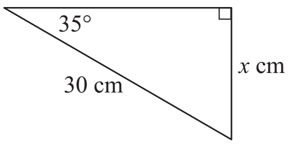
$\therefore \quad x = \dfrac{256}{\tan 32.4^o} \doteqdot 403.4$

$\therefore \quad x \doteqdot 403$

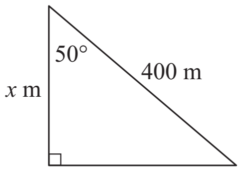
EXERCISE 10D

1 Find, correct to 3 significant figures, the value of the unknown in each of the following:

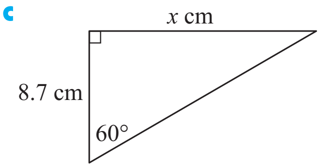
a



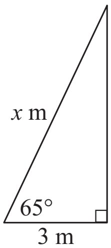
b



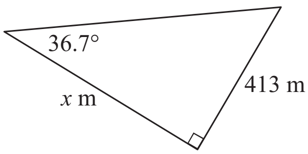
c



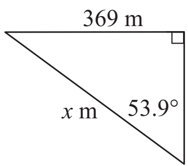
d



e



f



ANGLE FINDING

In a right angled triangle, if we wish to find the size of an acute angle we need to know the **lengths of two sides**.

Reminder:

- if $\sin\theta = \dfrac{a}{b}$ then $\theta = \sin^{-1}\left(\dfrac{a}{b}\right)$ which reads 'the angle with a sine of $\dfrac{a}{b}$'
- if $\cos\theta = \dfrac{a}{b}$ then $\theta = \cos^{-1}\left(\dfrac{a}{b}\right)$ which reads 'the angle with a cosine of $\dfrac{a}{b}$'
- if $\tan\theta = \dfrac{a}{b}$ then $\theta = \tan^{-1}\left(\dfrac{a}{b}\right)$ which reads 'the angle with a tangent of $\dfrac{a}{b}$'.

When using your calculator to find, for example $\sin^{-1}(\frac{3}{4})$

Use **2nd**, **INV** or **SHIFT** **before** **sin**.

depending on what calculator you are using

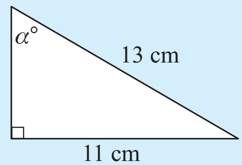
2 Use your calculator to find the acute angle θ in degrees to 3 significant figures:

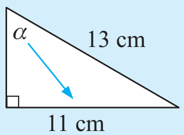
a $\sin\theta = 0.9364$ **b** $\cos\theta = 0.2381$ **c** $\tan\theta = 1.7321$ **d** $\cos\theta = \frac{2}{7}$

e $\sin\theta = \frac{1}{3}$ **f** $\tan\theta = \frac{14}{3}$ **g** $\sin\theta = \frac{\sqrt{3}}{11}$ **h** $\cos\theta = \frac{5}{\sqrt{37}}$

Example 7

Find α in degrees, correct to 3 significant figures:





For angle α, OPP = 11, HYP = 13.

So, $\sin\alpha = \frac{11}{13}$

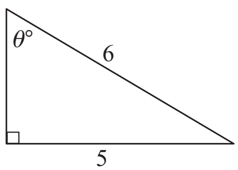
$\therefore \quad \alpha = \sin^{-1}(\frac{11}{13})$

$\therefore \quad \alpha \doteqdot 57.8$

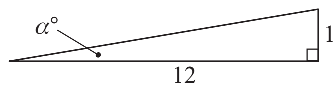
Steps: DEG mode, **2nd** **sin** 11 **÷** 13 **)** **ENTER**

3 Find, correct to 3 significant figures, the measure of the unknown angle in each of the following:

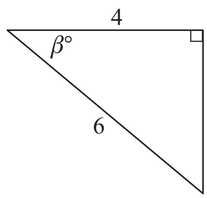
a



b

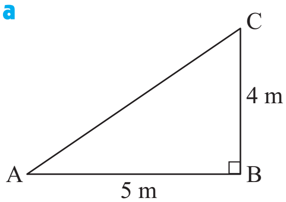


c

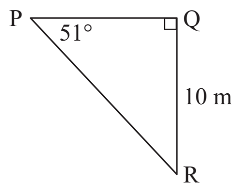


4 Solve the following triangles, i.e., find all unknown sides and angles:

a

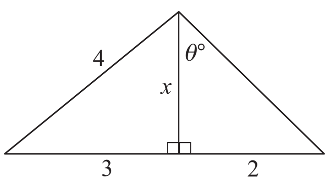


b

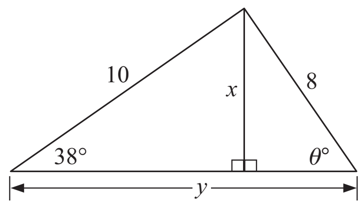


5 Find unknown sides and angles in the following figures:

a



b



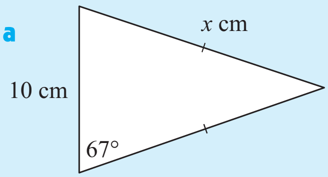
ISOSCELES TRIANGLES

To use trigonometry with isosceles triangles we invariably draw the **perpendicular** from the apex to the base. This altitude **bisects** the base.

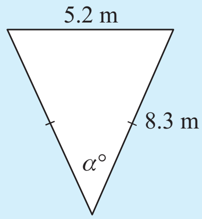
Example 8

Find, to 3 s.f. the unknowns in the following diagrams:

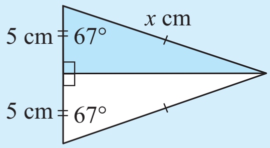
a



b



a

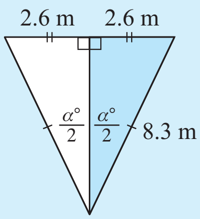


In the shaded right angled triangle

$$\cos 67^o = \frac{5}{x}$$

$$\therefore \quad x = \frac{5}{\cos 67^o} \doteqdot 12.8$$

b



In the shaded right angled triangle

$$\sin\left(\frac{\alpha}{2}\right) = \frac{2.6}{8.3}$$

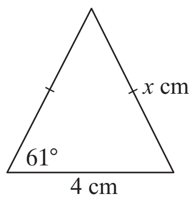
$$\therefore \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{2.6}{8.3}\right)$$

$$\therefore \quad \alpha = 2\sin^{-1}\left(\frac{2.6}{8.3}\right) \doteqdot 36.5$$

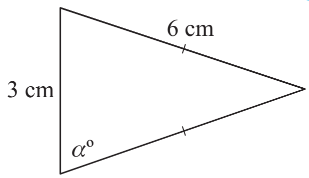
Steps: 2 [×] [2nd] [sin⁻¹] 2.6 [÷] 8.3 [)] [ENTER]

6 Find, correct to 3 significant figures, the unknowns in the following:

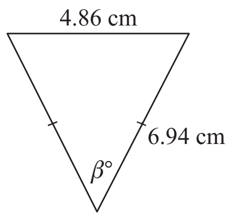
a



b

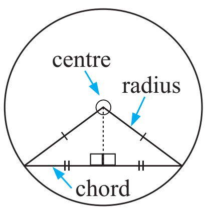


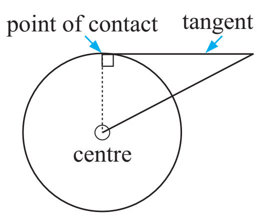
c

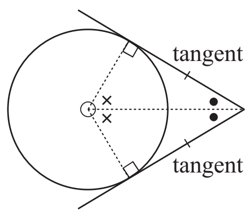


CHORDS AND TANGENTS

Right angled triangles occur in chord and tangent problems.

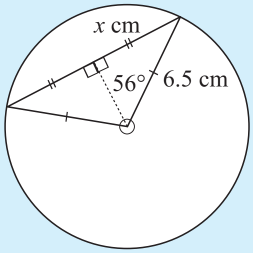






Example 9

A chord of a circle subtends an angle of 112^o at its centre. Find the length of the chord if the radius of the circle is 6.5 cm.



We complete an isosceles triangle and draw the line from the apex to the base.

For the 56^o angle, HYP $= 6.5$, OPP $= x$,

$$\sin 56^o = \frac{x}{6.5}$$

$$\therefore \quad 6.5 \times \sin 56^o = x$$

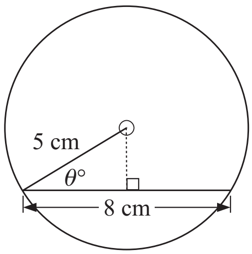
$$\therefore \quad x \doteqdot 5.389$$

$$\therefore \quad 2x \doteqdot 10.78$$

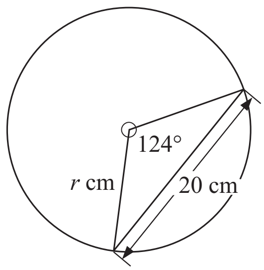
$\therefore$ chord is 10.8 cm long.

7 Find the value of the unknown in:

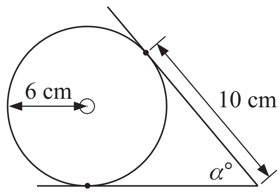
a



b



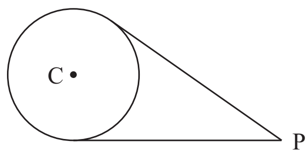
c



8 A chord of a circle subtends an angle of 89^o at its centre. Find the length of the chord given that the circle's diameter is 11.4 cm.

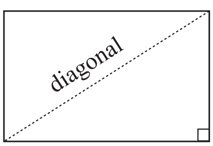
9 A chord of a circle is 13.2 cm long and the circle's radius is 9.4 cm. Find the angle subtended by the chord at the centre of the circle.

10 Point P is 10 cm from the centre of a circle of radius 4 cm. Tangents are drawn from P to the circle. Find the angle between the tangents.

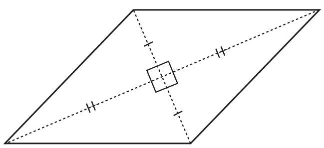


OTHER FIGURES

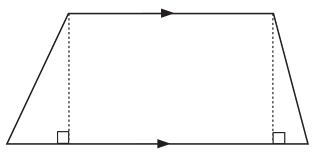
Sometimes right angled triangles can be found in other geometric figures such as rectangles, rhombi and trapezia.



rectangle



rhombus

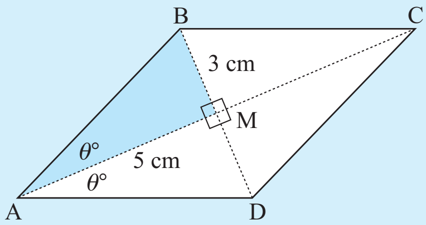


trapezium or trapezoid

Often right angled triangle trigonometry can be used in these figures if sufficient information is given.

Example 10

A rhombus has diagonals of length 10 cm and 6 cm respectively.
Find the smaller angle of the rhombus.



The diagonals bisect each other at right angles, so AM = 5 cm and BM = 3 cm.

In ΔABM, θ will be the smallest angle as it is opposite the shortest side.

$$\tan\theta = \tfrac{3}{5}$$
$$\therefore \quad \theta = \tan^{-1}(\tfrac{3}{5})$$
$$\text{i.e.,} \quad \theta \doteqdot 30.964$$

But the required angle is 2θ as the diagonals bisect the angles at each vertex,
$\therefore$ angle is 61.9^o.

11 A rectangle is 9.2 m by 3.8 m. What angle does its diagonal make with its longer side?

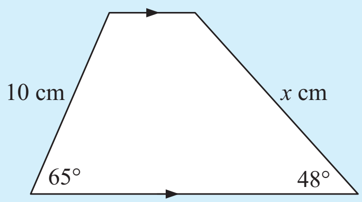
12 The diagonal and the longer side of a rectangle make an angle of 43.2^o. If the longer side is 12.6 cm, find the length of the shorter side.

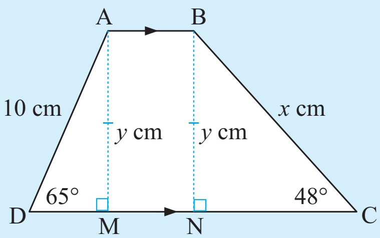
13 A rhombus has diagonals of length 12 cm and 7 cm respectively. Find the larger angle of the rhombus.

14 The smaller angle of a rhombus measures 21.8^o and the shorter diagonal is 13.8 cm. Find the lengths of the sides of the rhombus.

Example 11

Find x given:





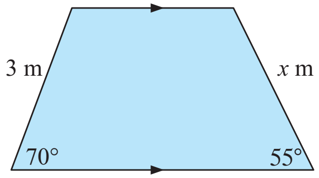
We draw perpendiculars AM and BN to DC creating right angled triangles and rectangle ABNM.

In ΔADM, $\sin 65^o = \dfrac{y}{10}$ and $\therefore\ y = 10\sin 65^o$

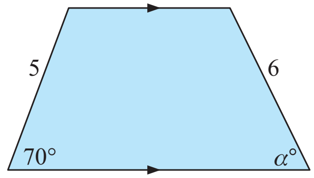
In ΔBCN, $\sin 48^o = \dfrac{y}{x} = \dfrac{10\sin 65^o}{x}$

$$\therefore \quad x = \frac{10\sin 65^o}{\sin 48^o} \doteqdot 12.2$$

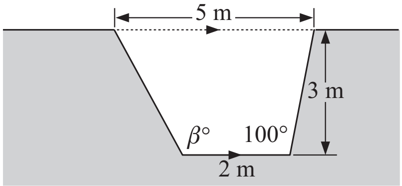
15 **a** Find the value of x in:



b Find the unknown angle in:



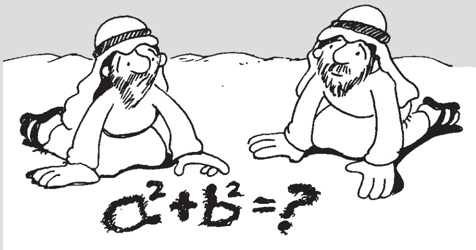
16 A stormwater drain is to have the shape as shown. Determine the angle the left hand side makes with the bottom of the drain.



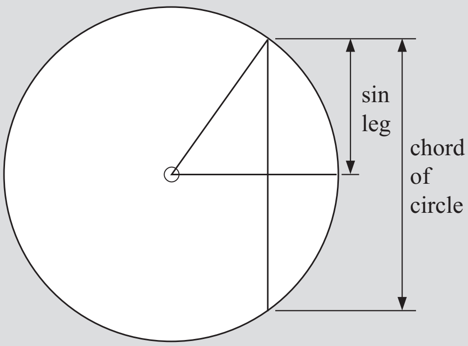
HISTORICAL NOTE

The origin of the term "sine" is quite fascinating. **Arbyabhata**, a Hindu mathematician who studied trigonometry in the 5th century AD, called the sine-leg of the circle diagram "ardha-jya" which means "half-chord".

This was eventually shortened to "jya". Arab scholars later translated Arbyabhata's work into Arabic and initially phonetically translated "jya" as "jiba" but since this meant nothing in Arabic they very shortly began writing the word as "jaib" which has the same letters but means "cove" or "bay".



Finally in 1150 an Italian, **Gerardo of Cremona**, translated this work into Latin and replaced "jaib" with "sinus" which means "bend" or "curve" but is commonly used in Latin to refer to a bay or gulf on a coastline. The term "sine" that we use today comes from this Latin word "sinus". The term "cosine" comes from the fact that the sine of an angle is equal to the cosine of its complement. In 1620, **Edmund Gunter** introduced the abbreviated "co sinus" for "complementary sine".

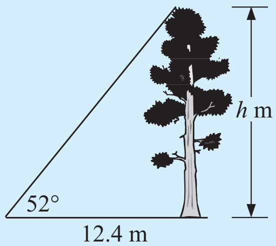


E PROBLEM SOLVING USING TRIGONOMETRY

Trigonometry is a very useful branch of mathematics. **Heights** and **distances** which are very difficult or even impossible to measure can often be found using **trigonometry**.

Example 12

Find the height of a tree which casts a shadow of 12.4 m when the sun makes an angle of 52^o to the horizon.



Let h m be the tree's height.

For the 52^o angle, OPP $= h$, ADJ $= 12.4$

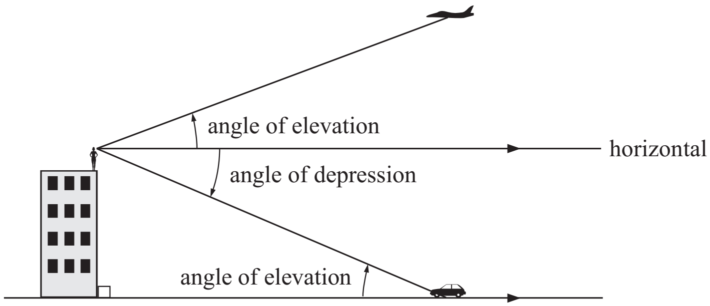
$$\therefore \quad \tan 52^o = \frac{h}{12.4}$$

$$\therefore \quad 12.4 \times \tan 52^o = h$$

$$\therefore \quad h \doteqdot 15.9$$

$\therefore$ tree is 15.9 m high.

Reminder:



EXERCISE 10E

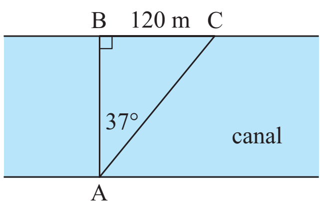
1 Find the height of a flagpole which casts a shadow of 9.32 m when the sun makes an angle of 63^o to the horizontal.

2 A hill is inclined at 18^o to the horizontal. If the base of the hill is at sea level find:

a my height above sea level if I walk 1.2 km up the hill

b how far I have walked up the hill if I am 500 metres above sea level.

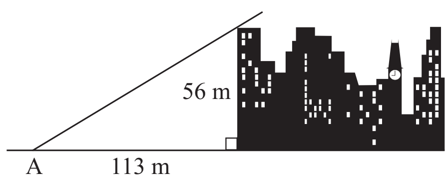
3



A surveyor standing at A notices two posts B and C on the opposite side of a canal. The posts are 120 m apart. If the angle of sight between the posts is 37^o, how wide is the canal?

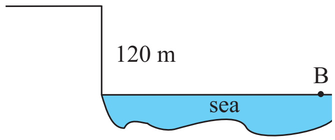
4 A train must climb a constant gradient of 5.5 m for every 200 m of track. Find the angle of incline.

5 Find the angle of elevation to the top of a 56 m high building from point A, which is 113 m from its base. What is the angle of depression from the top of the building to A?



6 The angle of depression from the top of a 120 m high vertical cliff to a boat B is 16^o.

Find how far the boat is from the base of the cliff.

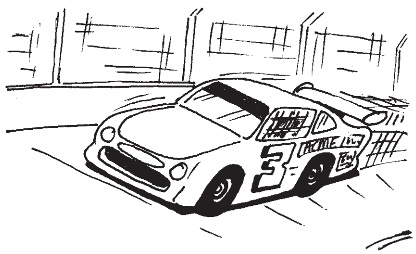


7 Sarah measures the angle of elevation to the top of a tree as 23.6^o from a point which is 250 m from its base. Her eye level, where the angle measurement was taken, is 1.5 m above the ground. Assuming the ground to be horizontal, find the height of the tree.

8 Kylie measures the angle of elevation from a point on level ground to the top of a building 120 metres high to be 32^o. She walks towards the building until the angle of elevation is 45^o. How far does she walk?

9 From a point A, 40 metres from the base of a building B, the angle of elevation to the top of the building C is 51^o, and to the top of the flagpole D on top of the building is 56^o. Find the height of the flagpole.

10 For a circular track of radius r metres, banked at θ degrees to the horizontal, the ideal velocity (the velocity that gives no tendency to sideslip) in metres per second is given by the formula: $v = \sqrt{gr\tan\theta}$, where $g = 9.8$ m/s^2.

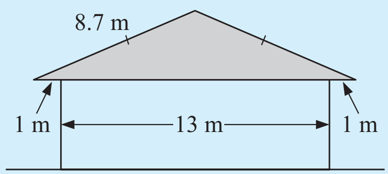


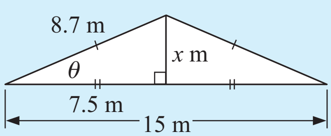
a What is the ideal velocity for a vehicle travelling on a circular track of radius 100 m, banked at an angle of 15^o?

b At what angle should a track of radius 200 m be banked, if it is designed for a vehicle travelling at 20 m/s?

Example 13

A builder designs a roof structure as illustrated. The pitch of the roof is the angle that the roof makes with the horizontal. Find the pitch of the roof.





Using the right angled triangle created from the isosceles triangle, for angle θ:

ADJ = 7.5, HYP = 8.7

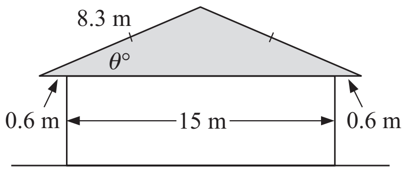
$\therefore \quad \cos\theta = \dfrac{7.5}{8.7}$

$\therefore \quad \theta = \cos^{-1}(\dfrac{7.5}{8.7})$

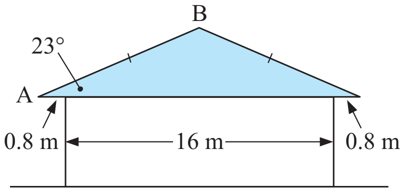
$\therefore \quad \theta \doteqdot 30.450....$

$\therefore$ the pitch is approximately $30\frac{1}{2}^o$.

11 Find θ, the pitch of the roof.



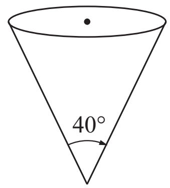
12 If the pitch of the given roof is 23^o, find the length of the timber beam AB.



13 An open right-circular cone has a vertical angle measuring 40^o and a base radius of 30 cm.

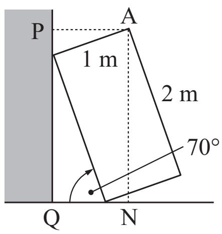
Find the capacity of the cone in litres.

$$(V = \tfrac{1}{3}\pi r^2 h)$$



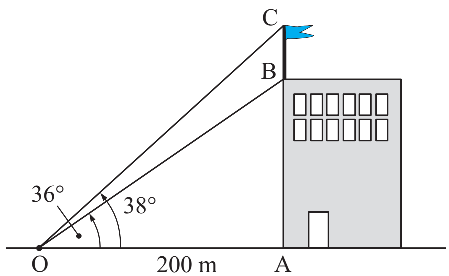
14 A refrigerator leans against a vertical wall making an angle of 70^o with the horizontal floor. How high is point A above the floor?

(**Hint:** AN = PQ)



15 From an observer O, the angles of elevation to the bottom and the top of a flagpole are 36^o and 38^o respectively. Find the height of the flagpole.

(**Hint:** Find AB and AC.)

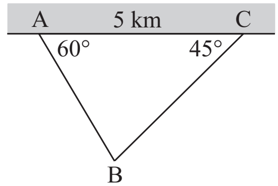


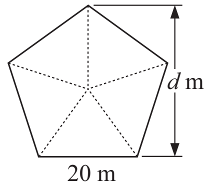
16 The angle of depression from the top of a 150 m high cliff to a boat at sea is 7^o. How much closer to the cliff must the boat move for the angle of depression to become 19^o?

17 A helicopter flies horizontally at 100 kmph. An observer notices that it took 20 seconds for the helicopter to fly from directly overhead to being at an angle of elevation of 60^o. Find the height of the helicopter above the ground.

18 A balloon travels horizontally at a distance h kilometres above the ground between two points A and B, which are two kilometres apart. From a point C on the ground, the angle of elevation of the balloon at A is 40^o and at B is 25^o. Assume that A, B and C are in the same plane and that A and B are on the same side of the observation point. Find the height h of the balloon.

19 AC is a straight shore line and B is a boat out at sea. Find the shortest distance from the boat to the shore if A and C are 5 km apart.



20 

A regular pentagonal garden plot is to be constructed. The sides are to be of length 20 m.

Find the width of land (d m) required for the plot.

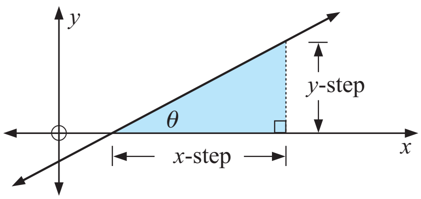
F THE SLOPE OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its slope $m = \tan\theta$.

Proof:

$$\text{slope } m = \frac{y\text{-step}}{x\text{-step}}$$

$$= \tan\theta \qquad \{\text{in shaded } \Delta\}$$



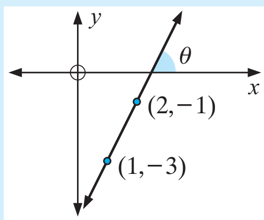
Example 14

Find the angle that the line through $P(2, -1)$ and $Q(1, -3)$ makes with the positive x-axis.

$$\tan\theta = m = \frac{(-3) - (-1)}{1 - 2}$$

$$\therefore \quad \tan\theta = \frac{-2}{-1} = 2$$

$$\therefore \quad \theta = \tan^{-1}(2) \doteqdot 63.4^o$$



EXERCISE 10F

1 What angle does the line through:

a $A(2, 5)$ and $B(-1, 4)$ make with the positive x-axis

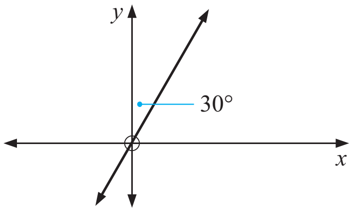
b $C(3, -2)$ and $D(-1, -4)$ make with the positive x-axis

c $E(-2, 1)$ and $F(1, -5)$ make with the positive x-axis

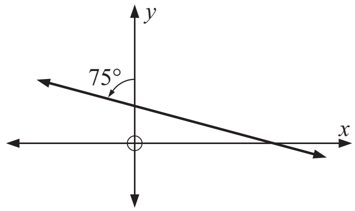
d $G(-7, 4)$ and $H(-2, -1)$ make with the positive x-axis?

2 Find the slope of the following lines:

a

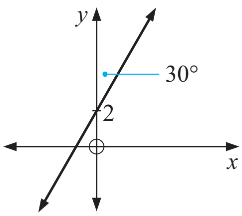


b

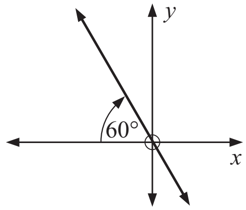


3 Find the equations of the following lines:

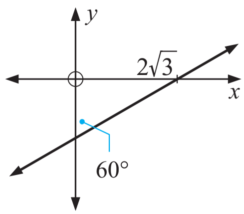
a



b

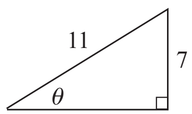


c

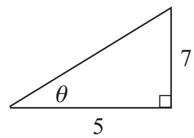


REVIEW SET 10A

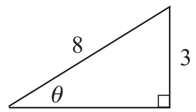
1 Find $\sin\theta$ for:



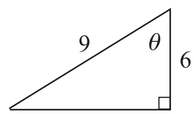
2 Find $\sin\theta$ and $\cos\theta$ for:



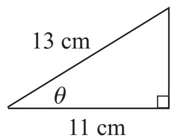
3 Find $\tan\theta$ for:



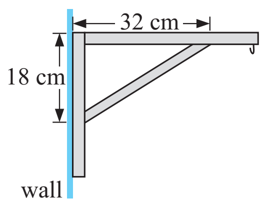
4 Find the unknown side and hence $\sin\theta$, $\cos\theta$, and $\tan\theta$:



5 Find the unknown side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:



6



Metal brackets as shown alongside are attached to a wall so that hanging baskets may be hung from them. Using the dimensions given, find:

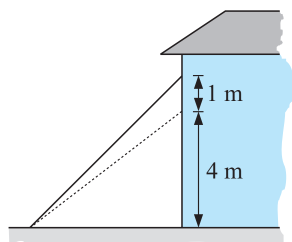
a the length of the diagonal support

b the angle the diagonal support makes with the wall.

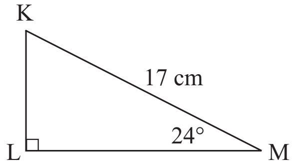
7 When an extension ladder rests against a wall it reaches 4 m up the wall. The ladder is extended a further 0.8 m without moving the foot of the ladder and it now rests against the wall 1 m further up. Find:

a the length of the extended ladder

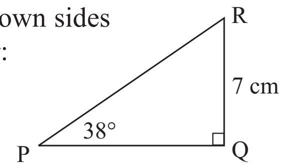
b the increase in the angle that the ladder makes with the ground now that the ladder is extended.



8 Find all unknown sides and angles.



9 Find the unknown sides and angles for:



10 Determine the height of a tree which casts a shadow of 13.7 m when the sun is at an angle of 28^o.

REVIEW SET 10B

1 A yacht sails 8.6 km due east and then 13.2 km south. Find the distance and bearing of the yacht from its starting point.

2 Find the angle of elevation to the top of a mountain 2300 m high from a point 5.6 km from its base.

3 An aeroplane flying at 10 000 m is at an angle of elevation of 36^o. If two minutes later, the angle of elevation is 21^o, determine the speed of the plane.

4 Find the acute angle θ, if:

a $\sin\theta = 0.8147$ **b** $\cos\theta = 0.0917$ **c** $\tan\theta = 5.23$

5 Find the acute angle θ, if:

a $\sin\theta = \frac{\sqrt{11}}{5}$ **b** $\cos\theta = \frac{5}{7}$ **c** $\tan\theta = 0.7452$

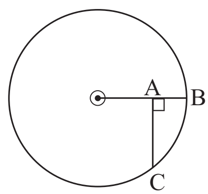
6 A chord of a circle subtends an angle of 114^o at its centre. Find the radius of the circle given that the length of the chord is 10.4 cm.

7 A tangent from a point P is drawn to a circle of radius 6.4 cm. Find the angle between the tangent and the line joining P to the centre of the circle if the tangent has length 13.6 cm.

8 In the given figure AB = 1 cm and AC = 3 cm. Find:

a the radius of the circle

b the angle subtended by chord BC at the centre of the circle.



9 The larger angle of a rhombus measures 114^o and the longer diagonal is 16.4 cm. Find the lengths of the sides of the rhombus.

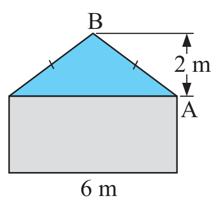
10 A flagpole 19.6 m high is supported by 3 wires which meet the ground at an angle of 56^o. Determine the total length of the three wires.

REVIEW SET 10C

1 In the illustrated roof structure:

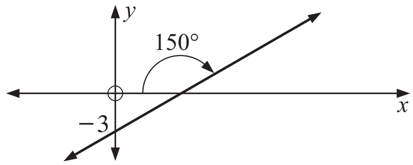
a how long is the timber beam AB

b at what angle is the beam inclined to the horizontal?

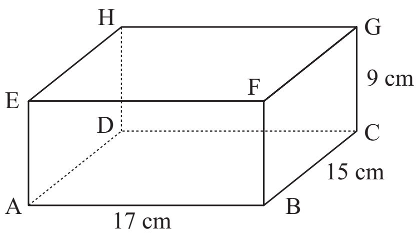


2 What angle does the line through P(7, -4) and Q(1, 6) make with the positive x-axis?

3 Find the equation of this line:



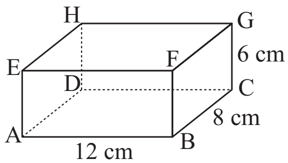
4



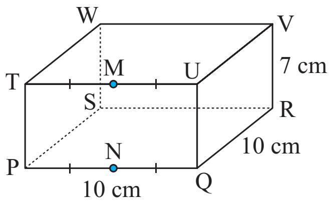
- **a** Sketch triangle DHE showing which angle is the right angle.
- **b** Find the measure of angle HDE.
- **c** Sketch triangle ACG showing which angle is the right angle.
- **d** Find the measure of angle AGC.

5 Find the angle that:

- **a** AG makes with BG
- **b** DF makes with DB.



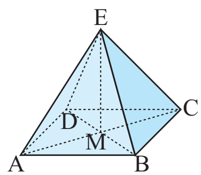
6



M and N are the midpoints of TU and PQ respectively.

- **a** Draw a sketch of triangle RMN showing which angle is the right angle.
- **b** Find the length of RN.
- **c** Find the measure of angle RMN.

7 ABCD is a square-based pyramid. E, the apex of the pyramid is vertically above M, the point of intersection of AC and BD. If an Egyptian Pharoah wished to build a square-based pyramid with all edges 200 m, find:

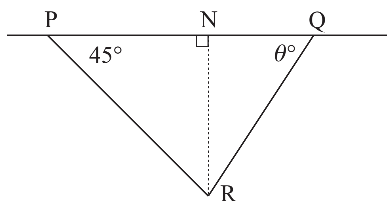


- **a** how high (to the nearest metre) the pyramid would reach above the desert sands
- **b** the measure of the angle between a slant edge and a base diagonal.

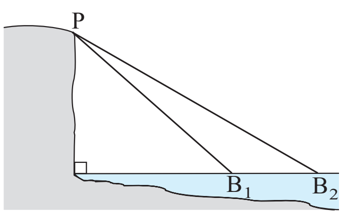
8 An isosceles triangle is drawn with base angles 32^o and base 24 cm. Find the base angles of an isosceles triangle with the same base but with double the area.

9 PQ is a straight shore line and R is a boat out at sea. Show that if P and Q are 5 km apart the shortest distance from the boat to shore is given by

$$RN = \frac{5\tan\theta}{1+\tan\theta}.$$



10



From the top of a cliff 200 m above sea level the angles of depression to two fishing boats are 6.7^o and 8.2^o respectively. How far apart are the boats?

Chapter 11

The unit circle

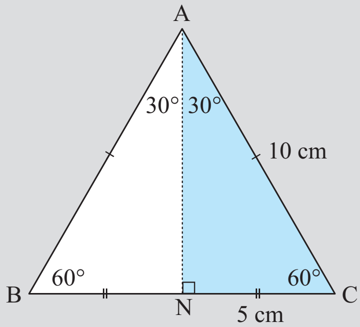
Contents:

A The unit quarter circle
B Obtuse angles
C The unit circle
Investigation: Parametric equations

Review set 11

OPENING PROBLEM

Consider an equilateral triangle with sides 10 cm long. All its angles are of size 60^o. Altitude AN bisects side BC and the vertical angle BAC.



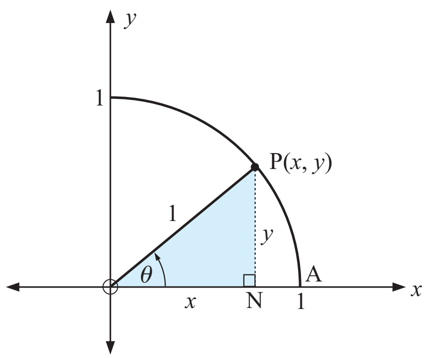
a Can you see from this figure that $\sin 30^o = \frac{1}{2}$?

b Use your calculator to find the values of $\sin 30^o$, $\sin 150^o$, $\sin 390^o$, $\sin 1110^o$ and $\sin(-330^o)$. What do you notice? Can you explain why this result occurs even though the angles are not between 0^o and 90^o?

By the end of this chapter you should be able to answer the above question.

A THE UNIT QUARTER CIRCLE

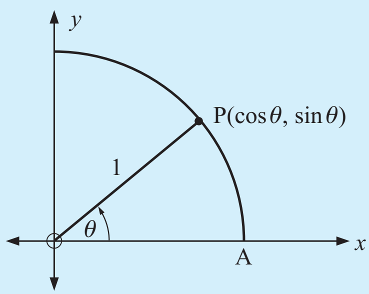
The unit quarter circle is the part of a circle centre (0, 0) and radius 1 unit that lies in the first quadrant. Suppose P(x, y) can move anywhere on this arc from A to B.



Notice that: $\cos\theta = \dfrac{\text{ON}}{\text{OP}} = \dfrac{x}{1} = x$

and $\sin\theta = \dfrac{\text{PN}}{\text{OP}} = \dfrac{y}{1} = y.$

So:



On the unit quarter circle, if $\measuredangle\text{AOP} = \theta^o$, then the coordinates of P are $(\cos\theta, \sin\theta)$.

DEMO

The x- and y-coordinates of P each have a special name.

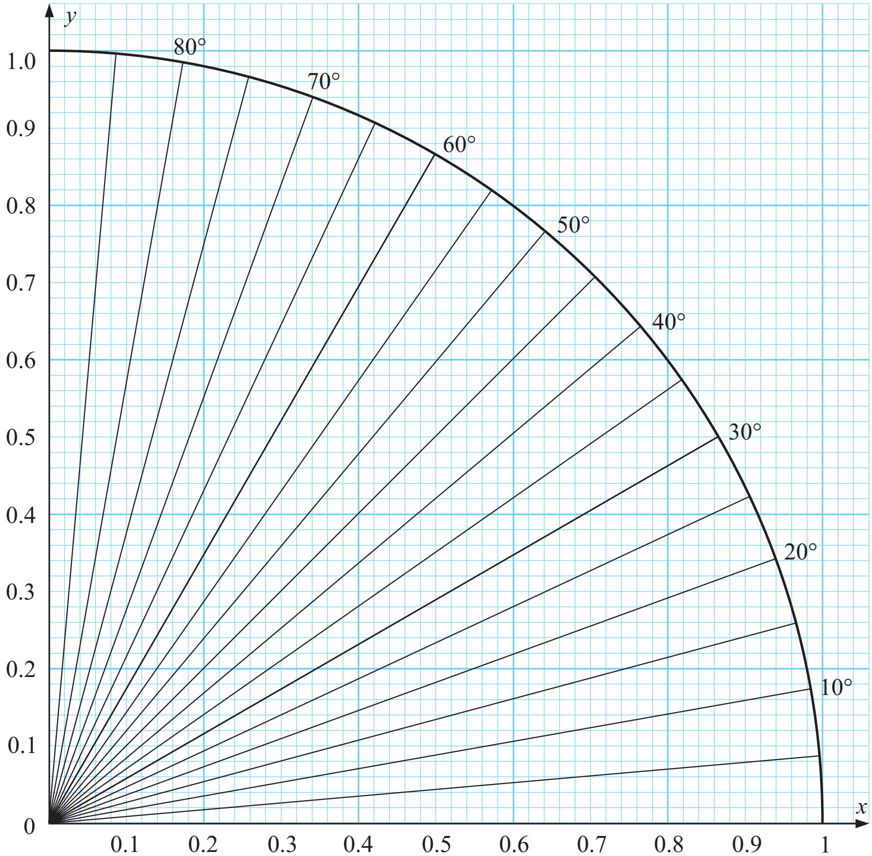
- The y-coordinate is called "the sine of angle θ" or $\sin\theta$.
- The x-coordinate is called "the cosine of angle θ" or $\cos\theta$.

Notice also that in ΔONP, $x^2 + y^2 = 1$ {Pythagoras}

and so $[\cos\theta]^2 + [\sin\theta]^2 = 1$

or $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

Note: We use $\cos^2\theta$ for $[\cos\theta]^2$ and $\sin^2\theta$ for $[\sin\theta]^2$.



Example 1

Use the unit circle to find:

a $\sin 40^o$ **b** $\cos 30^o$ **c** the coordinates of P if $\theta = 50^o$

a The y-coordinate at 40^o is about 0.64

$\therefore\ \sin 40^o \doteqdot 0.64$

b The x-coordinate at 30^o is about 0.87

$\therefore\ \cos 30^o \doteqdot 0.87$

c For $\theta = 50^o$, P is $(\cos 50^o, \sin 50^o) \doteqdot (0.64, 0.77)$

You have probably already noticed the difficulty of obtaining accurate values from the unit circle and the impossibility of estimating beyond 2 decimal places.

EXERCISE 11A

1 Use the unit quarter circle to find the value of:

a $\sin 0^o$ **b** $\sin 15^o$ **c** $\sin 25^o$ **d** $\sin 30^o$
e $\sin 45^o$ **f** $\sin 60^o$ **g** $\sin 75^o$ **h** $\sin 90^o$

2 Use your calculator to check your answers to question **1**.

3 Use the unit quarter circle diagram to find the value of:

a $\cos 0^o$ **b** $\cos 15^o$ **c** $\cos 25^o$ **d** $\cos 30^o$
e $\cos 45^o$ **f** $\cos 60^o$ **g** $\cos 75^o$ **h** $\cos 90^o$

4 Use your calculator to check your answers to question **3**.

5 Use the unit quarter circle diagram to find the coordinates of the point on the unit circle where OP makes an angle of 55^o with the x-axis. Use your calculator to check this answer.

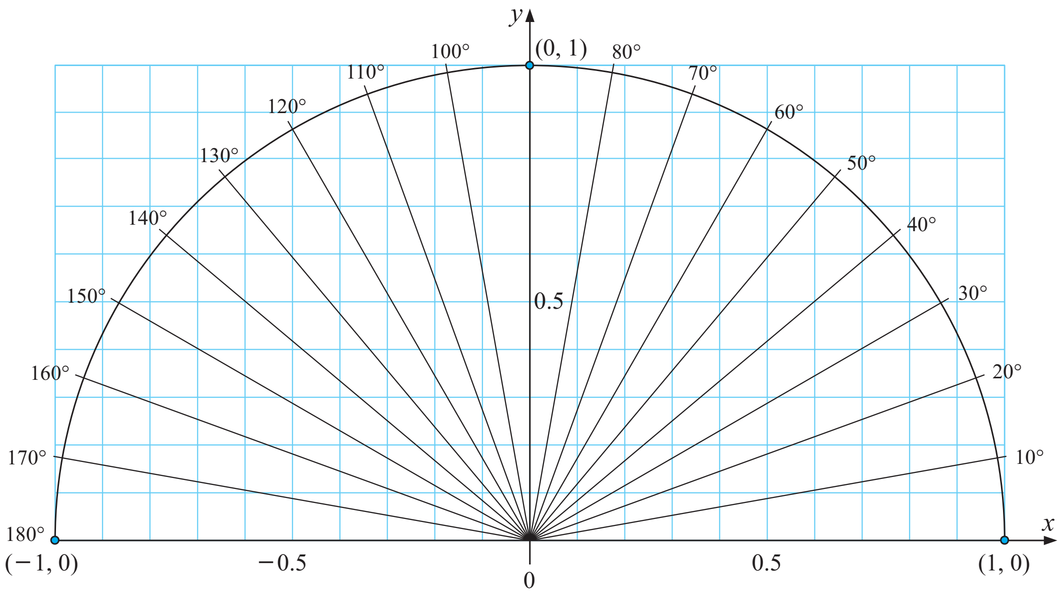
6 Draw a sketch of a unit quarter circle and on it show how to locate the point with coordinates:

a $(\cos 20^o, \sin 20^o)$ **b** $(\cos 75^o, \sin 75^o)$ **c** $(\cos\phi, \sin\phi)$

7 **a** If $\cos\theta = 0.8$ and $0^o < \theta < 90^o$, find $\sin\theta$.
b If $\sin\theta = 0.7$ and $0^o < \theta < 90^o$, find $\cos\theta$ correct to 3 significant figures.

B OBTUSE ANGLES

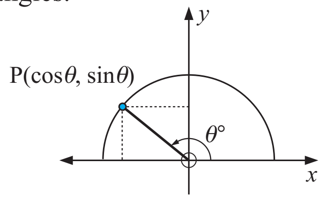
So far we have only considered angles between 0^o and 90^o, i.e., acute angles. **Obtuse angles** have measurement between 90^o and 180^o. In order to display obtuse angles we can extend the unit quarter circle as seen in the diagram below.



We will now apply the definitions for $\sin\theta$ and $\cos\theta$ to obtuse angles.

Definition: If P is any point on the unit circle and θ is the angle measured from the positive x-axis then

$\cos\theta$ is the x-coordinate of P and
$\sin\theta$ is the y-coordinate of P.



Example 2

Use the unit circle to find:

a $\sin 140^o$ **b** $\cos 150^o$ **c** the coordinates of P if $\theta = 160^o$

a The y-coordinate at 140^o is about 0.64 $\therefore$ $\sin 140^o \doteqdot 0.64$

b The x-coordinate at 150^o is about -0.87 $\therefore$ $\cos 150^o \doteqdot -0.87$

c For $\theta = 160^o$, P is $(\cos 160^o, \sin 160^o)$, i.e., $(-0.94, 0.34)$

EXERCISE 11B

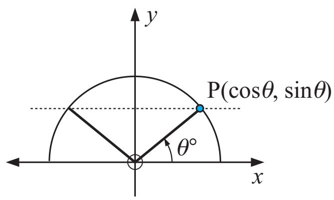
1 Use the unit half circle to find the value of:

a $\sin 100^o$ **b** $\sin 80^o$ **c** $\sin 120^o$ **d** $\sin 60^o$
e $\sin 150^o$ **f** $\sin 30^o$ **g** $\sin 180^o$ **h** $\sin 0^o$

2 Use your calculator to check your answers to question **1**.

3 **a** Use your results from question **1** to copy and complete: $\sin(180 - \theta)^o = ...$

b Justify your answer using the diagram alongside.



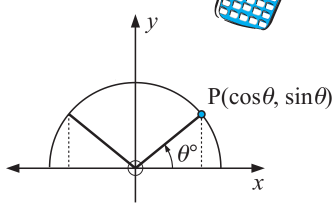
4 Use the unit half circle to find the value of:

a $\cos 110^o$ **b** $\cos 70^o$ **c** $\cos 130^o$ **d** $\cos 50^o$
e $\cos 140^o$ **f** $\cos 40^o$ **g** $\cos 180^o$ **h** $\cos 0^o$

5 Use your calculator to check your answers to quesion **4**.

6 **a** Use your results from question **4** to copy and complete: $\cos(180 - \theta)^o = ...$

b Justify your answer using the diagram alongside.



7 Find the obtuse angle which has the same sine as:

a 45^o **b** 51^o **c** 74^o **d** 82^o

8 Find the acute angle which has the same sine as:

a 130^o **b** 146^o **c** 162^o **d** 171^o

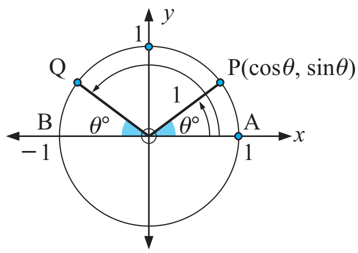
9 Without using your calculator find:

a $\sin 137^o$ if $\sin 43^o \doteqdot 0.6820$
b $\sin 59^o$ if $\sin 121^o \doteqdot 0.8572$
c $\cos 143^o$ if $\cos 37^o \doteqdot 0.7986$
d $\cos 24^o$ if $\cos 156^o \doteqdot -0.9135$
e $\sin 115^o$ if $\sin 65^o \doteqdot 0.9063$
f $\cos 132^o$ if $\cos 48^o \doteqdot 0.6691$

10 **a** If angle AOP $= \theta$ and angle BOQ $= \theta$ also, what is the measure of angle AOQ?

b Copy and complete:
OQ is a reflection of OP in the ......
and so Q has coordinates ......

c Now using **a** and **b**, what trigonometric formulae can be deduced?

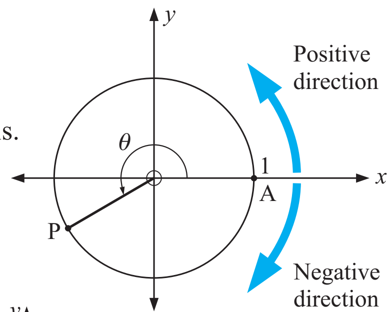


In the exercises above you should have discovered that:

- If θ is acute, then $\cos\theta$ and $\sin\theta$ are both positive.
- If θ is obtuse, then $\cos\theta$ is negative and $\sin\theta$ is positive.
- $\boldsymbol{\sin(180-\theta) = \sin\theta}$ and $\boldsymbol{\cos(180-\theta) = -\cos\theta}$

These facts are particularly important in the next chapter.

ANGLE MEASUREMENT

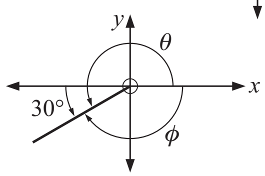


Suppose P lies anywhere on the unit circle and A is (1, 0).

Let θ be the angle measured from OA, on the positive x-axis.

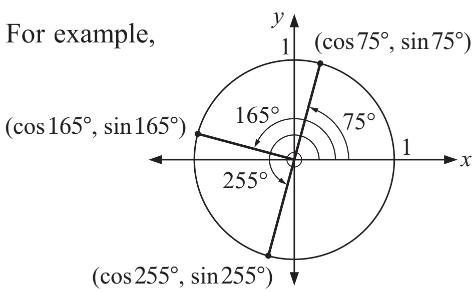
θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.

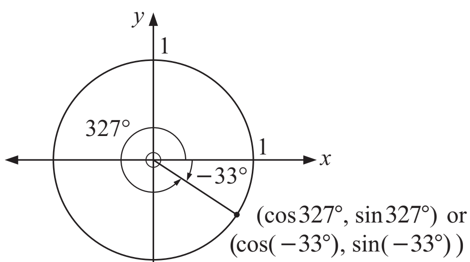
For example, $\theta = 210^o$ and $\phi = -150^o$.



Consequently, we can easily find the coordinates of any point on the unit circle for a particular angle measured from the positive x-axis.

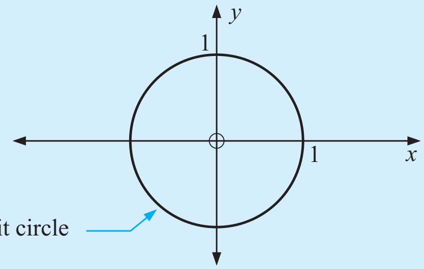
For example,



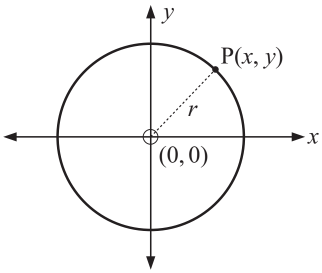


C THE UNIT CIRCLE

The **unit circle** is the circle with centre (0, 0) and radius 1 unit.



CIRCLES WITH CENTRE (0, 0)



Consider a circle with centre (0, 0) and radius r units, and suppose $P(x, y)$ is any point on this circle.

Since $OP = r$, then

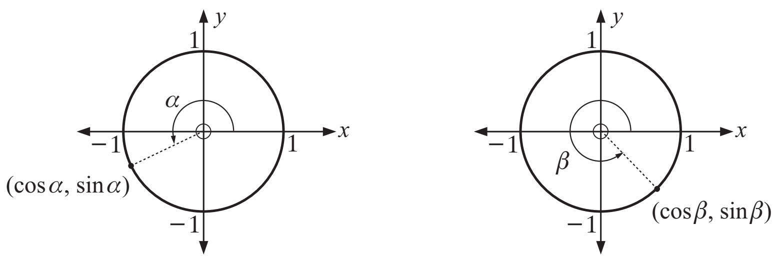
$$\sqrt{(x-0)^2 + (y-0)^2} = r \qquad \{\text{distance formula}\}$$

$$\therefore \quad x^2 + y^2 = r^2$$

We say that $x^2 + y^2 = r^2$ is the equation of a circle with centre (0, 0) and radius r.

So, the **equation** of the **unit circle** is $x^2 + y^2 = 1$. $\{$as $r = 1\}$

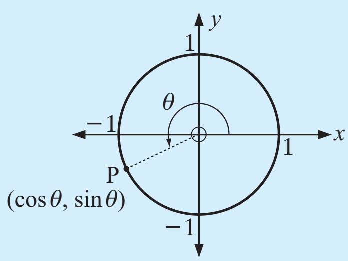
If we allow the definitions of $\cos\theta$ and $\sin\theta$ to apply to any angle we see that, for example:



So, as point P moves anywhere on the unit circle,

its x-coordinate is $\cos\theta$
its y-coordinate is $\sin\theta$

provided that θ is the angle made by OP with the positive x-axis.



Notice that: as $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$ for all points on the unit circle,

then $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$ for all $\boldsymbol{\theta}$.

EXERCISE 11C

1 Sketch the graph of the curve with equation:

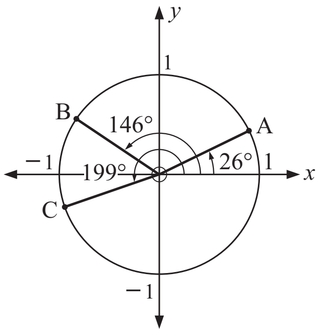
a $x^2+y^2=1$ **b** $x^2+y^2=4$ **c** $x^2+y^2=1, \quad y \geqslant 0$

2 For each angle illustrated:

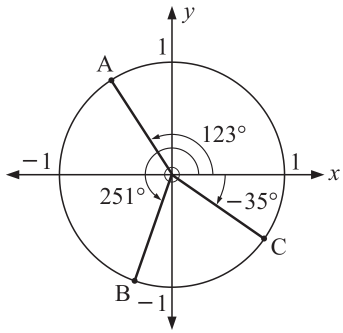
i write down the actual coordinates of points A, B and C

ii use your calculator to give the coordinates of A, B and C correct to 3 significant figures.

a

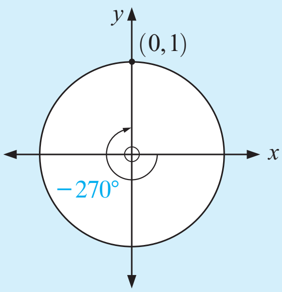


b



Example 3

Use a unit circle diagram to find the values of $\cos(-270^o)$ and $\sin(-270^o)$.



$\therefore \quad \cos(-270^o)=0 \quad$ {the x-coordinate}

and $\quad \sin(-270^o)=1 \quad$ {the y-coordinate}

3 Use a unit circle diagram to find:

a $\cos 0^o$ and $\sin 0^o$

b $\cos 90^o$ and $\sin 90^o$

c $\cos 180^o$ and $\sin 180^o$

d $\cos 270^o$ and $\sin 270^o$

e $\cos(-90^o)$ and $\sin(-90^o)$

f $\cos 450^o$ and $\sin 450^o$

INVESTIGATION PARAMETRIC EQUATIONS

Usually we write functions in the form $y=f(x)$.

For example: $y=3x+7$, $y=x^2-6x+8$, $y=\sin x$

However, sometimes it is useful to express **both** x and y in terms of one convenient variable, t say, called the **parameter**.

The purpose of this investigation is to use technology to graph a set of ordered pairs defined in **parametric form**, for example, $x = \cos t$ and $y = \sin t$.

What to do:

1. Either click on the icon, or use your graphics calculator (with the same scale on both axes) to plot $\{(x, y): \ x = \cos t, \ y = \sin t, \ 0^o \leqslant t \leqslant 360^o\}$

   Note: Set up your calculator in degrees.
2. Describe the resulting graph.
3. What is the equation of this graph? (Two possible answers).
4. If using a graphics calculator, use the *trace* key to move along the curve. What do you notice?

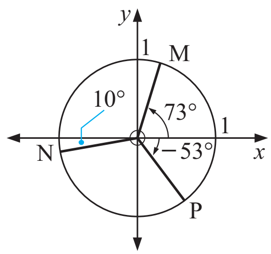
REVIEW SET 11

1. Use the unit quarter circle on page **227** to find:

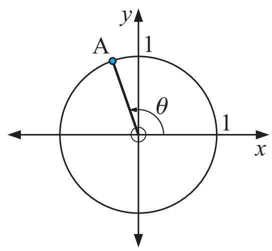
   a $\sin 70^o$ **b** $\cos 35^o$

   Check using your calculator.

2. Find the coordinates of the points M, N and P on the unit circle.



3. Find θ if the value of the x-coordinate of the point A on the unit circle is -0.222.



4. Find the acute angles that would have the same sine as:

   a 120^o **b** 165^o **c** 95^o

5. Find the obtuse angles which have the same sine as:

   a 47^o **b** 8^o **c** 86^o

6. Without using your calculator, find:

   a $\sin 159^o$ if $\sin 21^o \doteqdot 0.358$ **b** $\cos 92^o$ if $\cos 88^o \doteqdot 0.035$

   c $\cos 75^o$ if $\cos 105^o \doteqdot -0.259$ **d** $\sin 227^o$ if $\sin 47^o \doteqdot 0.731$

   e $\cos 320^o$ if $\cos 40^o \doteqdot 0.766$

7 Given that $\cos 60^o = 0.5$ and $\sin 60^o \doteqdot 0.866$ use a unit circle diagram to find:

a $\cos 240^o$ and $\sin 240^o$ **b** $\cos 480^o$ and $\sin 480^o$

c $\cos(-840^o)$ and $\sin(-840^o)$

8 Use a unit circle diagram to find:

a $\cos 360^o$ and $\sin 360^o$ **b** $\cos(-180^o)$ and $\sin(-180^o)$

c $\cos 630^o$ and $\sin 630$

9 Find the acute angles that would have the same sine as:

a 101^o **b** 127^o **c** 168^o

10 Find the acute angles that would have the same cosine as:

a 276^o **b** 298^o **c** 357^o

11 If $\sin 74^o \doteqdot 0.961$, without using a calculator, find the value of:

a $\sin 106^o$ **b** $\sin 254^o$ **c** $\sin 286^o$ **d** $\sin 646^o$

12 If $\cos 42^o \doteqdot 0.743$, without using a calculator, find the value of:

a $\cos 138^o$ **b** $\cos 222^o$ **c** $\cos 318^o$ **d** $\cos(-222^o)$

Chapter 12

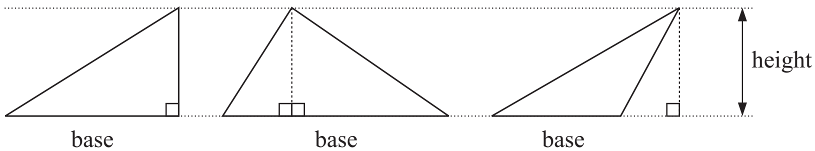
Non right angled triangle trigonometry

Contents:

A Areas of triangles
B Sectors and segments
C The Cosine rule
D The Sine rule
Investigation: The ambiguous case
E Using the Sine and Cosine rules

Review set 12A
Review set 12B

A AREAS OF TRIANGLES



DEMO

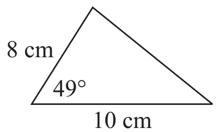
If we know the base and height measurements of a triangle we can calculate the area using **area** $= \frac{1}{2}$ **base** $\times$ **height**.

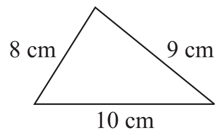
However, cases arise where we do not know the height but we can still calculate the area.

These cases are:

- knowing two sides and the angle between them (called the **included angle**)
- knowing all three sides

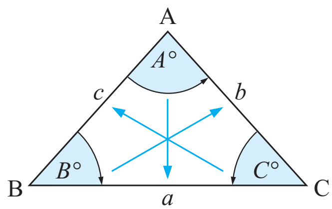
For example:



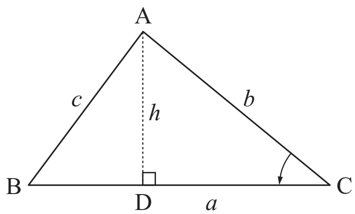


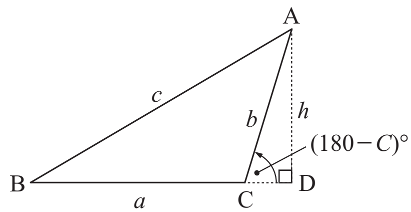
LABELLING TRIANGLES

If triangle ABC has angles of size A^o, B^o, C^o, the sides opposite these angles are labelled a, b and c respectively.



Using trigonometry, we can develop an alternative formula that does not depend on a perpendicular height. Any triangle that is not right angled must be either acute or obtuse. We will consider both cases.





In both triangles a perpendicular is constructed from A to D on BC (extended if necessary).

$$\sin C = \frac{h}{b} \qquad\qquad \sin(180 - C) = \frac{h}{b}$$

$$\therefore \quad h = b\sin C \qquad\qquad \therefore \quad h = b\sin(180 - C)$$

$$\text{but} \quad \sin(180 - C) = \sin C$$

$$\therefore \quad h = b\sin C$$

So, as area $= \frac{1}{2}ah$ then $\boxed{\text{area} = \frac{1}{2}ab\sin C.}$

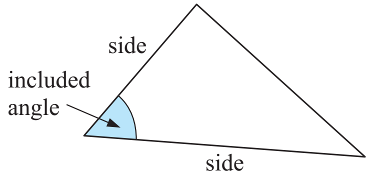
Using different altitudes we could also show that the area of ΔABC is given by

$$\tfrac{1}{2}bc\sin A \quad \text{or} \quad \tfrac{1}{2}ac\sin B.$$

Summary:

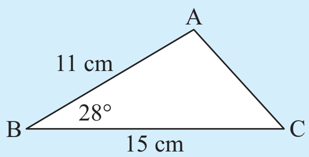
Given the lengths of two sides of a triangle and the angle between them (called the **included angle**), the area of the triangle is

a half of the product of two sides and the **sine of the included angle**.



Example 1

Find the area of triangle ABC:

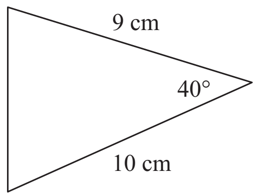


$$\begin{aligned}\text{Area} &= \tfrac{1}{2}ac\sin B \\ &= \tfrac{1}{2} \times 15 \times 11 \times \sin 28^o \\ &\doteqdot 38.7\ \text{cm}^2\end{aligned}$$

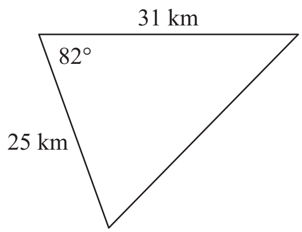
EXERCISE 12A

1 Find the area of:

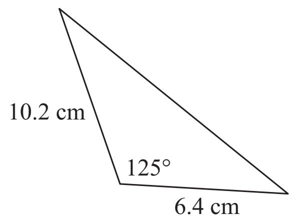
a



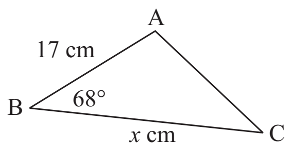
b



c



2 If triangle ABC has area 150 cm^2, find the value of x:



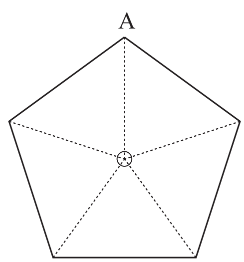
3 A parallelogram has two adjacent sides of length 4 cm and 6 cm respectively. If the included angle measures 52^o, find the area of the parallelogram.

4 A rhombus has side lengths 12 cm and an angle of 72^o. Find its area.

5 Find the area of a regular hexagon with sides of length 12 cm.

6 A rhombus has an area of 50 cm^2 and an internal angle of size 63^o. Find the length of its sides.

7 A regular pentagonal garden plot has centre of symmetry O and an area of 338 m^2. Find the distance OA.



Example 2

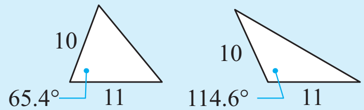
A triangle has sides of length 10 cm and 11 cm and an area of 50 cm^2.
Show that the included angle may take two different possible sizes.

If the included angle measures θ^o, then
$$\tfrac{1}{2} \times 10 \times 11 \times \sin\theta = 50$$
$$\therefore \quad 55\sin\theta = 50$$
$$\therefore \quad \sin\theta = \frac{50}{55}$$

Now $\sin^{-1}\left(\dfrac{50}{55}\right) \doteqdot 65.4$

$\therefore \quad \theta = 65.4 \quad \text{or} \quad 180 - 65.4$

i.e., $\quad \theta = 65.4 \quad \text{or} \quad 114.6$



So, the two different possible angles are 65.4^o and 114.6^o

Reminder: $\sin\theta = \sin(180 - \theta)$ was established in the previous chapter.

8 Find the possible values of the included angle of a triangle with:

a sides 5 cm and 8 cm, and area 15 cm^2

b sides 45 km and 53 km, and area 800 km^2.

9 The Australian 50 cent coin has the shape of a dodecagon (12 sides). Eight of these 50 cent coins will fit exactly on an Australian \$10 note as shown below. What fraction of the \$10 note is *not* covered?

10 Heron of Alexandria, in the first century A.D., showed that if a triangle has sides of length a, b and c, then its area can be calculated using $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$.

a Find the area of the right angled triangle with sides 3 cm, 4 cm and 5 cm:

i without using Heron's formula

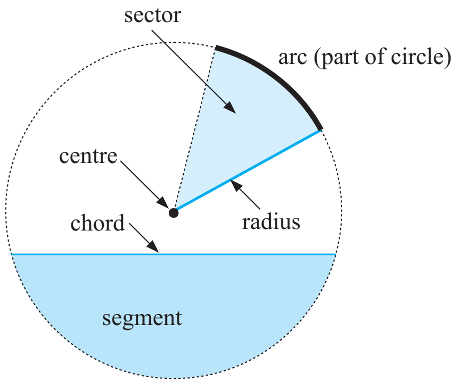
ii using Heron's formula.

b Find the area of a triangle with sides of length:

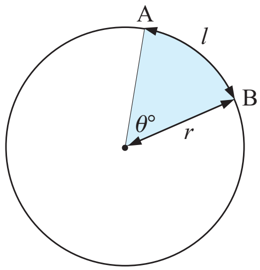
i 6 cm, 8 cm and 12 cm **ii** 7.2 cm, 8.9 cm and 9.7 cm

B SECTORS AND SEGMENTS

Reminder:



SECTORS



Consider a sector of a circle of radius r and angle θ^o at the centre.

If l is the length of the arc from A to B then:

- $\boldsymbol{l = \left(\frac{\theta}{360}\right) \times 2\pi r}$
- $\textbf{area} = \boldsymbol{\left(\frac{\theta}{360}\right) \times \pi r^2}$

Note: $\frac{\theta}{360}$ is the fraction of the full circle occupied by the sector.

Example 3

A sector has radius 12 cm and angle 65^o. Find:

a its arc length **b** its area

a arc length $= \left(\dfrac{\theta}{360}\right) \times 2\pi r$

$= \dfrac{65}{360} \times 2 \times \pi \times 12$

$\doteqdot 13.6$ cm

b area $= \left(\dfrac{\theta}{360}\right) \times \pi r^2$

$= \dfrac{65}{360} \times \pi \times 12^2$

$\doteqdot 81.7\text{ cm}^2$

EXERCISE 12B

1 Find **i** the arc length **ii** the area of a sector of a circle of:

a radius 9 cm and angle 41.6^o **b** radius 4.93 cm and angle 122^o

2 A sector has an angle of 107.9^o and an arc length of 5.92 m. Find:

a its radius **b** its area.

3 A sector has an angle of 68.2^o and an area of 20.8 cm^2. Find:

a its radius **b** its perimeter.

Example 4

A sector has radius 8.2 cm and arc length 13.3 cm. Find its angle.

$$\text{arc length} = \left(\frac{\theta}{360}\right) \times 2\pi r$$

$$\therefore \quad \frac{\theta}{360} \times 2 \times \pi \times 8.2 = 13.3$$

$$\therefore \quad \theta \times 2 \times \pi \times 8.2 = 360 \times 13.3 \qquad \{\text{multiply both sides by } 360\}$$

$$\therefore \quad \theta = \frac{360 \times 13.3}{(2 \times \pi \times 8.2)} \doteq 92.93$$

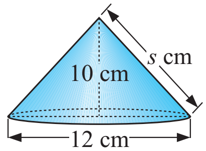
So, its angle is 92.9^o.

4 Find the angle of a sector of:

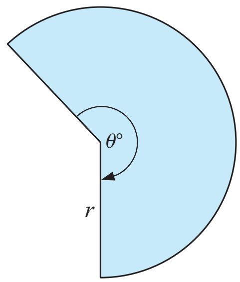
a radius 4.3 m and arc length 2.95 m

b radius 10 cm and area 30 cm^2.

5 This cone



is made from this sector



Find correct to 3 significant figures:

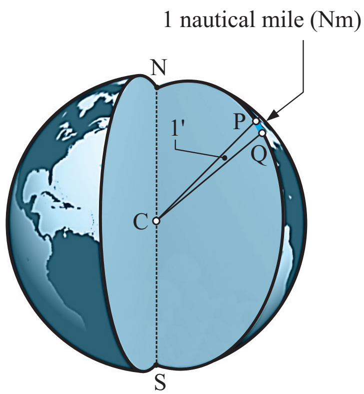
a the slant length (s cm)

b the value of r

c the arc length of the sector

d the sector angle (θ^o)

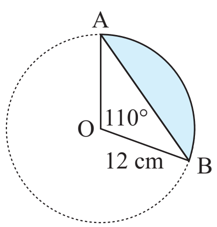
6 A **nautical mile** (n mile) is the distance on the Earth's surface that subtends an angle of 1 minute (where 1 minute $= \frac{1}{60}$ degree) of the Great Circle arc measured from the centre of the Earth. A **knot** is a speed of 1 nautical mile per hour.



a Given that the radius of the Earth is 6370 km, show that 1 n mile is approximately equal to 1.853 km.

b Calculate how long it would take a plane to fly from Perth to Adelaide (a distance of 2130 km) if the plane can fly at 480 knots.

7 Find:



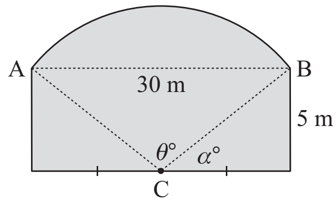
a the area of ΔOAB

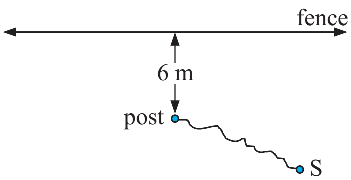
b the area of sector OAB

c the area of the shaded segment.

8 The end wall of a building has the shape illustrated, where the centre of arc AB is at C. Find:

- **a** α to 4 significant figures
- **b** θ to 4 significant figures
- **c** the area of the wall.

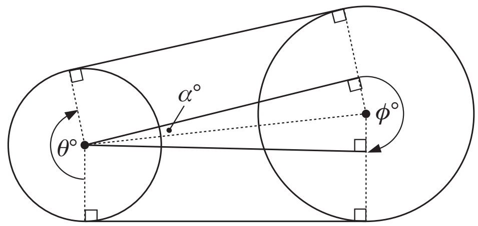


9 

A sheep is tethered to a post which is 6 m from a long fence. The length of rope is 9 m. Find the area which is available for the sheep to feed on.

10 A belt fits tightly around two pulleys of radii 4 cm and 6 cm respectively and the distance between their centres is 20 cm. Find, correct to 4 significant figures:

- **a** α
- **b** θ
- **c** ϕ
- **d** the length of the belt

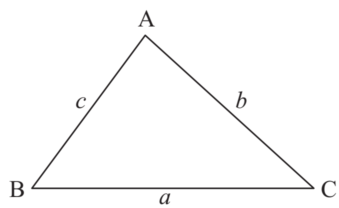


C THE COSINE RULE

The **cosine rule** involves the sides and angles of a triangle.

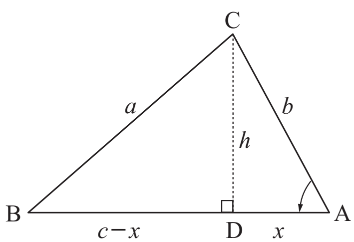
In any ΔABC:

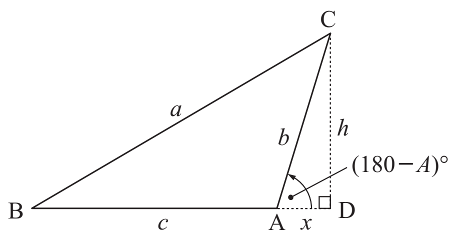
$$\begin{aligned} & a^2 = b^2 + c^2 - 2bc\cos A \\ \textit{or} \quad & b^2 = a^2 + c^2 - 2ac\cos B \\ \textit{or} \quad & c^2 = a^2 + b^2 - 2ab\cos C \end{aligned}$$



We will develop the first formula for both an acute and an obtuse triangle.

Proof:





In both triangles drop a perpendicular from C to meet AB (extended if necessary) at D.

Let AD $= x$ and let CD $= h$.

Apply the theorem of Pythagoras in ΔBCD:

$$a^2 = h^2 + (c-x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 - 2cx + x^2$$

$$a^2 = h^2 + (c+x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 + 2cx + x^2$$

In both cases, applying Pythagoras to ΔADC: $h^2 + x^2 = b^2$ and substitute for h^2.

$\therefore \quad a^2 = b^2 + c^2 - 2cx$

In ADC: $\cos A = \dfrac{x}{b}$

$\therefore \quad b\cos A = x$

$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$

$\therefore \quad a^2 = b^2 + c^2 + 2cx$

Now $\cos(180 - A) = \dfrac{x}{b}$

$\therefore \quad b\cos(180 - A) = x$

But, $\cos(180 - A) = -\cos A$

$\therefore \quad -b\cos A = x$

$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$

The other variations of the cosine rule could be developed by rearranging the vertices of ΔABC.

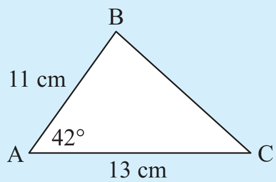
Note that if $A = 90^o$, $\cos A = 0$ and $a^2 = b^2 + c^2 - 2bc\cos A$ reduces to $a^2 = b^2 + c^2$, the Pythagoras' Rule.

The **cosine rule** can be used to solve triangles given:
- two sides and an included angle
- three sides.

There is no ambiguity possible using the cosine rule.

Example 5

Find, correct to 2 decimal places, the length of BC.



By the cosine rule:

$$BC^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o$$

$$\therefore \quad BC \doteqdot \sqrt{(11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o)}$$

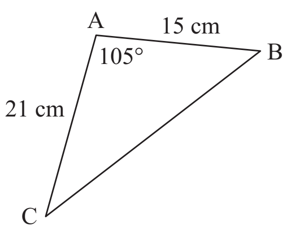
$$\therefore \quad BC \doteqdot 8.801.....$$

$\therefore$ BC is 8.80 cm in length.

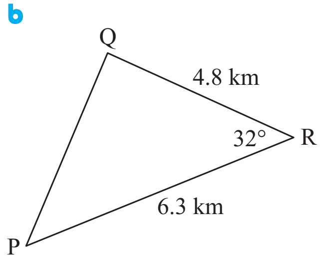
EXERCISE 12C

1 Find the length of the remaining side in the given triangle:

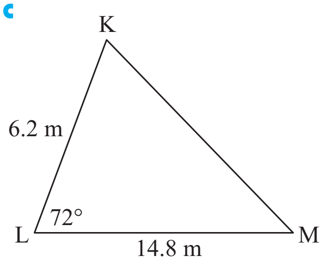
a



b



c



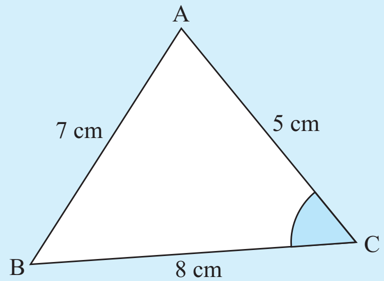
Rearrangement of the original cosine rule formulae can be used for angle finding if we know all three sides.

The formulae for finding the angles are:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 6

In triangle ABC, if AB $= 7$ cm, BC $= 8$ cm and CA $= 5$ cm, find the measure of angle BCA.



By the cosine rule:

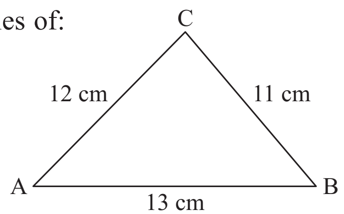
$$\cos C = \frac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}$$

$$\therefore \quad C = \cos^{-1}\left(\frac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}\right)$$

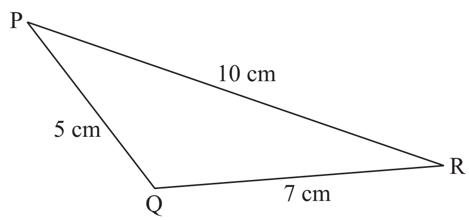
$$\therefore \quad C = 60$$

So, angle BCA measures 60^o.

2 Find the measure of all angles of:



3 Find the measure of obtuse angle PQR.



4 Find:

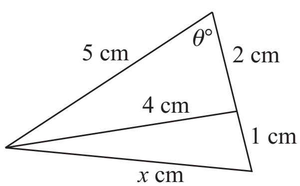
a the smallest angle of a triangle with sides 11 cm, 13 cm and 17 cm

b the largest angle of a triangle with sides 4 cm, 7 cm and 9 cm.

5 Find:

a $\cos\theta$ but not θ

b the value of x.



D THE SINE RULE

The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of the angles of the triangle. The triangle does not have to be right angled for the sine rule to be used.

In any triangle ABC with sides a, b and c units in length, and opposite angles A, B and C respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof: The area of any triangle ABC is given by

$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

Dividing each expression by $\frac{1}{2}abc$ gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

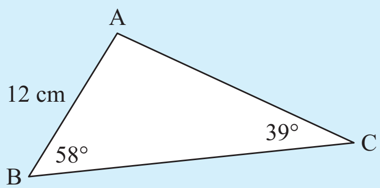
Note: The sine rule is used to solve problems involving triangles given either:

- **two angles** and **one side**, or
- **two sides** and a **non-included** angle.

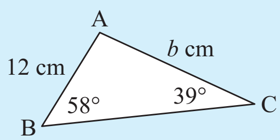
FINDING SIDES

Example 7

Find the length of AC correct to two decimal places.



By the sine rule



$\therefore \quad \dfrac{b}{\sin 58^o} = \dfrac{12}{\sin 39^o}$

$\therefore \quad b = \dfrac{12 \times \sin 58^o}{\sin 39^o}$

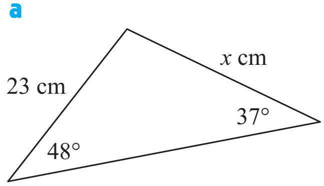
$\therefore \quad b \doteqdot 16.170\,74$

$\therefore$ AC is 16.2 cm long.

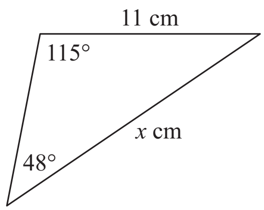
EXERCISE 12D.1

1 Find the value of x:

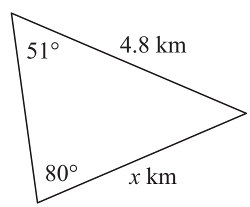
a



b



c



2 In triangle ABC find:

a a if $A = 63^o$, $B = 49^o$ and $b = 18$ cm

b b if $A = 82^o$, $C = 25^o$ and $c = 34$ cm

c c if $B = 21^o$, $C = 48^o$ and $a = 6.4$ cm

FINDING ANGLES

The problem of finding angles using the sine rule is more complicated because there may be two possible answers.

INVESTIGATION — THE AMBIGUOUS CASE

You will need a blank sheet of paper, a ruler, a protractor and a compass for the tasks that follow. In each task you will be required to construct triangles from given information.

Task 1: Draw AB = 10 cm. At A construct an angle of 30^o. Using B as centre, draw an arc of a circle of radius 6 cm. Let the arc intersect the ray from A at C. How many different positions may C have and therefore how many different triangles ABC may be constructed?

Task 2: As before, draw AB = 10 cm and construct a 30^o angle at A. This time draw an arc of radius 5 cm based on B. How many different triangles are possible?

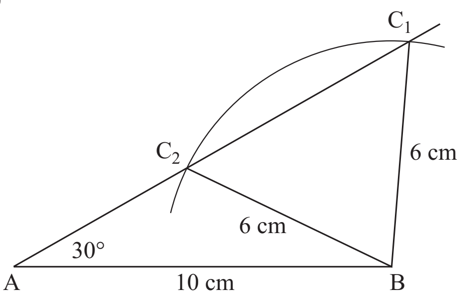
Task 3: Repeat, but this time draw an arc of radius 3 cm on B. How many different triangles are possible?

Task 4: Repeat with an arc of radius 12 cm from B. How many possible triangles?

In this investigation you should have discovered that when you are given two sides and a non-included angle there are a number of different possibilities. You could get two triangles, one triangle or it may be impossible to draw any triangles from the given data.

Let us consider the calculations involved in each of the cases of the investigation.

Task 1: Given: $c = 10$ cm, $a = 6$ cm, $A = 30^o$



Finding C: $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$$\therefore \quad \sin C = \frac{c \sin A}{a}$$

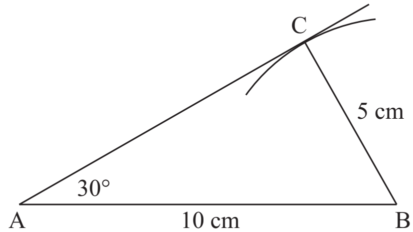
$$\therefore \quad \sin C = \frac{10 \times \sin 30^o}{6} = 0.8333$$

Because $\sin\theta = \sin(180^o - \theta)$ there are two possible angles:

$$C = 56.44^o \quad or \quad 180^o - 56.44^o = 123.56^o$$

On your calculator check that the sin ratio of both of these angles is 0.8333.

Task 2: Given: $c = 10$ cm, $a = 5$ cm, $A = 30^o$



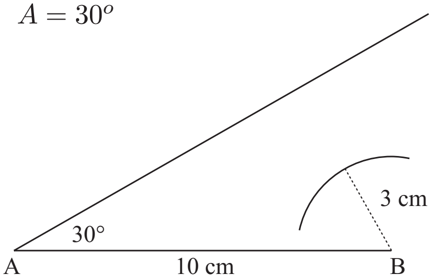
Finding C: $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$\therefore \quad \sin C = \dfrac{c \sin A}{a}$

$\therefore \quad \sin C = \dfrac{10 \times \sin 30^o}{5} = 1$

There is only one possible solution for C in the range from 0^o to 180^o and that is $C = 90^o$. So only one triangle (i.e. one set of solutions) is possible. Complete the solution of the triangle yourself.

Task 3: Given: $c = 10$ cm, $a = 3$ cm, $A = 30^o$



Finding C: $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$\therefore \quad \sin C = \dfrac{c \sin A}{a}$

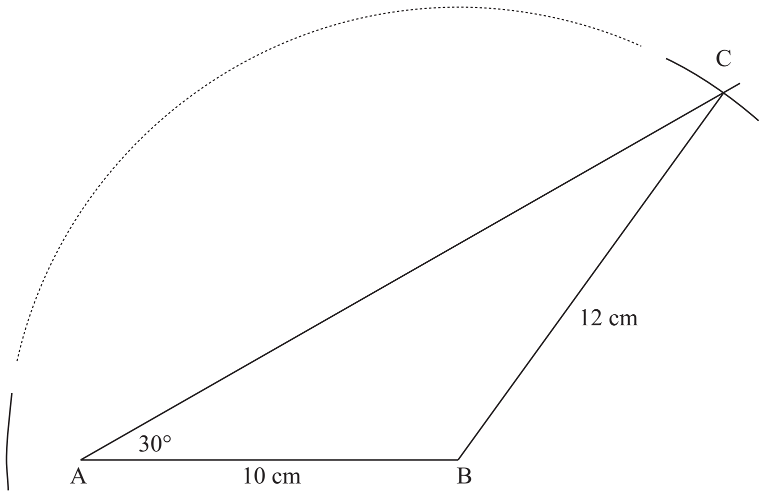
$\therefore \quad \sin C = \dfrac{10 \times \sin 30^o}{3}$

$\therefore \quad \sin C = 1.6667$

There is no angle that has a sin ratio > 1. Therefore there is *no solution* for this given data, i.e., *no possible* triangle can be drawn.

Task 4: Given: $c = 10$ cm, $a = 12$ cm, $A = 30^o$

Finding C:



$\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$\therefore \quad \sin C = \dfrac{c \sin A}{a}$

$\therefore \quad \sin C = \dfrac{10 \times \sin 30^o}{12}$

$\therefore \quad \sin C = 0.4167$

Two angles have a sin ratio of 0.4167

$C = 24.62^o$ *or* $180^o - 24.62^o$

$C = 24.62^o$ *or* 155.38^o

However, in this case only one of these two angles is valid.

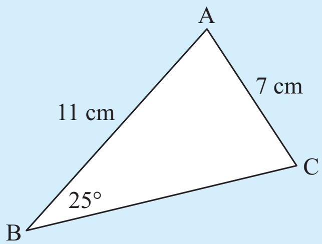
If $A = 30^o$ then C cannot possibly equal 155.38^o because $30^o + 155.38^o > 180^o$.

Therefore, there is only one solution, $C = 24.62^o$. Once again, you may wish to carry on and complete the solution.

Conclusion: Each situation using the sine rule with two sides and a non-included angle must be examined very carefully.

Example 8

Find the measure of angle C in triangle ABC if AC is 7 cm, AB is 11 cm and angle B measures 25^o.



By the sine rule

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\therefore \quad \frac{\sin C}{11} = \frac{\sin 25^o}{7}$$

$$\therefore \quad \sin C = \frac{11 \times \sin 25^o}{7}$$

$$\therefore \quad C = \sin^{-1}\left(\frac{11 \times \sin 25^o}{7}\right) \quad \text{or its supplement}$$

$$\therefore \quad C \doteqdot 41.6^o \text{ or } 180^o - 41.6^o$$

{as C may be obtuse}

$$\therefore \quad C \doteqdot 41.6^o \text{ or } 138.4^o$$

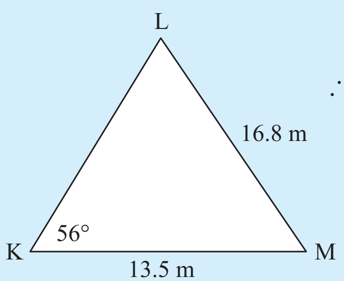
$\therefore$ C measures 41.6^o if angle C is acute
or C measures 138.4^o if angle C is obtuse.

In this example there is insufficient information to determine the actual shape of the triangle.

Note: Sometimes there is information in the question which enables us to **reject** one of the answers.

Example 9

Find the measure of angle L in triangle KLM given that angle LKM measures 56^o, LM $= 16.8$ m and KM $= 13.5$ m.



$$\frac{\sin L}{13.5} = \frac{\sin 56^o}{16.8} \qquad \text{\{the sine rule\}}$$

$$\therefore \quad \sin L = \frac{13.5 \times \sin 56^o}{16.8}$$

$$\therefore \quad L = \sin^{-1}\left(\frac{13.5 \times \sin 56^o}{16.8}\right) \quad \text{or its supplement}$$

$$\therefore \quad L \doteqdot 41.8^o \quad \text{or} \quad 180^o - 41.8^o$$

$$\therefore \quad L \doteqdot 41.8^o \quad \text{or} \quad 138.2^o$$

But reject $L = 138.2^o$ as $138.2^o + 56^o > 180^o$ which is impossible.

$$\therefore \quad \measuredangle L \doteqdot 41.8^o.$$

EXERCISE 12D.2

1 Triangle ABC has $\measuredangle B = 40^o$, $b = 8$ cm and $c = 11$ cm. Find the two possible values for angle C.

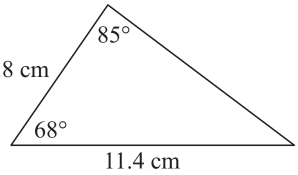
2 In triangle ABC, find the measure of:

a angle A if $a = 14.6$ cm, $b = 17.4$ cm and $\measuredangle \text{ABC} = 65^o$

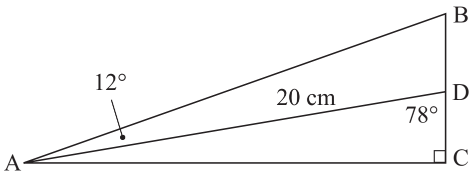
b angle B if $b = 43.8$ cm, $c = 31.4$ cm and $\measuredangle \text{ACB} = 43^o$

c angle C if $a = 6.5$ km, $c = 4.8$ km and $\measuredangle \text{BAC} = 71^o$.

3 Is it possible to have a triangle with measurements as shown? Explain!

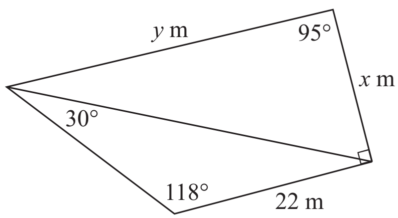


4



Find the magnitude of the angle ABC and hence BD in the given figure.

5 Find x and y in the given figure.



E USING THE SINE AND COSINE RULES

First decide which rule to use.

If the triangle is right angled then the trigonometric ratios or Pythagoras' Theorem can be used, and for some problems adding an extra line or two to the diagram may result in a right triangle.

However, if you have to choose between the sine and cosine rules, the following checklist may assist you.

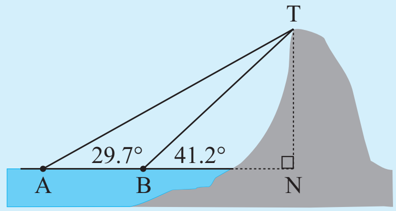
Use the **cosine rule** when given	• three sides • two sides and an included angle.
Use the **sine rule** when given	• one side and two angles • two sides and a non-included angle (but beware of the *ambiguous case* which can occur when the smaller of the two given sides is opposite the given angle).

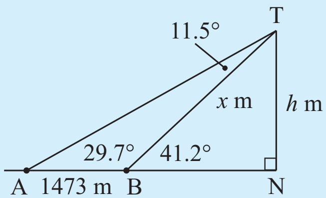
Example 10

The angles of elevation to the top of a mountain are measured from two beacons A and B, at sea.

These angles are as shown on the diagram.

If the beacons are 1473 m apart, how high is the mountain?





$\measuredangle ATB = 41.2^o - 29.7^o$ {exterior angle of Δ}

$= 11.5^o$

We can now find x in ΔABT using the sine rule,

i.e., $\dfrac{x}{\sin 29.7} = \dfrac{1473}{\sin 11.5}$

$\therefore \quad x = \dfrac{1473}{\sin 11.5} \times \sin 29.7$

$\doteqdot 3660.62.....$

Now, in ΔBNT, $\sin 41.2^o = \dfrac{h}{x} = \dfrac{h}{3660.62....}$

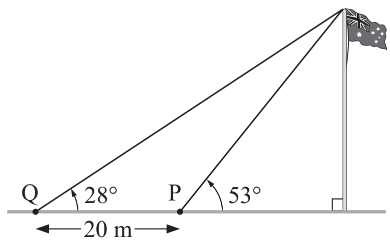
$\therefore \quad h = \sin 41.2^o \times 3660.62....$

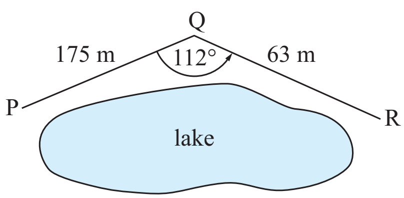
$\therefore \quad h \doteqdot 2410$

So, the mountain is about 2410 m high.

EXERCISE 12E

1 Manny wishes to determine the height of a flag pole. He takes a sighting of the top of the flagpole from point P. He then moves further away from the flagpole by 20 metres to point Q and takes a second sighting. The information is shown in the diagram alongside. How high is the flagpole?

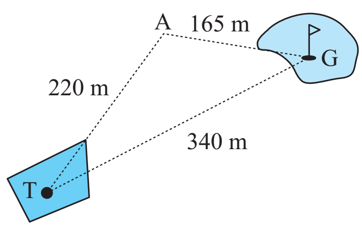


2 

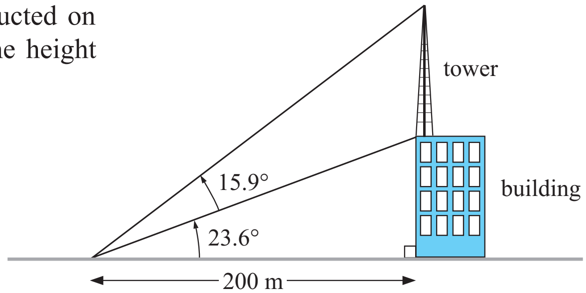
To get from P to R, a park ranger had to walk along a path to Q and then to R as shown.

What is the distance in a straight line from P to R?

3 A golfer played his tee shot a distance of 220 m to a point A. He then played a 165 m six iron to the green. If the distance from tee to green is 340 m, determine the number of degrees the golfer was off line with his tee shot.



4 A Communications Tower is constructed on top of a building as shown. Find the height of the tower.



5 A soccer goal is 5 metres wide. When a player is 21 metres from one goal post and 19 metres from the other, he shoots for goal. What is the angle of view of the goals that the player sees?

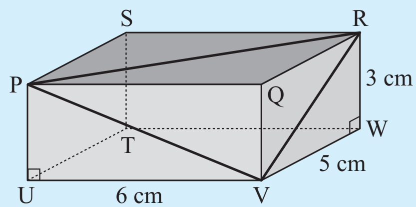
6 A tower 42 metres high, stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 13.2^o. From the same point the angle of elevation to the bottom of the tower is 8.3^o. Find the height of the hill.

7 From the foot of a building I have to look upwards at an angle of 22^o to sight the top of a tree. From the top of the building, 150 metres above ground level, I have to look down at an angle of 50^o below the horizontal to sight the tree top.

a How high is the tree? **b** How far from the building is this tree?

Example 11

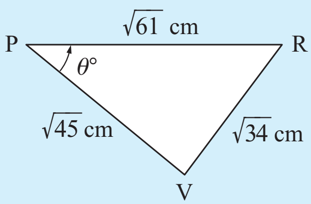
Find the measure of angle RPV.



In ΔRVW, $\text{RV} = \sqrt{5^2 + 3^2} = \sqrt{34}$ cm. {Pythagoras}

In ΔPUV, $\text{PV} = \sqrt{6^2 + 3^2} = \sqrt{45}$ cm. {Pythagoras}

Likewise in ΔPQR, $\text{PR} = \sqrt{6^2 + 5^2} = \sqrt{61}$ cm.



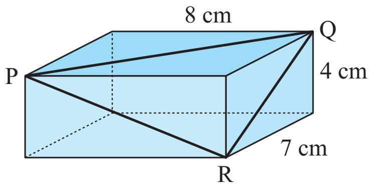
$$\cos\theta = \frac{(\sqrt{61})^2 + (\sqrt{45})^2 - (\sqrt{34})^2}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{61 + 45 - 34}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{72}{2\sqrt{61}\sqrt{45}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{\sqrt{61}\sqrt{45}}\right) \doteqdot 46.6$$

i.e., angle RPV measures 46.6^o.

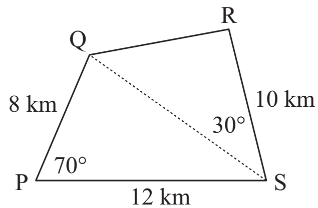
8 

Find the measure of angle PQR in the rectangular box shown.

9 Two observation posts are 12 km apart at A and B. From A, a third observation post C is located such that angle CAB is 42^o while angle CBA is 67^o. Find the distance of C from both A and B.

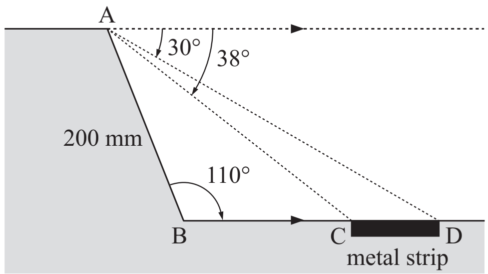
10 Stan and Olga are considering buying a sheep farm and the land agent supplies them with the given accurate sketch. Find the area of the property giving your answer in:

a km^2 **b** hectares.



11 Thabo and Palesa start at point A. They each walk in a straight line at an angle of 120^o to each other. Thabo walks at 6 kmph and Palesa walks at 8 kmph. How far apart are they after 45 minutes?

12 The design of the kerbing cross-section for a driverless-bus roadway is given. The metal strip is inlaid into the concrete and is used to control the direction of travel and speed of the bus. Find the width of the metal strip.

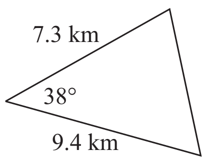


13 An orienteer runs for $4\frac{1}{2}$ km and then turns through an angle of 32^o and runs another 6 km. How far is she from her starting point?

14 Sam and Markus are standing on level ground 100 metres apart. A large tree is due North of Markus and on a bearing of 065^o from Sam. The top of the tree appears at an angle of elevation of 25^o to Sam and 15^o to Markus. Find the height of the tree.

REVIEW SET 12A

1 Determine the area of:



2 Determine the area of:

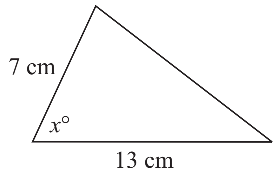
a a sector of angle 80^o and radius 13 cm

b a triangle with sides 11 cm, 9 cm and included angle 65^o.

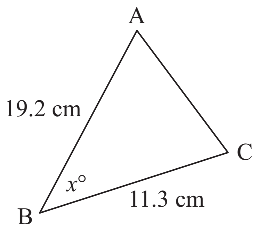
3 Determine the perimeter and area of a sector of radius 11 cm and angle 63^o.

4 Determine the radius and hence the area of a sector of perimeter 36 cm if the angle is 120^o.

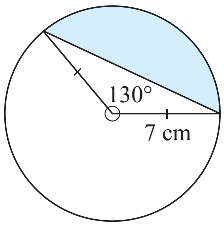
5 Find the value of x if the area is 42 cm^2:



6 Find the value of x if the area is 80 cm^2.
Hence, find the length of AC.

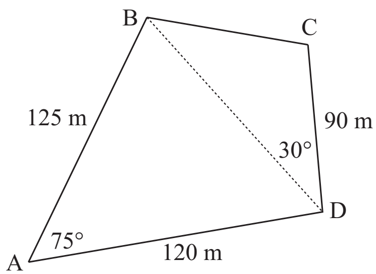


7 Determine the shaded area:



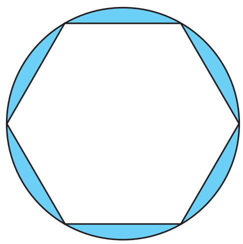
8 Anke and Lukas are considering buying a block of land and the land agent supplies them with the given accurate sketch. Find the area of the property giving your answer in:

a m^2
b hectares.



9 The diagram alongside shows a circular entertainment area. It has a paved hexagonal area with plants growing in the garden (shown as the shaded sectors).

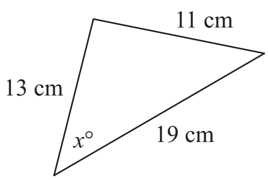
If the radius of the circle is 7 metres, find the area of the garden.



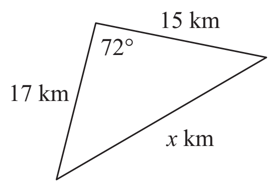
REVIEW SET 12B

1 Determine the value of x:

a

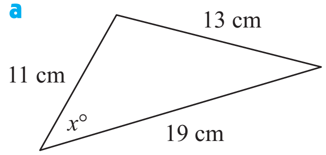


b

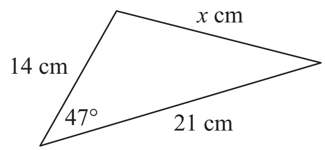


2 Find the value of x:

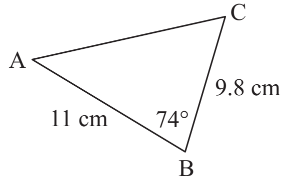
a



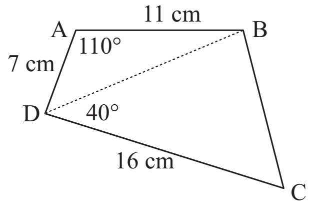
b



3 Find the unknown sides and angles:



4 Find the area of quadrilateral ABCD:



5 A vertical tree is growing on the side of a hill with slope of 10^o to the horizontal. From a point 50 m downhill from the tree, the angle of elevation to the top of the tree is 18^o. Find the height of the tree.

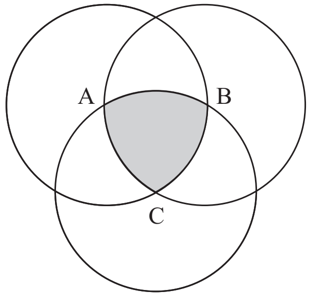
6 From point A, the angle of elevation to the top of a tall building is 20^o. On walking 80 m towards the building the angle of elevation is now 23^o. How tall is the building?

7 Peter, Sue and Alix are sea-kayaking. Peter is 430 m from Sue on a bearing of 113^o while Alix is on a bearing of 203^o and a distance 310 m from Sue. Find the distance and bearing of Peter from Alix.

8 A rally car drives at 140 kmph for 45 minutes on a bearing of 032^o and then 180 kmph for 40 minutes on a bearing 317^o. Find the distance and bearing of the car from its starting point.

9 Three equal circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B and C are the centres of the three circles. Prove that an expression for the area of the shaded region is:

$$A = \frac{r^2}{2}(\pi - \sqrt{3})$$



Chapter 13

Periodic phenomena

Contents:

A Observing periodic behaviour
B Radian measure and Periodic properties of circles
C The unit circle (revisited)
D The sine function
Investigation 1: The family $y = A\sin x$
Investigation 2: The family $y = \sin Bx,\ B > 0$
Investigation 3: The families $y = \sin(x - C)$ and $y = \sin x + D$
E Modelling using sine functions
F Equations involving sine
Investigation 4: The area under an arch of $y = \sin\theta$
G The cosine function
H Solving cosine equations
I Trigonometric relationships
J Double angle formulae
Investigation 5: Double angle formulae
K The tangent function
L Tangent equations
M Other equations involving $\tan x$

Review sets 13A, B, C, D, E

INTRODUCTION

Periodic phenomena occur in the physical world in:

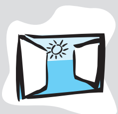
- seasonal variations in our climate
- variations in the average maximum and minimum monthly temperatures at a place
- the number of daylight hours at a place
- variations in the depth of water in a harbour due to tidal movement
- the phases of the moon etc.

Periodic phenomena also occur in the living world in animal populations.

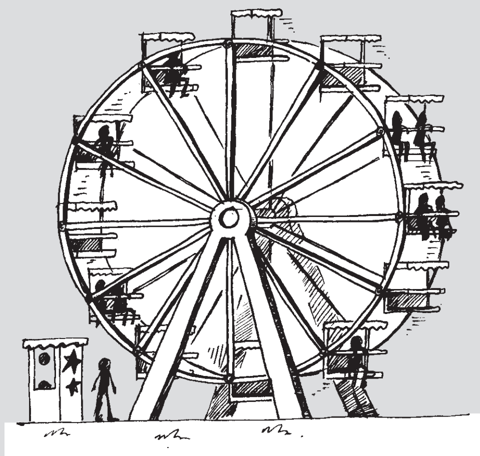
These phenomena illustrate variable behaviour which is repeated over time. This repetition may be called periodic, oscillatory or cyclic in different situations.

In this topic we will consider various data sets which display periodic behaviour.

OPENING PROBLEM



A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He then considers how high the light is above ground level at two second intervals and draws a scatterplot of his results.



- What would his scatterplot look like?
- Could a known function be used to model the data?
- How could this function be used to find the light's position at any point in time?
- How could this function be used to find the time when the light is at a maximum (or minimum) height?
- What part of the function would indicate the time interval over which one complete cycle occurs?

Click on the icon to visit a simulation of the Ferris wheel.

You are to view the light on the Ferris wheel:

- from a position in front of the wheel
- from a side-on position
- from above the wheel.

Now observe the graph of height above (or below) the wheel's axis over time as the wheel rotates at a constant rate.

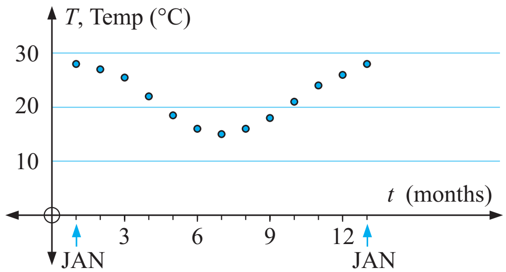
A OBSERVING PERIODIC BEHAVIOUR

Consider the table below which shows the mean monthly maximum temperature (^{o}C) for Cape Town.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	21	24	26

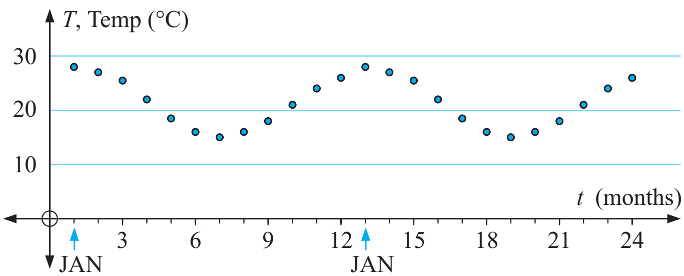
If this data is graphed using a scatterplot, assigning January = 1, February = 2 etc., for the 12 months of the year, the graph shown is obtained.

(**Note:** The points are not joined as interpolation has no meaning here.)

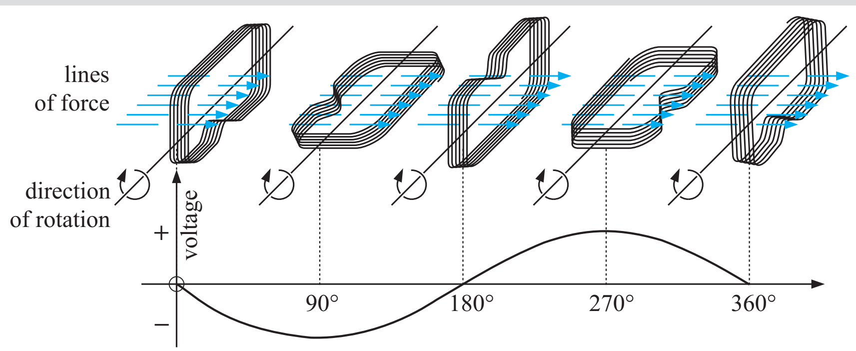


The temperature shows a variation from an average of 28^{o}C in January through a range of values across the months and the cycle will repeat itself for the next 12 months.

It is worthwhile noting that later we will be able to establish a function which approximately fits this set of points.



HISTORICAL NOTE



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through 360^o.

Graphs which have this basic shape where the cycle is repeated over and over are called **sine waves**.

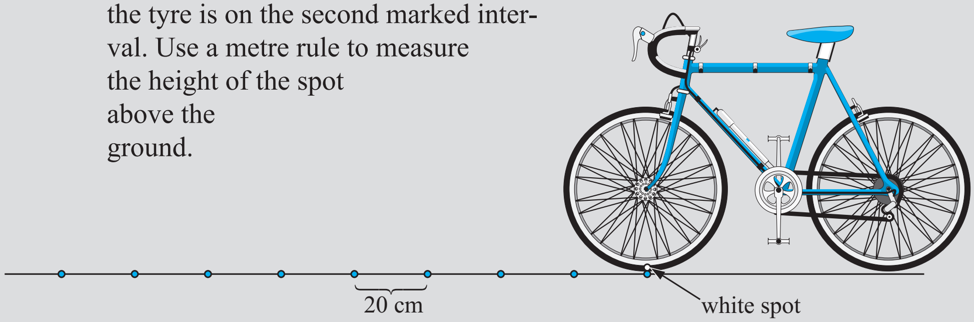
GATHERING PERIODIC DATA

- Maximum and minimum monthly temperatures are obtained from appropriate internet sites. (e.g. http://www.bom.gov.au/silo/)
- Tidal details can be obtained from daily newspapers or http://www.ntf.flinders.edu.au/TEXT/TIDES/tides.html

ACTIVITY — BICYCLE DATA

VIDEO CLIP

On a flat surface such as a tennis court mark a chalk line with equal intervals of 20 cm. On a tyre of a bicycle wheel mark a white spot using correcting fluid. Start with the spot at the bottom of the tyre on the first marked interval. Wheel the bike until the bottom of the tyre is on the second marked interval. Use a metre rule to measure the height of the spot above the ground.



a Record your result and continue until you have 20 or more data values.

b Plot this data on a set of axes.

c Are you entitled to fit a smooth curve through these points or should they be left as discrete points? Keep your results for future analysis.

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction.

The **period** of a periodic function is the length of one repetition or cycle.

If $f(x)$ is a periodic function with period p then $f(x+p) = f(x)$ for all x and p is the smallest positive value for this to be true.

Use a **graphing package** to examine the following function:

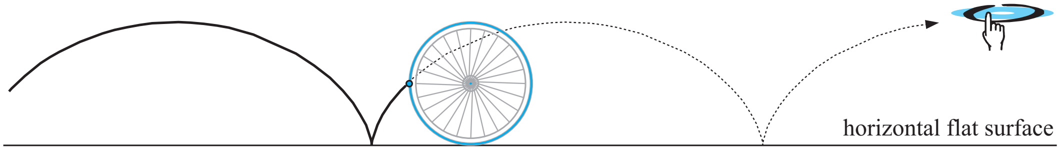
$$f : x \longmapsto x - [x]$$

GRAPHING PACKAGE

where $[x]$ is the largest integer less than or equal to x.

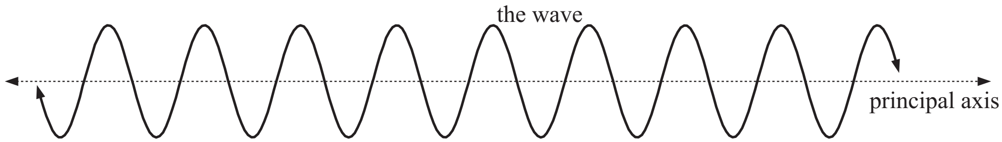
Is $f(x)$ periodic? What is its period?

A **cardioid** is also an example of a periodic function. It is the curve traced out by a point on a circle as the circle moves along a flat surface.

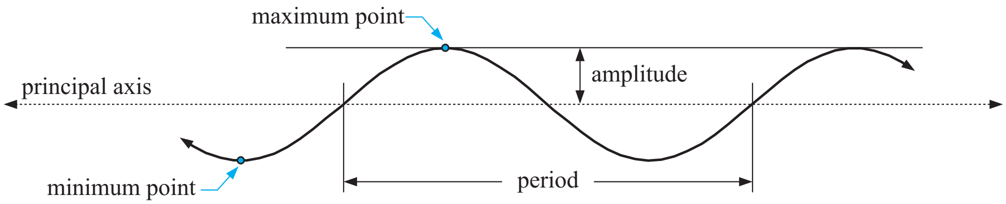


Unfortunately the cardioid function cannot be written as $y = ...$ or $f(x) = ...$

In this course we are mainly concerned with periodic phenomena which show a wave pattern when graphed.



The wave oscillates about a horizontal line called the **principal axis** (or **mean line**).



A **maximum point** occurs at the top of a crest and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

EXERCISE 13A

1 For each set of data below, draw a scatterplot and decide whether or not the data exhibits approximately periodic behaviour.

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0

b

x	0	1	2	3	4
y	4	1	0	1	4

c

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
y	0	1.9	3.5	4.5	4.7	4.3	3.4	2.4

d

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

2 The following tabled values show actual bicycle wheel data as determined by the method described earlier.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160
Height above ground (cm)	0	6	23	42	57	64	59	43	23
Distance travelled (cm)	180	200	220	240	260	280	300	320	340
Height above ground (cm)	7	1	5	27	40	55	63	60	44
Distance travelled (cm)	360	380	400						
Height above ground (cm)	24	9	3						

a Plot the graph of height against distance.

b Is the data periodic, and if so find estimates of:

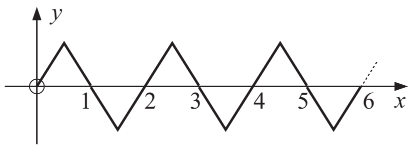
  i the equation of the principal axis **ii** the maximum value

  iii the period **iv** the amplitude

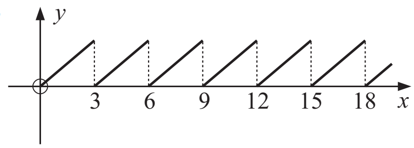
c Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

3 Which of these graphs show periodic behaviour?

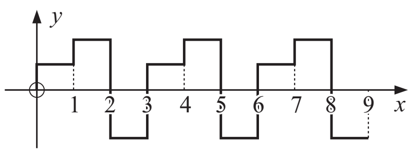
a



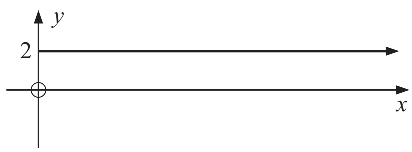
b



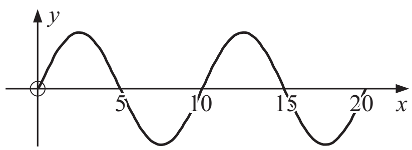
c



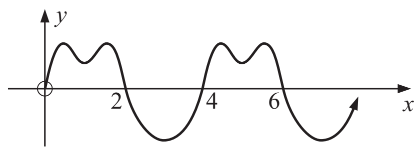
d



e



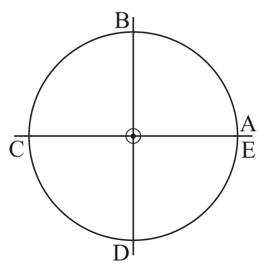
f



B RADIAN MEASURE AND PERIODIC PROPERTIES OF CIRCLES

Before looking at periodic phenomena involving circles we will first consider another way of measuring angle size.

RADIAN MEASURE



Consider the Ferris wheel again, but this time with unknown radius r.

It is of interest to consider how far a person at A actually travels in one full revolution of the wheel.

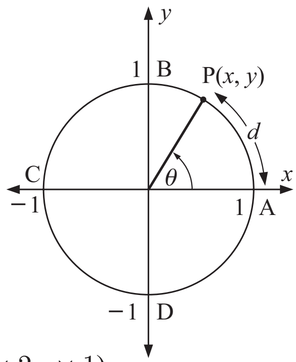
If the radius is r units, then the distance travelled is $2\pi \times r$ units (i.e., the circumference of the circle).

At B, the distance is $\frac{1}{4} \times 2\pi r = \frac{\pi}{2}$ lots of $r = \frac{\pi}{2}$ radius units

and at C the distance is $\frac{1}{2} \times 2\pi r = \pi$ radius units

Similarly the distance for D is $\frac{3\pi}{2}$ radius units

and for E is 2π radius units.



For ease of reference, consider a circle of radius 1 unit, and a point P on the circle with coordinates (x, y) as shown. The point P moves around the circle in an anti-clockwise direction.

The cartesian equation of this circle is $x^2 + y^2 = 1$.

Let the distance moved from A to P be d units, and $\measuredangle$POA $= \theta^o$.

At B, $\theta = 90^o$, $d = \frac{\pi}{2}$ ($\frac{1}{4}$ of circumference $= \frac{1}{4} \times 2\pi \times 1$)

at C, $\theta = 180^o$, $d = \pi$

at D, $\theta = 270^o$, $d = \frac{3\pi}{2}$

at A, $\theta = 0^o$ or 360^o and $d = 0$ or 2π units.

Note:
- In one revolution x varies between 1 and -1, i.e., $-1 \leqslant x \leqslant 1$, and y varies between 1 and -1 i.e., $-1 \leqslant y \leqslant 1$.
- For our convenience if P moved in a clockwise direction then d would be negative to signify the direction change and θ similarly could have negative values.

EXERCISE 13B.1

1 Find the angle θ which is equivalent to:

a $d = \frac{\pi}{4}$ **b** $d = -\frac{\pi}{4}$ **c** $d = \frac{3\pi}{2}$

d $d = -\frac{3\pi}{2}$ **e** $d = \frac{\pi}{3}$ **f** $d = \frac{\pi}{6}$

Example 1

What distance d is equivalent to $\theta = 120^o$?

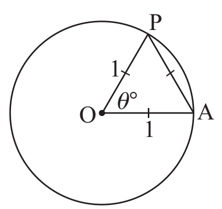
Since 120^o is $\frac{1}{3}$ of 360^o, the distance $d = \frac{1}{3}$ of $2\pi = \frac{2\pi}{3}$ units.

2 Find the distance d which is equivalent to:

a $\theta = 30^o$ **b** $\theta = 60^o$ **c** $\theta = 150^o$ **d** $\theta = 135^o$

e $\theta = 240^o$ **f** $\theta = -45^o$ **g** $\theta = -135^o$ **h** $\theta = -270^o$

3



a What is the value of θ in the given diagram?

b Find d for the value of θ in **a**.

c If θ changes so that arc AP $= d = 1$, will $\theta = 60$ increase or decrease?

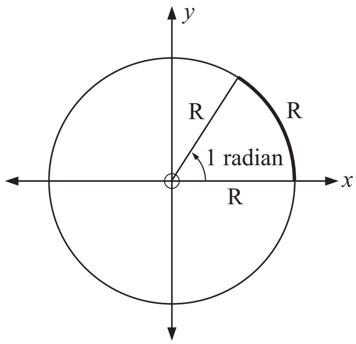
d If $d = 1$, find θ correct to 1 decimal place.

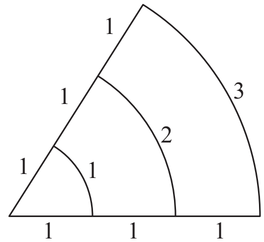
For convenience we introduce a new measure of angle size called a radian or radius angle.

One **radian** is the angle swept out by one radius unit around a circle and is approximately equivalent to 57.3^o.

It is the angle subtended at the centre of a circle by an arc of length equal to the radius.

Notice that 1 radian is exactly the same angle regardless of how the radius of the circle changes.





Notation:

One radian could be written as 1^R or 1^c or just 1.

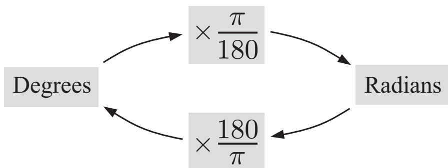
DEGREE-RADIAN CONVERSIONS

A full revolution is measured as 360^o using degrees and 2π when using radians.

So, 2π radians is equivalent to 360^o and consequently

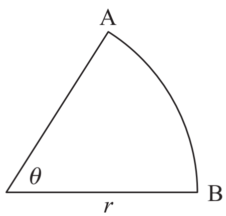
π radians is equivalent to 180^o.

The following diagram is useful for converting from one system of measure to the other:



In higher mathematics radian measure only is used as it is more convenient. Results using radian measure are usually simpler.

For example:

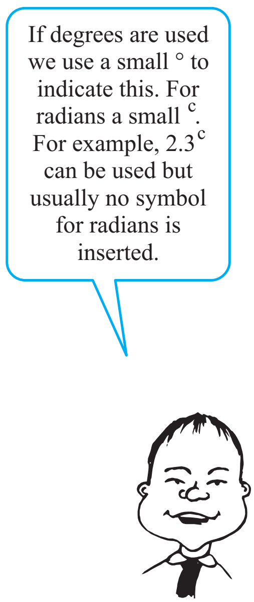


using degrees

arc AB $= \left(\dfrac{\theta}{360}\right) \times 2\pi r$,

using radians

arc AB $= r\theta$. Why?



Example 2

Convert 45^o to radians in terms of π.

$45^o = (45 \times \frac{\pi}{180})$ radians
$= \frac{\pi}{4}$ radians

or

$180^o = \pi$ radians
$\therefore \left(\frac{180}{4}\right)^o = \frac{\pi}{4}$ radians
i.e., $45^o = \frac{\pi}{4}$ radians

EXERCISE 13B.2

1 Convert to radians, in terms of π:

a 90^o **b** 60^o **c** 30^o **d** 18^o **e** 9^o
f 135^o **g** 225^o **h** 270^o **i** 360^o **j** 720^o
k 315^o **l** 540^o **m** 36^o **n** 80^o **o** 230^o

Example 3

Convert 126.5^o to radians.

126.5^o
$= (126.5 \times \frac{\pi}{180})$ radians
$\doteqdot 2.21$ radians (3 s.f.)

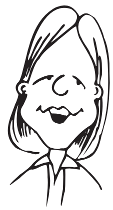
2 Convert to radians (correct to 3 s.f.):

a 36.7^o **b** 137.2^o **c** 317.9^o **d** 219.6^o **e** 396.7^o

Example 4

Convert $\frac{5\pi}{6}$ to degrees.

$\frac{5\pi}{6}$
$= \frac{5\times 180^o}{6}$
$= 150^o$



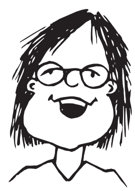
3 Convert the following radian measure to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{5\pi}{6}$ **j** $\frac{\pi}{8}$

Example 5

Convert 0.638 radians to degrees.

0.638 radians
$= (0.638 \times \frac{180}{\pi})^o$
$\doteqdot 36.55^o$



4 Convert the following radians to degrees (to 2 decimal places):

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete:

a

Degrees	0	45	90	135	180	225	270	315	360
Radians									

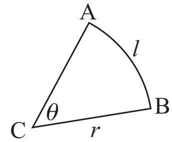
b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

6 **a** Show that for θ in radians:

i arc length AB, $l = r\theta$

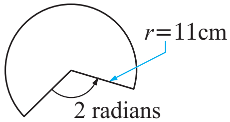
ii sector area, $A = \frac{1}{2}r^2\theta$.



b Find the arc length l for a sector with:

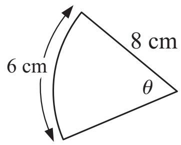
i radius 12 cm and angle 0.8^c **ii** radius 8 cm and angle 1.75^c

iii

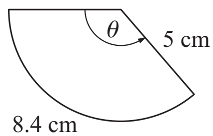


c Find θ (in radians) for each of:

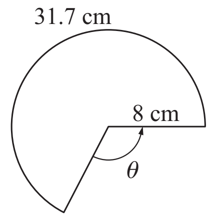
i



ii

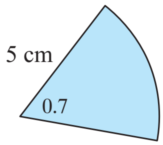


iii

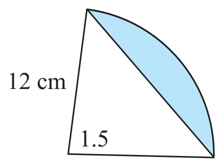


d Find the area of:

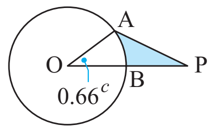
i



ii



iii



OA $=$ 12 cm, BP $=$ 18 cm

e Find the arc length and area of a sector of radius 5 cm and angle 2 radians.

f If a sector has radius 10 cm and arc length 13 cm, find its area.

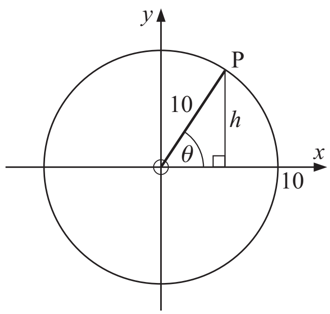
PERIODIC FUNCTIONS FROM CIRCLES

In previous studies of trigonometry we have only considered right angled triangles, or static situations, where the angle θ is fixed.

However, when an object moves in a circle the situation is dynamic, with θ (the angle between the radius OP and the horizontal axis) continually changing.

Once again consider the Ferris wheel of radius 10 m revolving at constant speed.

The height of P, the point representing the person on the wheel relative to the principal axis at any given time, can be determined by using right angle triangle trigonometry.



As $\sin\theta = \dfrac{h}{10}$, then $h = 10\sin\theta$.

From this it is obvious that as time goes by θ changes and so does h.

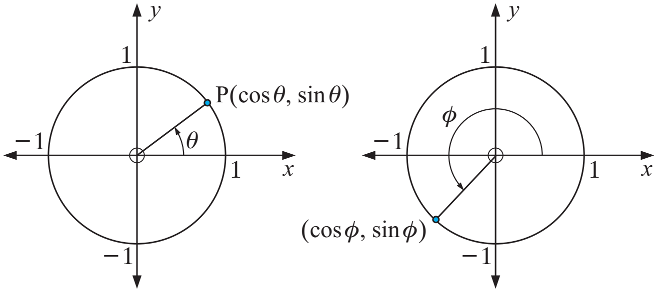
So, h is a function of θ, but more importantly h is a function of time t.

C THE UNIT CIRCLE (REVISITED)

From **Chapter 11**, remember that:

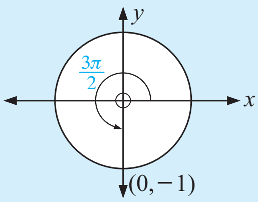
If P$(x,\ y)$ moves around the unit circle (circle centre $(0,\ 0)$, radius 1) such that OP makes an angle of θ with the positive x-axis then:

the x-coordinate of P is $\cos\theta$
and
the y-coordinate of P is $\sin\theta$.



Example 6

Use a unit circle diagram to find the values of $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$.



$\therefore\quad \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \{x\text{-coordinate}\}$

$\sin\left(\frac{3\pi}{2}\right) = -1 \qquad \{y\text{-coordinate}\}$

EXERCISE 13C.1

1 Use a unit circle diagram to find:

a $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$
b $\cos 2\pi$ and $\sin 2\pi$
c $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$
d $\cos\left(\frac{7\pi}{2}\right)$ and $\sin\left(\frac{7\pi}{2}\right)$

Example 7

Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate.

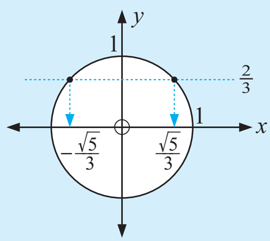
Since $\cos^2\theta + \sin^2\theta = 1$, then

$$\cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$

$$\therefore\quad \cos^2\theta + \tfrac{4}{9} = 1$$

$$\therefore\quad \cos^2\theta = \tfrac{5}{9}$$

$$\therefore\quad \cos\theta = \pm\tfrac{\sqrt{5}}{3}$$



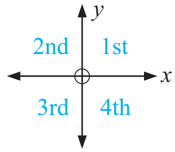
2 Find the possible values of $\cos\theta$ for:

a $\sin\theta = \frac{1}{2}$
b $\sin\theta = -\frac{1}{3}$
c $\sin\theta = 0$
d $\sin\theta = -1$

3 Find the possible values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

4 The diagram alongside shows the 4 quadrants. They are numbered anticlockwise.



a Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	positive	positive
2				
3				
4				

b In which quadrants are the following true?

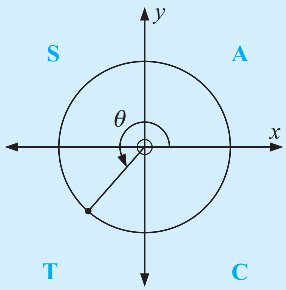
i $\cos\theta$ is positive **ii** $\cos\theta$ is negative

iii $\cos\theta$ and $\sin\theta$ are both negative

iv $\cos\theta$ is negative and $\sin\theta$ is positive

Example 8

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ without using a calculator.



Since $\pi < \theta < \frac{3\pi}{2}$, then $180^o < \theta < 270^o$.

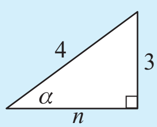
So, θ is a quad. 3 angle and $\therefore$ $\cos\theta$ is *negative*.

$$\text{Now} \quad \cos^2\theta + \sin^2\theta = 1$$
$$\therefore \quad \cos^2\theta + (-\tfrac{3}{4})^2 = 1$$
$$\therefore \quad \cos^2\theta + \tfrac{9}{16} = 1$$
$$\therefore \quad \cos^2\theta = \tfrac{7}{16}$$
$$\therefore \quad \cos\theta = \pm\tfrac{\sqrt{7}}{4}$$

and since $\cos\theta$ is negative, $\cos\theta = -\frac{\sqrt{7}}{4}$.

or using a **working angle**

Suppose α is a quad 1 where $\sin\alpha = \frac{3}{4}$



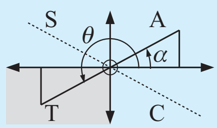
then $n^2 = 4^2 - 3^2 = 7$ {Pythagoras}

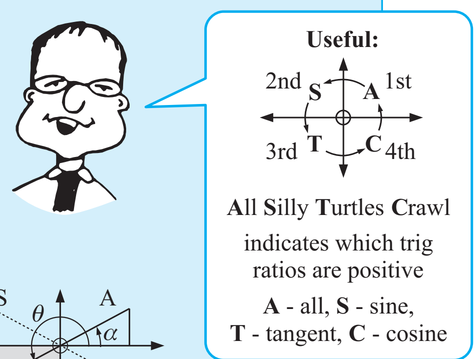
$\therefore$ $n = \sqrt{7}$

$\therefore$ $\cos\alpha = \frac{\sqrt{7}}{4}$

But θ is in quad 3 where $\cos\theta$ is negative.

so $\cos\theta = -\frac{\sqrt{7}}{4}$





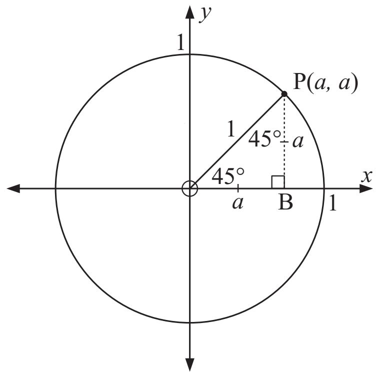
5 Without using a calculator, find:

- **a** $\sin\theta$ if $\cos\theta = \frac{2}{3}$, $0 < \theta < \frac{\pi}{2}$
- **b** $\cos\theta$ if $\sin\theta = \frac{2}{5}$, $\frac{\pi}{2} < \theta < \pi$
- **c** $\cos\theta$ if $\sin\theta = -\frac{3}{5}$, $\frac{3\pi}{2} < \theta < 2\pi$
- **d** $\sin\theta$ if $\cos\theta = -\frac{5}{13}$, $\pi < \theta < \frac{3\pi}{2}$
- **e** $\sin\theta$ if $\cos\theta = \frac{1}{2}$, $\frac{3\pi}{2} < \theta < 2\pi$
- **f** $\cos\theta$ if $\sin\theta = -\frac{1}{\sqrt{2}}$, $\pi < \theta < \frac{3\pi}{2}$.

MULTIPLES OF 30° AND 45°

MULTIPLES OF 45° OR $\frac{\pi}{4}$

Consider $\theta = 45^o$:



Triangle OBP is isosceles as angle OPB measures 45^o also.

$\therefore$ OB $=$ BP $= a$, say

and $a^2 + a^2 = 1^2$ {Pythagoras}

$\therefore$ $2a^2 = 1$

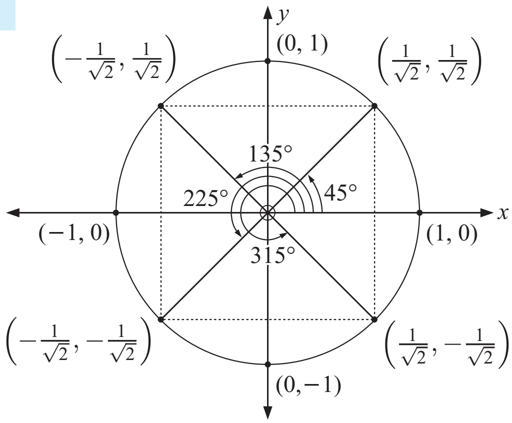
$\therefore$ $a^2 = \frac{1}{2}$

$\therefore$ $a = \frac{1}{\sqrt{2}}$ as $a > 0$

Hence, P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \doteq 0.7$.

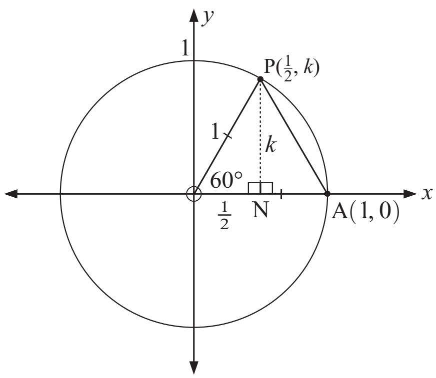
Consequently we can find the coordinates corresponding to angles of 135^o, 225^o and 315^o using suitable rotations and reflections.

So, we have:



MULTIPLES OF 30° OR $\frac{\pi}{6}$

Consider $\theta = 60^o$:



Triangle OAP is isosceles with vertical angle 60^o. The remaining angles are therefore 60^o and so triangle AOP is equilateral.
The altitude PN bisects base OA,

$\therefore$ ON $= \frac{1}{2}$.

If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

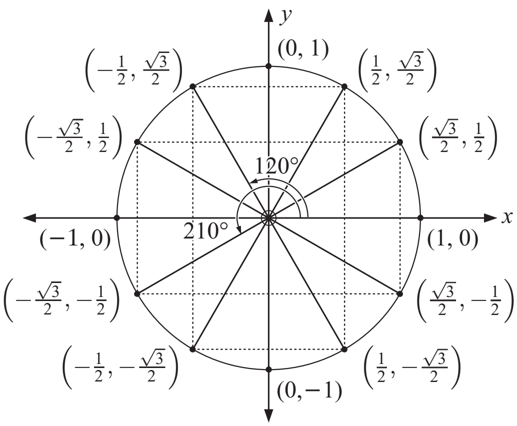
$\therefore$ $k^2 = \frac{3}{4}$

$\therefore$ $k = \frac{\sqrt{3}}{2}$ {as $k > 0$}

$\therefore$ P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $\frac{\sqrt{3}}{2} \doteq 0.9$.

Consequently, we can find the coordinates of all points on the unit circle corresponding to multiples of 30^o using rotations/reflections.

So we have:



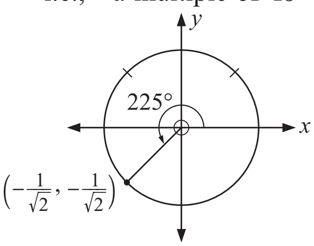
Summary:

- If θ is a **multiple of 90^o**, the coordinates of the points on the unit circle involve 0 and ± 1.
- If θ is a **multiple of 45^o**, (but not a multiple of 90^o), the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- If θ is a **multiple of 30^o**, (but not a multiple of 90^o), the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

You should not try to memorise the coordinates on the above circles for multiples of 30^o and 45^o, but rather use the summary.

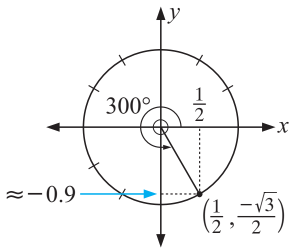
For example:

for 215^o
$215^o = 5 \times 45^o$
i.e., a multiple of 45^o



as in quad 3, signs are both negative and both have $\frac{1}{\sqrt{2}}$ size

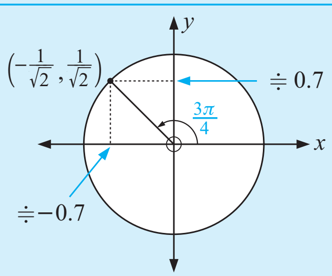
for 300^o
$300^o = 10 \times 30^o$
i.e., a multiple of 30^o



as in quad 4, signs are $(+, -)$ and from the diagram the x-value is $\frac{1}{2}$

Example 9

Use a unit circle to find the exact values of $\sin\alpha$ and $\cos\alpha$ for $\alpha = \frac{3\pi}{4}$.



$\alpha = \frac{3\pi}{4} = \frac{3}{4}$ of $180^o = 135^o$

$\therefore \quad \cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}, \quad \sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$

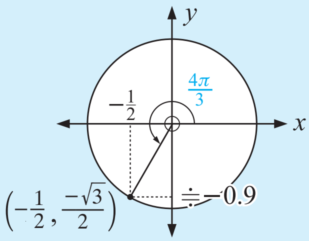
EXERCISE 13C.2

1 Use a unit circle diagram to find $\sin\theta$ and $\cos\theta$ for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{4}$ **d** π **e** $\frac{-3\pi}{4}$

Example 10

Use a unit circle diagram to find the exact values of $\sin A$ and $\cos A$ for $A = \frac{4\pi}{3}$.



$\frac{4\pi}{3} = \frac{4}{3} \times 180^o = 240^o$

$\therefore \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

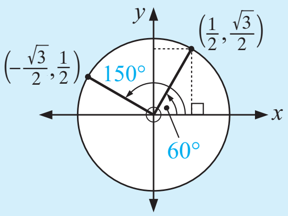
and $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

2 Use a unit circle diagram to find $\sin\beta$ and $\cos\beta$ for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

Example 11

Without using a calculator, find the value of $8\sin 60^o \cos 150^o$.



$\sin 60^o = \frac{\sqrt{3}}{2}$ and $\cos 150^o = -\frac{\sqrt{3}}{2}$

$$\begin{aligned} \therefore\quad 8\sin 60^o \cos 150^o &= 8\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right) \\ &= 2(-3) \\ &= -6 \end{aligned}$$

3 Without using a calculator, evaluate:

a $\sin^2 60^o$ **b** $\sin 30^o \cos 60^o$ **c** $4\sin 60^o \cos 30^o$

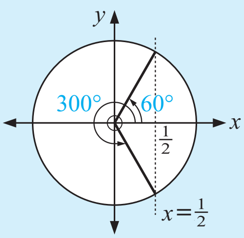
d $1 - \cos^2\left(\frac{\pi}{6}\right)$ **e** $\sin^2\left(\frac{2\pi}{3}\right) - 1$ **f** $\cos^2\left(\frac{\pi}{4}\right) - \sin\left(\frac{7\pi}{6}\right)$

g $\sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)$ **h** $1 - 2\sin^2\left(\frac{7\pi}{6}\right)$ **i** $\cos^2\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)$

Check all answers using your calculator.

Example 12

Use a unit circle diagram to find all angles between 0^o and 360^o with a cosine of $\frac{1}{2}$.



As the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved we know the required angles are multiples of 30^o.

They are 60^o and 300^o.

4 Use a unit circle diagram to find all angles between 0^o and 360^o with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

5 Use a unit circle diagram to find all angles between 0^o and 720^o with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

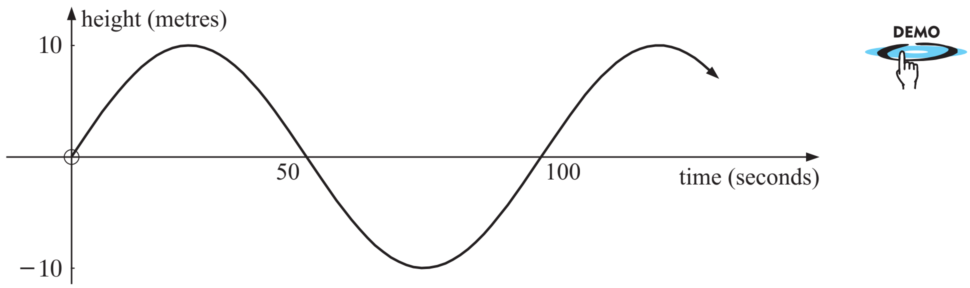
6 Find θ in radians if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

D THE SINE FUNCTION

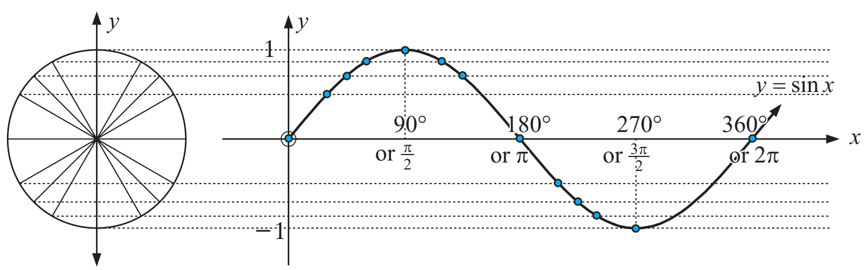
Returning to the Ferris wheel we will examine the graph obtained when plotting the height of the light above or below the principal axis against the time in seconds. We do this for a wheel of radius 10 m which takes 100 seconds for one full revolution.



We observe that the amplitude is 10 and the period is 100 seconds.

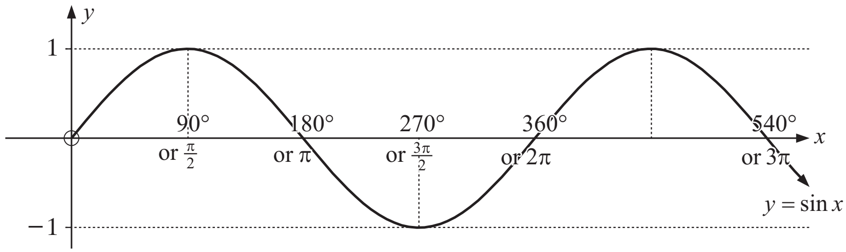
The family of sine curves can have different amplitudes and different periods. We will examine such families in this section.

THE BASIC SINE CURVE



If we project the values of $\sin\theta$ from the unit circle to the set of axes on the right we obtain the graph of $y = \sin x$.

The wave of course can be continued beyond $0 \leqslant x \leqslant 2\pi$.



We expect the *period* to be 2π, as for example, the Ferris wheel repeats its positioning after one full revolution.

The *maximum* value is 1 and the *minimum* is -1 as $-1 \leqslant y \leqslant 1$ on the unit circle.

The *amplitude* is 1.

Use your **graphics calculator** or **graphing package** to obtain the graph of $y = \sin x$ to check these features.

GRAPHING PACKAGE

When patterns of variation can be identified and quantified in terms of a formula (or equation) predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of the full moon in the future.

INVESTIGATION 1 THE FAMILY $y = A\sin x$

PORTFOLIO OPPORTUNITY

What to do:

1. Use technology to graph on the same set of axes:
   a $y = \sin x$ and $y = 2\sin x$ **b** $y = \sin x$ and $y = 0.5\sin x$
   c $y = \sin x$ and $y = -\sin x$ $(A = -1)$

   If using a graphics calculator, make sure that the mode is set in **radians** and that your viewing window is appropriate.

   GRAPHING PACKAGE

2. For each of $y = \sin x$, $y = 2\sin x$, $y = 0.5\sin x$, $y = -\sin x$ record the maximum and minimum values and state the period and amplitude. If using a calculator use the built in functions to find the maximum and minimum values.
3. How does A affect the function $y = A\sin x$?
4. State the amplitude of: **a** $y = 3\sin x$ **b** $y = \sqrt{7}\sin x$ **c** $y = -2\sin x$

INVESTIGATION 2 THE FAMILY $y = \sin Bx,\ B > 0$

PORTFOLIO OPPORTUNITY

What to do:

1. Use technology to graph on the same set of axes:
   a $y = \sin x$ and $y = \sin 2x$ **b** $y = \sin x$ and $\sin(\frac{1}{2}x)$
2. For each of $y = \sin x$, $y = \sin 2x$, $y = \sin(\frac{1}{2}x)$ record the maximum and minimum values and state the period and amplitude.

   GRAPHING PACKAGE

3 How does B affect the function $y = \sin Bx$?

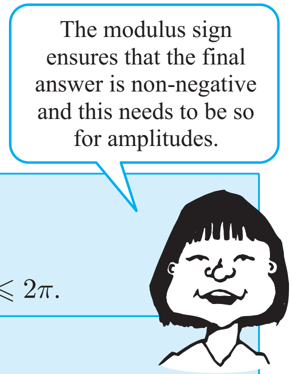
4 State the period of:

a $y = \sin 3x$ **b** $y = \sin(\frac{1}{3}x)$ **c** $y = \sin(1.2x)$ **d** $y = \sin Bx$

From the previous investigations you should have observed that:

- in $y = A\sin x$, A affects the amplitude and the amplitude is $|A|$
- in $y = \sin Bx$, $B > 0$, B affects the period and the period is $\dfrac{2\pi}{B}$.

Recall $|x|$ is the modulus of x, the size of x ignoring its sign.

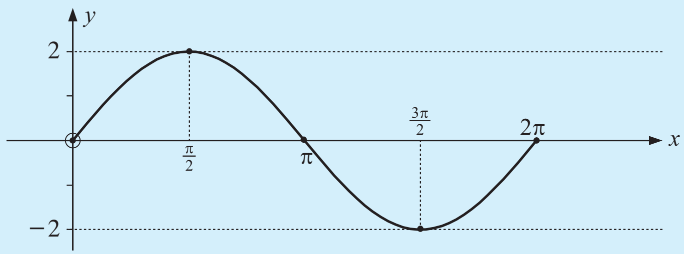


Example 13

Without using technology sketch the graphs of:

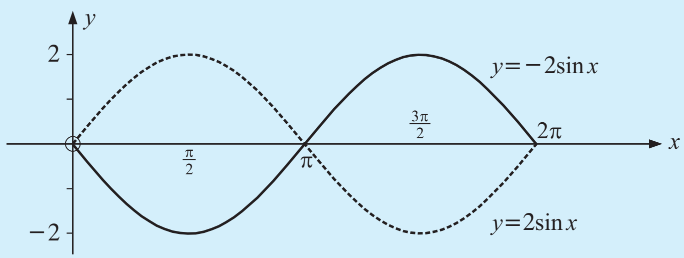
a $y = 2\sin x$ **b** $y = -2\sin x$ for $0 \leqslant x \leqslant 2\pi$.

a The amplitude is 2, and the period is 2π.



We place the 5 points as shown and fit the sine wave to them.

b The amplitude is 2, the period is 2π, and it is the reflection of $y = 2\sin x$ in the x-axis.



EXERCISE 13D.1

1 Without using technology draw the graphs of the following for $0 \leqslant x \leqslant 2\pi$:

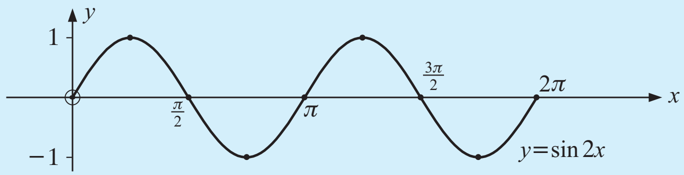
a $y = 3\sin x$ **b** $y = -3\sin x$ **c** $y = \frac{3}{2}\sin x$ **d** $y = -\frac{3}{2}\sin x$

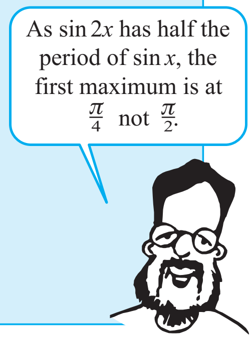
Example 14

Without using technology sketch the graph of $y = \sin 2x$, $0 \leqslant x \leqslant 2\pi$.

The period is $\frac{2\pi}{2} = \pi$.

So, for example, the maximum values are π units apart.





2 Without using technology sketch the graphs of the following for $0 \leqslant x \leqslant 3\pi$:

a $y = \sin 3x$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \sin(-2x)$

3 State the period of:

a $y = \sin 4x$ **b** $y = \sin(-4x)$ **c** $y = \sin\left(\frac{x}{3}\right)$ **d** $y = \sin(0.6x)$

4 Find B given that the function $y = \sin Bx$, $B > 0$ has period:

a 5π **b** $\frac{2\pi}{3}$ **c** 12π **d** 4 **e** 100

5 Use a **graphics calculator** or **graphing package** to help you graph for $0 \leqslant x \leqslant 720$:

a $y = 2\sin x + \sin 2x$ **b** $y = \sin x + \sin 2x + \sin 3x$ **c** $y = \dfrac{1}{\sin x}$

6 Use a **graphing package** or **graphics calculator** to graph:

a $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$

b $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9} + \dfrac{\sin 11x}{11}$

Predict the graph of $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \ldots\ldots + \dfrac{\sin 1001x}{1001}$

INVESTIGATION 3 THE FAMILIES $y = \sin(x - C)$ AND $y = \sin x + D$

PORTFOLIO OPPORTUNITY

What to do:

1 Use technology to graph on the same set of axes:

a $y = \sin x$ and $y = \sin(x - 2)$
b $y = \sin x$ and $y = \sin(x + 2)$
c $y = \sin x$ and $y = \sin\left(x - \frac{\pi}{3}\right)$

GRAPHING PACKAGE

2 For each of $y = \sin x$, $y = \sin(x - 2)$, $y = \sin(x + 2)$, $y = \sin\left(x - \frac{\pi}{3}\right)$ record the maximum and minimum values and state the period and amplitude.

3 What transformation moves $y = \sin x$ to $y = \sin(x - C)$?

4 Use technology to graph on the same set of axes:

a $y = \sin x$ and $y = \sin x + 3$ **b** $y = \sin x$ and $y = \sin x - 2$

5 For each of $y = \sin x$, $y = \sin x + 3$ and $y = \sin x - 2$ record the maximum and minimum values and state the period and amplitude.

6 What transformation moves $y = \sin x$ to $y = \sin x + D$?

7 What transformation would move $y = \sin x$ to $y = \sin(x - C) + D$?

From **Investigation 3** we observe that:

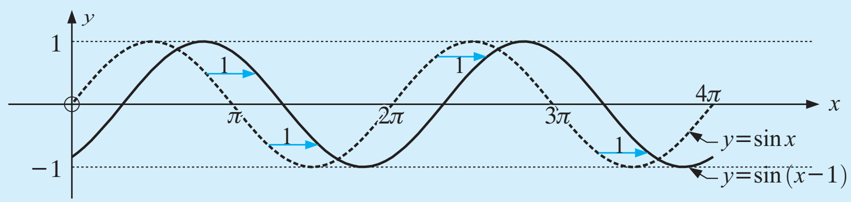
- $y = \sin(x - C)$ is a **horizontal translation** of $y = \sin x$ through C units.
- $y = \sin x + D$ is a **vertical translation** of $y = \sin x$ through D units.
- $y = \sin(x - C) + D$ is a **translation** of $y = \sin x$ through vector $\begin{bmatrix} C \\ D \end{bmatrix}$.

Example 15

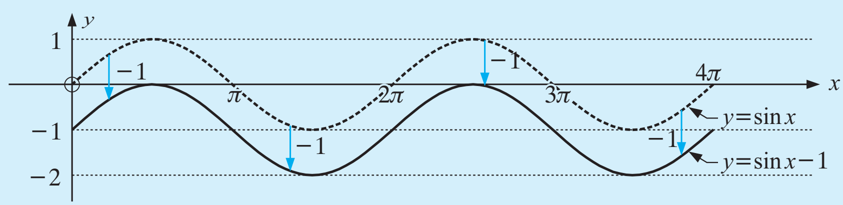
On the same set of axes graph for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x$ and $y = \sin(x - 1)$ **b** $y = \sin x$ and $y = \sin x - 1$

a



b



THE GENERAL SINE FUNCTION

$y = A \sin B(x - C) + D$ is called the **general sine function**.

affects **amplitude** | affects **period** | affects **horizontal translation** | affects **vertical translation**

Note: The **principal axis** of $y = A \sin B(x - C) + D$ is $y = D$.

Consider $y = 2\sin 3(x - \frac{\pi}{4}) + 1$. It is a translation of $y = 2\sin 3x$ under $\begin{bmatrix} \frac{\pi}{4} \\ 1 \end{bmatrix}$.

So starting with $y = \sin x$ we would:

- first double the amplitude to produce $y = 2\sin x$, then
- the period is divided by 3 to produce $y = 2\sin 3x$, then
- translate $\begin{bmatrix} \frac{\pi}{4} \\ 1 \end{bmatrix}$ to produce $y = 2\sin 3\left(x - \frac{\pi}{4}\right) + 1$.

Actually doing these multiple transformations is unimportant compared with using the facts in modelling data which is periodic.

GRAPHING PACKAGE

EXERCISE 13D.2

1 Draw sketch graphs of:

a $y = \sin x - 2$ **b** $y = \sin(x - 2)$ **c** $y = \sin(x + 2)$

d $y = \sin x + 2$ **e** $y = \sin(x + \frac{\pi}{4})$ **f** $y = \sin(x - \frac{\pi}{6}) + 1$

2 Check your answers to **1** using technology.

3 State the period of:

a $y = \sin 5t$ **b** $y = \sin\left(\frac{t}{4}\right)$ **c** $y = \sin(-2t)$

4 Find B where $B > 0$, in $y = \sin Bx$ if the period is:

a 3π **b** $\frac{\pi}{10}$ **c** 100π **d** 50

5 State the transformation(s) which maps:

a $y = \sin x$ onto $y = \sin x - 1$ **b** $y = \sin x$ onto $y = \sin(x - \frac{\pi}{4})$

c $y = \sin x$ onto $y = 2\sin x$ **d** $y = \sin x$ onto $y = \sin 4x$

e $y = \sin x$ onto $y = \frac{1}{2}\sin x$ **f** $y = \sin x$ onto $y = \sin\left(\frac{x}{4}\right)$

g $y = \sin x$ onto $y = -\sin x$ **h** $y = \sin x$ onto $y = -3 + \sin(x + 2)$

i $y = \sin x$ onto $y = 2\sin 3x$ **j** $y = \sin x$ onto $y = \sin(x - \frac{\pi}{3}) + 2$

E MODELLING USING SINE FUNCTIONS

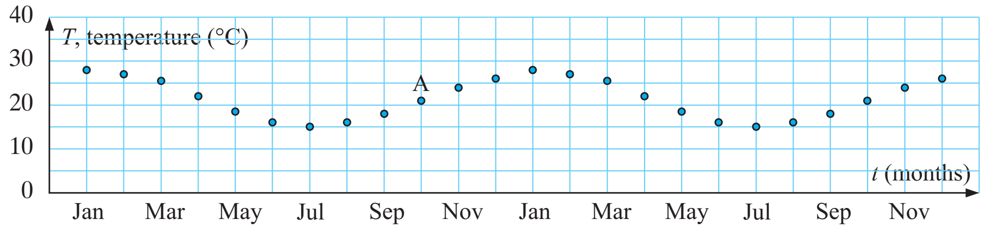
Sine functions can be useful for modelling certain biological and physical phenomena in nature which are approximately periodic.

MEAN MONTHLY TEMPERATURE

The mean monthly maximum temperature (^{o}C) for Cape Town is as shown in the given table

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

and the graph over a two year period is as follows:



We will attempt to model this data to $y = A\sin B(x - C) + D$

i.e., $T = A\sin B(t - C) + D$.

Now the period is 12 months, so $\dfrac{2\pi}{B} = 12$ and $\therefore$ $B = \dfrac{\pi}{6}$.

The amplitude $= \dfrac{\text{max.} - \text{min.}}{2} \doteqdot \dfrac{28 - 15}{2} \doteqdot 6.5$, so $A = 6.5$.

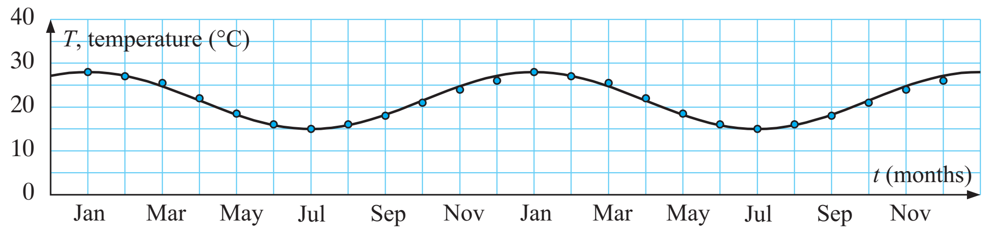
The principal axis is midway between max. and min. $\therefore$ $D = \dfrac{28 + 15}{2} = 21.5$.

So, the model is $T \doteqdot 6.5\sin\frac{\pi}{6}(t - C) + 21.5$

Viewing A on the original graph as (10, 21.5) means that C is 10.

So $T \doteqdot 6.5\sin\frac{\pi}{6}(t - 10) + 21.5$ is the model.

The model is therefore $T \doteqdot 6.5\sin\frac{\pi}{6}(t - 10) + 21.5$ and is superimposed on the original data below.



TIDAL MODELS

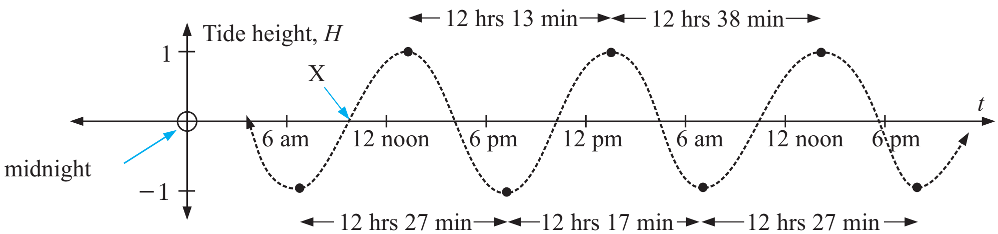
At Juneau, in Alaska, on one day it was noticed that:

high tide occurred at 1.18 pm
low tides occurred at 6.46 am and at 7.13 pm,

and on the next day high tides occurred at 1.31 am and 2.09 pm
low tides occurred at 7.30 am and 7.57 pm.

Suppose high tide corresponds to 1 and low tide to -1.

Plotting these times (where t is the time after midnight before the first low tide), we get:



We will attempt to model this periodic data to $y = A\sin B(x - C) + D$

or $H = A\sin B(t - C) + D$.

Since the principal axis appears to be $H = 0$, then $D = 0$.

The amplitude is 1, so $A = 1$.

The graph shows that the 'average' period is about 12 hours 24 min $\doteqdot$ 12.4 hours.

But the period is $\dfrac{2\pi}{B}$. $\therefore$ $\dfrac{2\pi}{B} \doteqdot 12.4$ and so $B \doteqdot \dfrac{2\pi}{12.4} \doteqdot 0.507$.

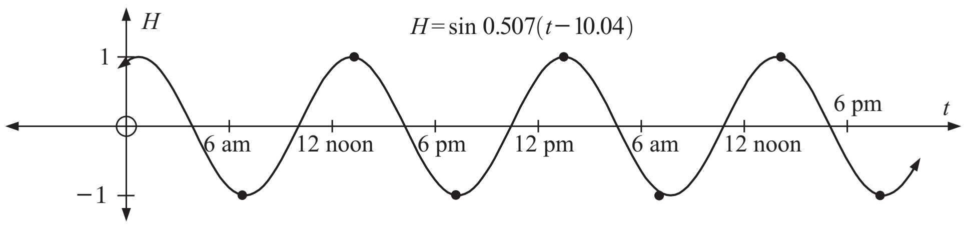
The model is now $H \doteqdot \sin 0.507(t - C)$ and so we have to find C.

Point X is midway between a maximum and a minimum value,

i.e., between $t = 6.77$ and $t = 13.3$ $\therefore$ $C = \dfrac{13.3 + 6.77}{2} \doteqdot 10.0$.

So, finally the model is $H \doteqdot \sin 0.507(t - 10.0)$.

Following is our original graph of seven plotted points and our model which attempts to fit them.



Use your **graphics calculator** to check this result.
Times must be given in hours after midnight,
i.e., (6.77, −1), (13.3, 1), (19.22, −1), etc.

TI
C

EXERCISE 13E

1 Below is a table which shows the mean monthly maximum temperature (^{o}C) for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temp	15	14	15	18	21	25	27	26	24	20	18	16

a A sine function of the form $T \doteqdot A \sin B(t - C) + D$ is used to model the data. Find good estimates of the constants A, B, C and D without using technology. Use Jan $\equiv$ 1, Feb $\equiv$ 2, etc.

b Use technology to check your answer to **a**. How well does your model fit?

2 The data in the table is of the mean monthly minimum temperature for Christchurch.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temp	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a sine model for this data in the form $T \doteqdot A \sin B(t - C) + D$. Do not use technology and assume Jan $\equiv$ 1, Feb $\equiv$ 2, etc.

b Use technology to check your answer to **a**.

3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are as follows:

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temp	0	–4	–10	–15	–16	–17	–18	–19	–17	–13	–6	–1

Find a sine model for this data using your calculator. Use Jan $\equiv 1$, Feb $\equiv 2$, etc. How appropriate is the model?

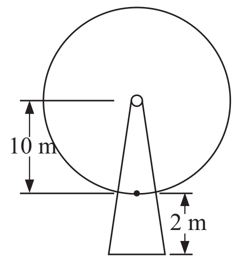
4 Obtain mean monthly data appropriate for your city or nearest city. The data could be tidal, temperature or any other data of periodic nature. Determine a sine model for the data.

5 In Canada's Bay of Fundy, some of the largest tides are observed. The difference between high and low tide is 14 metres and the average time difference is about 12.4 hours.

a Find a sine model for the height of the tide H, in terms of the time t.
b Sketch the graph of the model over one period.

6 Revisit the **Opening Problem** on page **256**.

The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume at time $t = 0$, the light is at its lowest point.



F EQUATIONS INVOLVING SINE

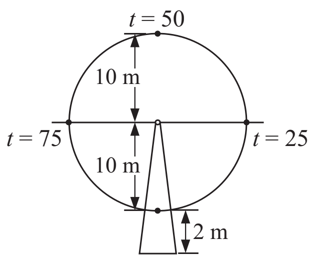
Linear equations such as $2x + 3 = 11$ have exactly one solution and quadratic equations, i.e., equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ have at most two real solutions.

Trigonometric equations generally have infinitely many solutions unless a restrictive domain such as $0 \leqslant x \leqslant 3\pi$ is given.

We will examine solving sine equations using:
- preprepared graphs
- technology
- algebraic methods.

For the Ferris Wheel **Opening Problem** the model is $H = 10 \sin \frac{\pi}{50}(t - 25) + 12$.

We can easily check this by substituting $t = 0, 25, 50, 75$



$H(0) = 10 \sin\left(-\frac{\pi}{2}\right) + 12 = -10 + 12 = 2$ ✓

$H(25) = 10 \sin 0 + 12 = 12$ ✓

$H(50) = 10 \sin\left(\frac{\pi}{2}\right) + 12 = 22$ ✓

etc.

However, we may be interested in the times when the light is 16 m above the ground, which means that we need to solve the equation

$10 \sin \frac{\pi}{50}(t - 25) + 12 = 16$ which is of course a **sine equation**.

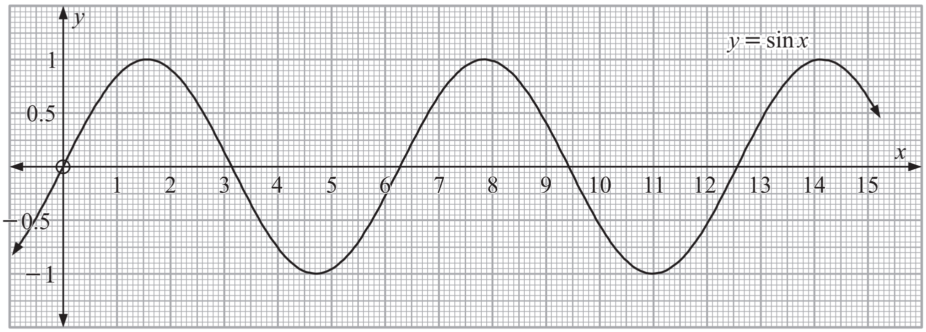
GRAPHICAL SOLUTION OF SINE EQUATIONS

Sometimes simple sine graphs on grid paper are available and estimates of solutions can be obtained.

To solve $\sin x = 0.3$, we observe where the horizontal line $y = 0.3$ meets the graph $y = \sin x$.

EXERCISE 13F.1

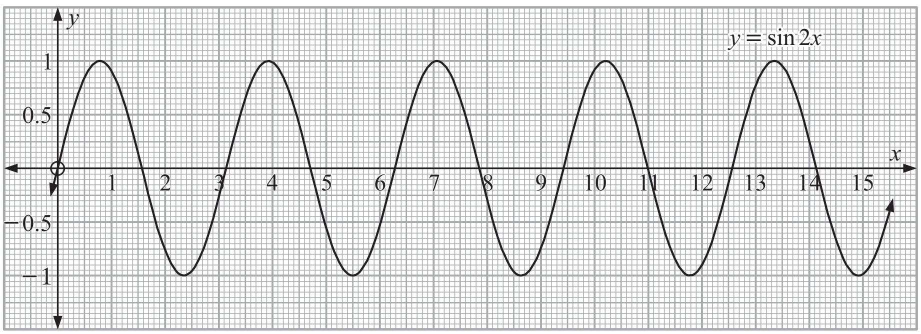
1



Use the graph of $y = \sin x$ to find correct to 1 decimal place the solutions of:

a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$ **b** $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$

2



Use the graph of $y = \sin 2x$ to find correct to 1 decimal place the solutions of:

a $\sin 2x = 0.7$ **b** $\sin 2x = -0.3$

SOLVING SINE EQUATIONS USING TECHNOLOGY

To solve $\sin x = 0.3$ we could use either a **graphing package** or **graphics calculator**.

If using a graphics calculator make sure the **mode** is set to **radians**.

TI C GRAPHING PACKAGE

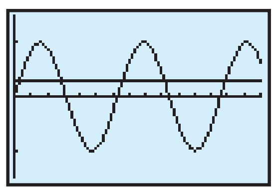
Graph $Y_1 = \sin X$ and $Y_2 = 0.3$

Use the built-in functions to find the first two points of intersection.

These are $X = 0.3047$ and $X = 2.8369$.

So, as $\sin x$ has period 2π, the general solution is

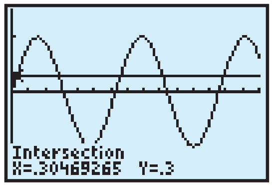
$$x = \left.\begin{matrix} 0.3047 \\ 2.8369 \end{matrix}\right\} + k2\pi, \quad k \text{ any integer.}$$



Note: We are entitled to substitute any integers for k, i.e., $k = 0, \pm1, \pm2$, etc.

For a restricted domain like $0 \leqslant x \leqslant 15$ the solutions would be

$$x = 0.3047,\quad 2.8369,\quad \underset{\substack{\uparrow\\ k=1}}{6.5879},\quad \underset{\substack{\uparrow\\ k=1}}{9.1201},\quad \underset{\substack{\uparrow\\ k=2}}{12.8711},\quad \underset{\substack{\uparrow\\ k=2}}{\cancel{15.4033}}$$



So, we have five solutions in this domain.

EXERCISE 13F.2

1 Use technology to solve for $0 \leqslant x \leqslant 8$; giving answers to 4 significant figures:

a $\sin x = 0.414$ **b** $\sin x = -0.673$ **c** $\sin x = 1.289$

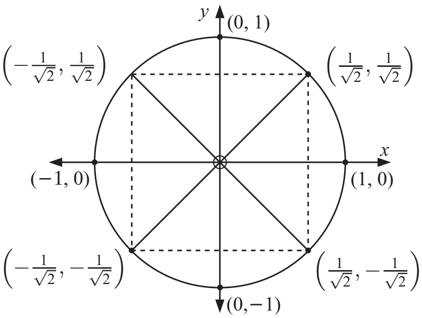
d $\sin 2x = 0.162$ **e** $\sin\left(\frac{x}{2}\right) = -0.606$ **f** $\sin(x+2) = 0.0652$

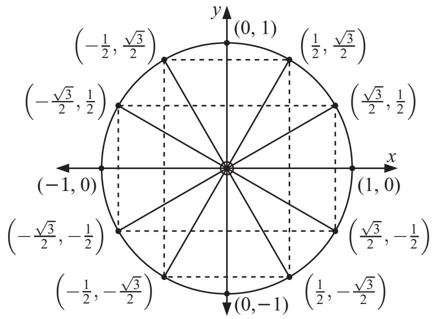
g $\sin(x - 1.3) = 0.866$ **h** $\sin\left(x - \frac{\pi}{3}\right) = 0.7063$ **i** $\sin\left(\frac{2x}{3}\right) = -0.9367$

SOLVING SINE EQUATIONS ALGEBRAICALLY

Using a calculator we get approximate decimal solutions to trigonometric equations.

Sometimes exact solutions are needed in terms of π, and these arise when the solutions are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. **Reminder:**





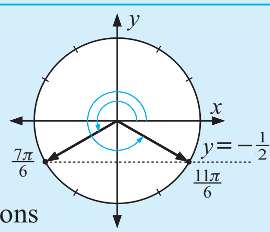
Example 16

Use the unit circle to find the exact solutions of x, $0 \leqslant x \leqslant 3\pi$ for:

a $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$ **c** $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

a $\sin x = -\frac{1}{2}$ so, from the unit circle

$$x = \left.\begin{matrix}\frac{7\pi}{6}\\[4pt] \frac{11\pi}{6}\end{matrix}\right\} + k2\pi, \quad k \text{ an integer}$$



$$\therefore\ x = \underset{\substack{\uparrow\\ k=0}}{\frac{7\pi}{6}},\quad \underset{\substack{\uparrow\\ k=0}}{\frac{11\pi}{6}},\quad \underset{\substack{\uparrow\\ k=1}}{\cancel{\frac{19\pi}{6}}}$$

i.e., 2 solutions

Substituting $k = 1, 2, 3, ...$ gives answers outside the required domain.
Likewise $k = -1, -2, ...$ gives answers outside the required domain.

b $\sin 2x = -\frac{1}{2}$ is solved exactly the same way only this time

$$2x = \left.\begin{matrix} \frac{7\pi}{6} \\ \frac{11\pi}{6} \end{matrix}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore \quad x = \left.\begin{matrix} \frac{7\pi}{12} \\ \frac{11\pi}{12} \end{matrix}\right\} + k\pi \qquad \{\text{divide each term by } 2\}$$

$$\therefore \quad x = \tfrac{7\pi}{12}, \tfrac{11\pi}{12}, \tfrac{19\pi}{12}, \tfrac{23\pi}{12}, \tfrac{31\pi}{12}, \tfrac{35\pi}{12} \qquad \{\text{obtained by letting } k = 0, 1, 2, 3\}$$

c $\sin(x - \frac{\pi}{6}) = -\frac{1}{2}$ is solved the same way, but this time

$$x - \tfrac{\pi}{6} = \left.\begin{matrix} \frac{7\pi}{6} \\ \frac{11\pi}{6} \end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \left.\begin{matrix} \frac{8\pi}{6} \\ 2\pi \end{matrix}\right\} + k2\pi \qquad \{\text{adding } \tfrac{\pi}{6} \text{ to both sides}\}$$

$$\therefore \quad x = \underset{k=0}{\underset{\uparrow}{\tfrac{4\pi}{3}}}, \quad \underset{k=0}{\underset{\uparrow}{2\pi}}, \quad \underset{k=1}{\underset{\uparrow}{\cancel{\tfrac{10\pi}{3}}}} \text{ too big}, \quad \underset{k=-1}{\underset{\uparrow}{0}}$$

So, $x = 0, \ \frac{4\pi}{3}, \ 2\pi$ which is *three solutions.*

EXERCISE 13F.3

1 List the possible answers if k is an integer and:

a $x = \frac{\pi}{6} + k2\pi, \quad 0 \leqslant x \leqslant 6\pi$

b $x = -\frac{\pi}{3} + k2\pi, \quad -2\pi \leqslant x \leqslant 2\pi$

c $x = -\frac{\pi}{2} + k\pi, \quad -4\pi \leqslant x \leqslant 4\pi$

d $x = \frac{5\pi}{6} + k\left(\frac{\pi}{2}\right), \quad 0 \leqslant x \leqslant 4\pi$

2 Solve algebraically giving answers in terms of π:

a $2\sin x = 1, \quad 0 \leqslant x \leqslant 6\pi$

b $\sqrt{2}\sin x = 1, \quad 0 \leqslant x \leqslant 4\pi$

c $2\sin x - 1 = 0, \quad -2\pi \leqslant x \leqslant 2\pi$

d $\sqrt{2}\sin x - 1 = 0, \quad -4\pi \leqslant x \leqslant 0$

e $\sin x = -1, \quad 0 \leqslant x \leqslant 6\pi$

f $\sin^2 x = 1, \quad 0 \leqslant x \leqslant 4\pi$

g $\sin 2x = \frac{1}{2}, \quad 0 \leqslant x \leqslant 3\pi$

h $\sqrt{2}\sin 3x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

i $2\sin 2x - \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$

j $2\sin\left(x + \frac{\pi}{3}\right) = 1, \quad -3\pi \leqslant x \leqslant 3\pi$

3 Solve algebraically giving answers in terms of π:

a $\sin^2 x + \sin x - 2 = 0$

b $4\sin^2 x = 3$

c $2\sin^2 x = \sin x + 1$

d $2\sin^2 x + 1 = 3\sin x$

4 Find the zeros of: (The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0$.)

a $y = \sin 2x$ between 0 and π (inclusive)

b $y = \sin(x - \frac{\pi}{4})$ between 0 and 3π (inclusive)

USING SINE MODELS

Example 17

The height $h(t)$ metres of the tide above mean sea level on January 24th at Cape Town is modelled approximately by $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ where t is the number of hours after midnight.

a Graph $y = h(t)$ for $0 \leqslant t \leqslant 24$.

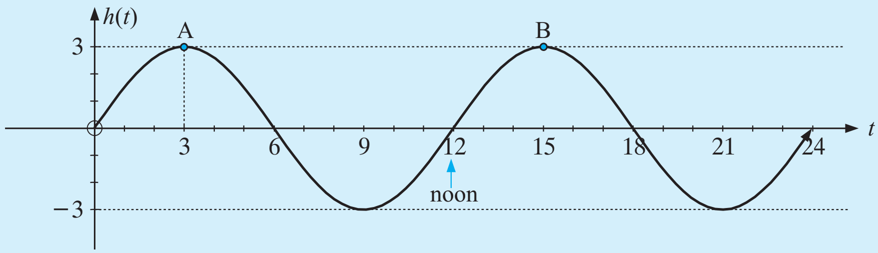
b When was high tide and what was the maximum height?

c What was the height at 2 pm?

d If a ship can cross the harbour provided the tide is at least 2 m above mean sea level, when is crossing possible on January 24?

a $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ has period $= \dfrac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$ hours

$h(0) = 0$

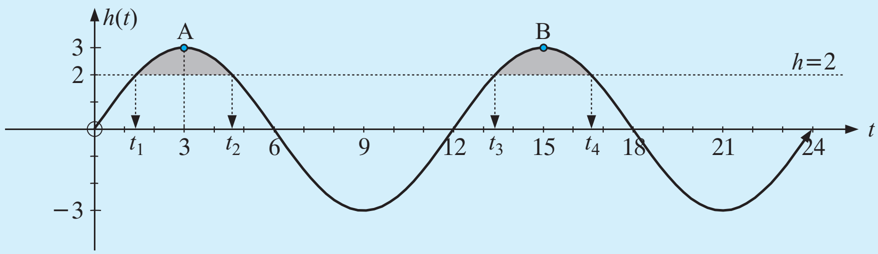


b High tide is at 3 am and 3 pm, and maximum height is 3 m above the mean as seen at points A and B.

c At 2 pm, $t = 14$ and $h(14) = 3\sin\left(\frac{14\pi}{6}\right) \doteqdot 2.60$

So the tide is 2.6 m above the mean.

d



We need to solve $h(t) = 2$ i.e., $3\sin\left(\frac{\pi t}{6}\right) = 2$.

Using a graphics calculator with $Y_1 = 3\sin\left(\frac{\pi X}{6}\right)$ and $Y_2 = 2$

we obtain $t_1 = 1.39$, $t_2 = 4.61$, $t_3 = 13.39$, $t_4 = 16.61$

or you could **trace** across the graph to find these values.

Now 1.39 hours = 1 hour 23 minutes, etc.

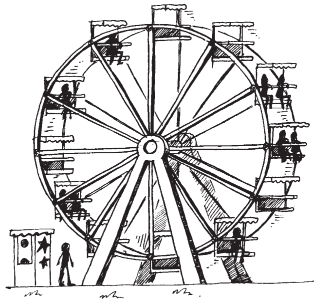
$\therefore$ can cross between 1:23 am and 4:37 am or 1:23 pm and 4:37 pm.

EXERCISE 13F.4

1 The population estimate of grass-hoppers after t weeks where $0 \leqslant t \leqslant 12$ is given by $P(t) = 7500 + 3000\sin\left(\frac{\pi t}{8}\right)$.

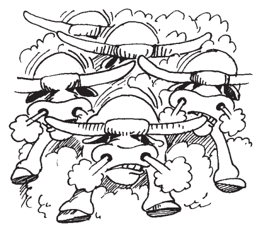
- **a** What was:
  - **i** the initial estimate
  - **ii** the estimate after 5 weeks?
- **b** What was the greatest population size over this interval and when did it occur?
- **c** When is the population **i** 9000 **ii** 6000?
- **d** During what time interval(s) does the population size exceed 10 000?

2 The model for the height of a light on a Ferris Wheel is $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$, where H is the height in metres above the ground, t is in minutes.



- **a** Where is the light at time $t = 0$?
- **b** At what time was the light at its lowest in the first revolution of the wheel?
- **c** How long does the wheel take to complete one revolution?
- **d** Sketch the graph of the $H(t)$ function over one revolution.

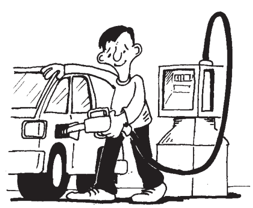
3 The population of water buffalo is given by $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$ where t is the number of years since the first estimate was made.



- **a** What was the initial estimate?
- **b** What was the population size after
  - **i** 6 months **ii** two years?
- **c** Find $P(1)$. What is the significance of this value?
- **d** Find the smallest population size and when it first occurs.
- **e** Find the first time interval when the herd exceeds 500.

4 Over a 28 day period, the cost per litre of petrol is modelled by

$C(t) = 9.2\sin\frac{\pi}{7}(t-4) + 107.8$ cents/L.



- **a** True or false?
  - **i** "The cost/litre oscillates about 107.8 cents with maximum price \$1.17."
  - **ii** "Every 14 days, the cycle repeats itself."
- **b** What is the cost at day 7?
- **c** On what days was the petrol priced at \$1.10/L?
- **d** What is the minimum cost per litre and when does it occur?

5 The temperature within a building t hours after midnight is given by

$$S(t) = 3\sin\tfrac{\pi}{12}(t-6) + 23^o\text{C} \quad \text{for} \quad 0 \leqslant t \leqslant 24.$$

The temperature outside the building t hours after midnight is given by

$$T(t) = 5\sin\tfrac{\pi}{12}(t-6) + 24^o\text{C} \quad \text{for} \quad 0 \leqslant t \leqslant 24.$$

a Find the temperature inside and outside the building at 10 am.

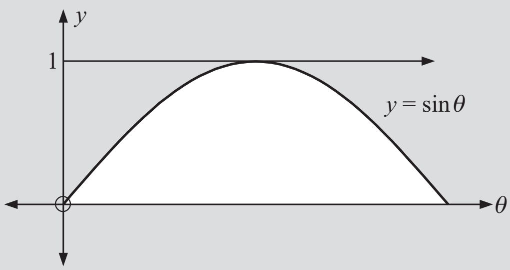
b Let $D = S - T$ which is the difference between the inside and outside temperatures. Graph D against t.

c When are the inside and outside temperatures the same?

INVESTIGATION 4 THE AREA UNDER AN ARCH OF $y = \sin\theta$

PORTFOLIO OPPORTUNITY

Have you ever thought what the area of the unshaded (white) region might be?

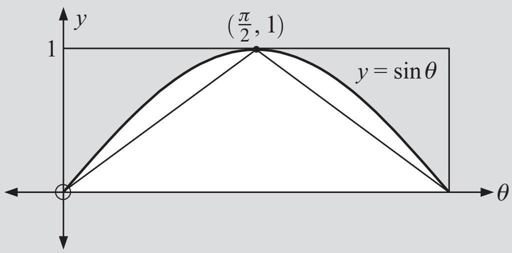


What to do:

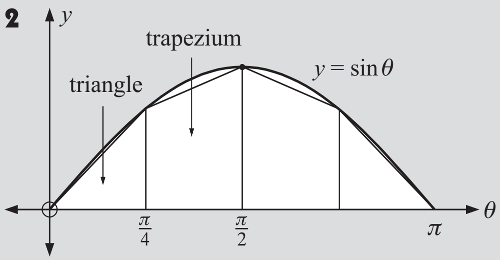
1 Use the given figure to show that

$$\tfrac{\pi}{2} < \text{area} < \pi$$

i.e., $1.571 < \text{area} < 3.142$



2



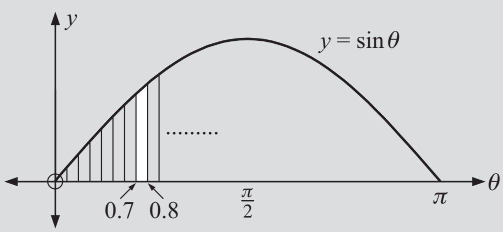
Use the given figure to show that area

$$\doteqdot \tfrac{\pi}{4}(1+\sqrt{2})$$
$$\doteqdot 1.896$$

3 Explain why the area unshaded (white)

$$\doteqdot 0.1 \times \left[\frac{\sin(0.7)+\sin(0.8)}{2}\right]$$

$$\doteqdot \tfrac{1}{20}[\sin(0.7)+\sin(0.8)]$$

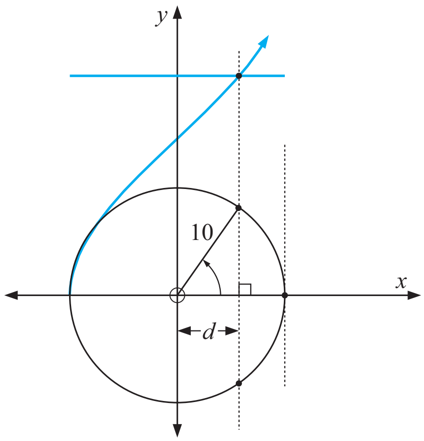


Hence find the approximate area under one arch of the sine function by adding the areas of all such strips of width 0.1 units.

Use a **spreadsheet** to do this using the appropriate sine function.

G THE COSINE FUNCTION

We return to the Ferris wheel to see the cosine function being generated.



Click on the icon to inspect a simulation of the view from above the wheel.

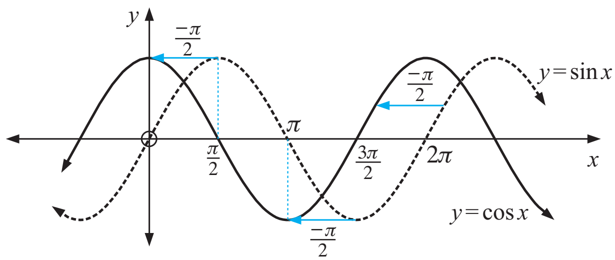
The graph being generated over time is a **cosine function**.

This is no surprise as $\cos\theta = \dfrac{d}{10}$

i.e., $d = 10\cos\theta$.

DEMO

Now view the relationship between the sine and cosine functions. Notice that the functions are identical in shape, but the cosine function is $\frac{\pi}{2}$ units left of the sine function under a horizontal translation.



DEMO

This suggests that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

Use your graphing package or graphics calculator to check this by graphing $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$.

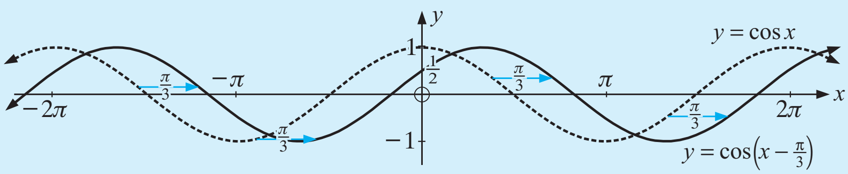
GRAPHING PACKAGE

Example 18

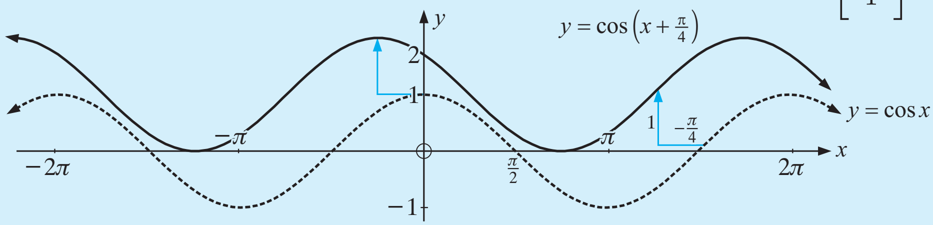
On the same set of axes graph:

a $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{3}\right)$ **b** $y = \cos x$ and $y = \cos\left(x + \frac{\pi}{4}\right) + 1$

a $y = \cos\left(x - \frac{\pi}{3}\right)$ comes from $y = \cos x$ under a horizontal translation through $\frac{\pi}{3}$.



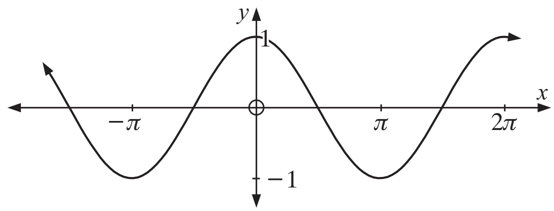
b $y = \cos\left(x + \frac{\pi}{4}\right) + 1$ comes from $y = \cos x$ under a translation of $\begin{bmatrix} -\frac{\pi}{4} \\ 1 \end{bmatrix}$.



Note: You could use technology to help draw your sketch graphs as in **Example 18**.

EXERCISE 13G

1 Given the graph of $y = \cos x$, sketch the graphs of:



a $y = \cos x + 2$ **b** $y = \cos x - 1$ **c** $y = \cos(x - \frac{\pi}{4})$

d $y = \cos(x + \frac{\pi}{6})$ **e** $y = \frac{2}{3}\cos x$ **f** $y = \frac{3}{2}\cos x$

g $y = -\cos x$ **h** $y = \cos(x - \frac{\pi}{6}) + 1$ **i** $y = \cos(x + \frac{\pi}{4}) - 1$

j $y = \cos 2x$ **k** $y = \cos\left(\frac{x}{2}\right)$ **l** $y = 3\cos 2x$

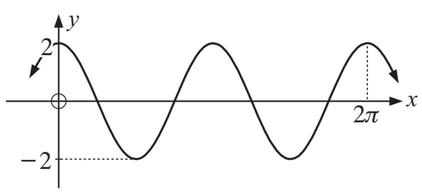
2 Without graphing them, state the periods of:

a $y = \cos 3x$ **b** $y = \cos\left(\frac{x}{3}\right)$ **c** $y = \cos\left(\frac{\pi}{50}x\right)$

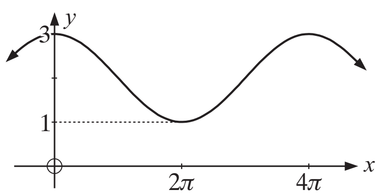
3 The general cosine function is $y = A\cos B(x - C) + D$.
State the geometrical significance of A, B, C and D.

4 For the following graphs, find the cosine function representing them:

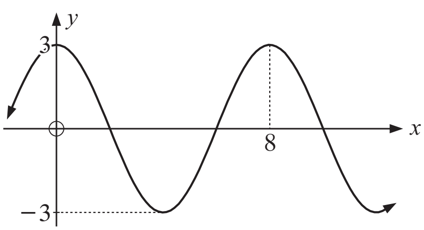
a



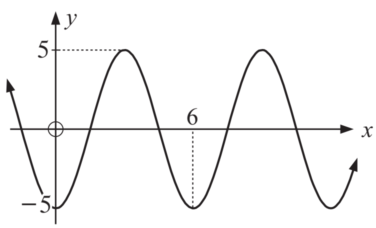
b



c

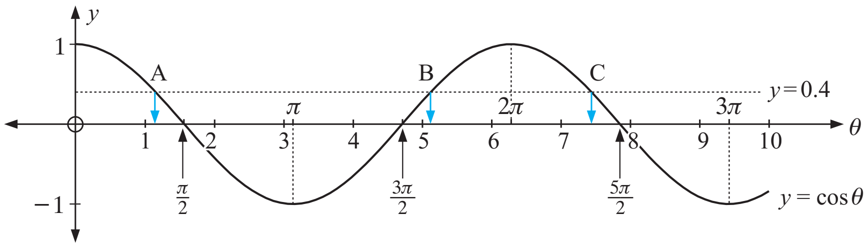


d



H SOLVING COSINE EQUATIONS

We could use a graph to find approximate solutions for trigonometric equations such as $\cos\theta = 0.4$ for $0 \leqslant \theta \leqslant 10$ radians. We draw the graph of $y = \cos\theta$ for $0 \leqslant \theta \leqslant 10$ and find all values of θ where the y-coordinate of any point of the graph is 0.4.



$y = 0.4$ meets $y = \cos\theta$ at A, B and C and hence $\theta \doteqdot 1.2,\ 5.1$ or 7.4.

So, the solutions of $\cos\theta = 0.4$ for $0 \leqslant \theta \leqslant 10$ radians are $1.2,\ 5.1$ and 7.4.

DISCUSSION

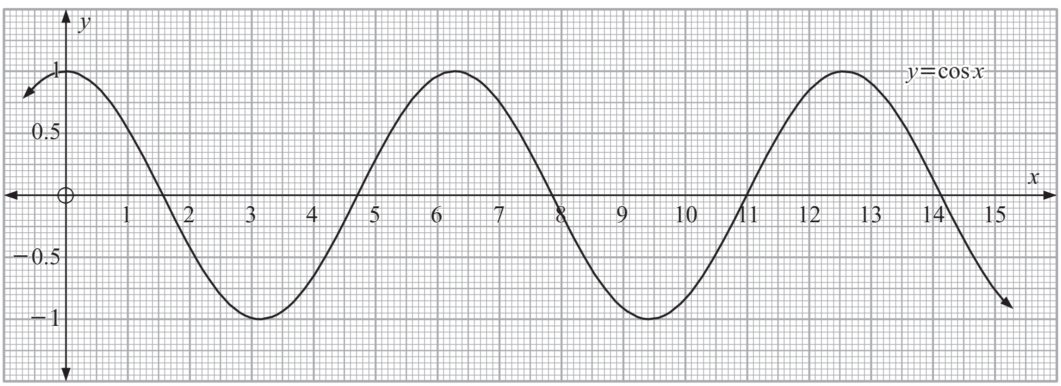
- How many solutions does $\cos\theta = 1.3$ have for $0 \leqslant \theta \leqslant 10$?
- How many solutions does $\cos\theta = 0.4$ have with no restrictions for θ?

Once again we could solve cosine equations:
- from given graphs
- using technology
- algebraically.

The techniques are the same as those used for sine equations.

EXERCISE 13H

1

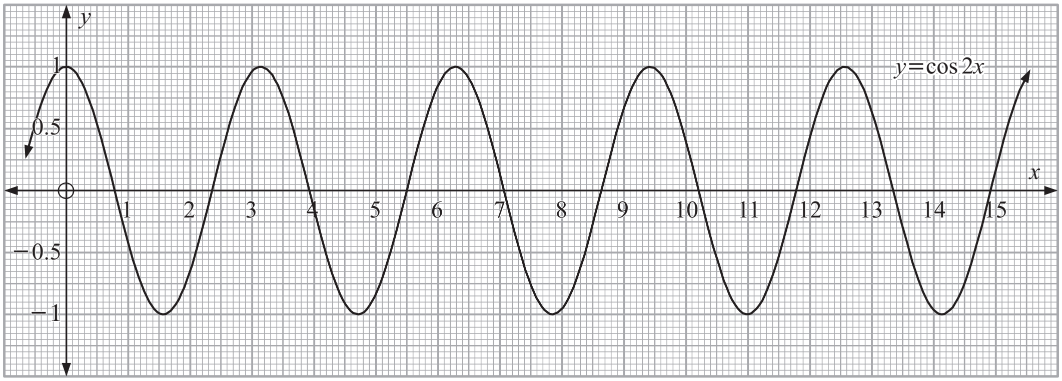


Use the graph of $y = \cos x$ to find to 1 decimal place, the approximate solutions of:

a $\cos x = 0.4$ for $0 \leqslant \theta \leqslant 10$ **b** $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$

c $\cos x = 0.9$ for $0 \leqslant x \leqslant 8$

2



Use the graph of $y = \cos 2x$ to find to 1 decimal place, the approximate solutions of:

a $\cos 2x = 0.55$ **b** $2\cos 2x + 0.6 = 0$

3 Use technology to solve the following to 3 decimal places:

a $\cos x = 0.561$ for $0 \leqslant x \leqslant 10$

b $\cos 2x = 0.782$ for $0 \leqslant x \leqslant 6$

c $\cos(x - 1.3) = -0.609$ for $0 \leqslant x \leqslant 12$

d $4\cos 3x + 1 = 0$ for $0 \leqslant x \leqslant 5$

e $5\cos 2x + 2 = 0$ for all x.

Example 19

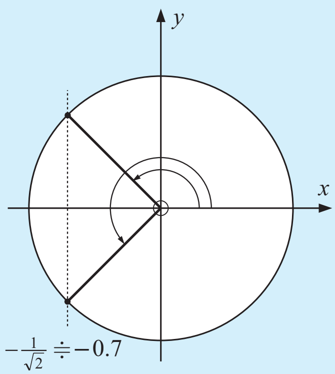
Find exact solutions of $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$ for $0 \leqslant x \leqslant 6\pi$.

As $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$

then $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) = -1$

$\therefore\ \cos\left(x - \frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

We recognise the $\frac{1}{\sqrt{2}}$ as a special fraction (for multiples of $\frac{\pi}{4}$)



$$\therefore\ x - \frac{3\pi}{4} = \left.\begin{matrix} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{matrix}\right\} + k2\pi$$

$$\therefore\ x = \left.\begin{matrix} \frac{3\pi}{2} \\ 2\pi \end{matrix}\right\} + k2\pi$$

If $k = -1$, $x = -\frac{\pi}{2}$ or 0. If $k = 0$, $x = \frac{3\pi}{2}$ or 2π.

If $k = 1$, $x = \frac{7\pi}{2}$ or 4π. If $k = 2$, $x = \frac{11\pi}{2}$ or 6π.

If $k = 3$, the answers are greater than 6π.

So, the solutions are: $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi, \frac{11\pi}{2}$ or 6π.

4 Find the exact solutions of:

a $\cos x = \frac{1}{\sqrt{2}}$ for $0 \leqslant x \leqslant 4\pi$

b $\cos x = -\frac{1}{2}$ for $0 \leqslant x \leqslant 5\pi$

c $2\cos x + \sqrt{3} = 0$ for $0 \leqslant x \leqslant 3\pi$

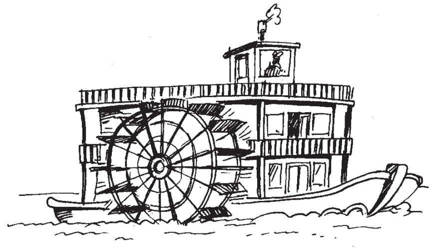
d $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}$ for $-2\pi \leqslant x \leqslant 2\pi$

e $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1 = 0$ for $0 \leqslant x \leqslant 3\pi$

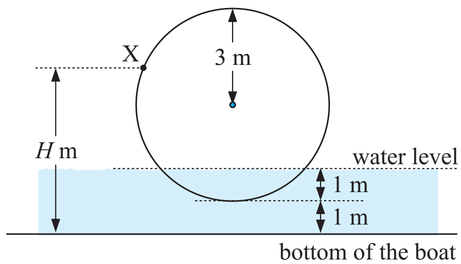
f $\cos 2x + 1 = 0$ for $0 \leqslant x \leqslant 2\pi$

g $2\cos 3x + 1 = 0$ for $0 \leqslant x \leqslant \pi$

h $2\cos^2 x = \cos x + 1$ for $0 \leqslant x \leqslant 3\pi$



5 A paint spot X lies on the outer rim of the wheel of a paddle-steamer. The wheel has radius 3 m and as it rotates at a constant rate, X is seen entering the water every 4 seconds. H is the distance of X above the bottom of the boat. At time $t = 0$, X is at its highest point.



a Find the cosine model, $H(t) = A\cos B(t - C) + D$.

b At what time does X first enter the water?

I TRIGONOMETRIC RELATIONSHIPS

There are a vast number of trigonometric relationships. However, we will use only a few of them. First of all we will look at how to simplify trigonometric expressions.

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

Since for a given angle θ, $\sin\theta$, and $\cos\theta$ are real numbers, the algebra of trigonometry is identical to the algebra of real numbers.

Consequently, expressions like $2\sin\theta + 3\sin\theta$ compare with $2x + 3x$ when we wish to do simplification.

So, $2\sin\theta + 3\sin\theta = 5\sin\theta$.

Example 20

Simplify: **a** $3\cos\theta + 4\cos\theta$ **b** $\sin\alpha - 3\sin\alpha$

a $3\cos\theta + 4\cos\theta$
$= 7\cos\theta$
$\{3x + 4x = 7x\}$

b $\sin\alpha - 3\sin\alpha$
$= -2\sin\alpha$
$\{x - 3x = -2x\}$

EXERCISE 13I

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$ **e** $\cos\theta - 3\cos\theta$ **f** $2\cos\theta - 5\cos\theta$

To simplify more complicated trigonometric expressions involving $\sin\theta$ and $\cos\theta$ we often use

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

(See pages **226** and **227**)

It is worth graphing $y = \sin^2\theta$, $y = \cos^2\theta$ and $y = \sin^2\theta + \cos^2\theta$ using technology.

GRAPHING PACKAGE

Notice that:

$\sin^2\theta + \cos^2\theta$	could be replaced by	1
1	could be replaced by	$\sin^2\theta + \cos^2\theta$
$\sin^2\theta$	could be replaced by	$1 - \cos^2\theta$
$1 - \cos^2\theta$	could be replaced by	$\sin^2\theta$
$\cos^2\theta$	could be replaced by	$1 - \sin^2\theta$
$1 - \sin^2\theta$	could be replaced by	$\cos^2\theta$.

Example 21

Simplify: **a** $2 - 2\cos^2\theta$ **b** $\sin^2\theta\cos\theta + \cos^3\theta$

a
$$\begin{aligned} & 2 - 2\cos^2\theta \\ &= 2(1 - \cos^2\theta) \\ &= 2\sin^2\theta \\ & \{\text{as } \cos^2\theta + \sin^2\theta = 1\} \end{aligned}$$

b
$$\begin{aligned} & \sin^2\theta\cos\theta + \cos^3\theta \\ &= \cos\theta\,(\sin^2\theta + \cos^2\theta) \\ &= \cos\theta \times 1 \\ &= \cos\theta \end{aligned}$$

2 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$ **b** $-2\sin^2\theta - 2\cos^2\theta$ **c** $-\cos^2\theta - \sin^2\theta$

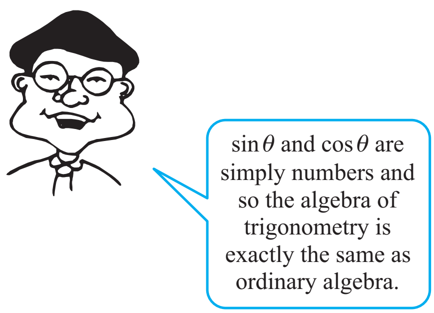
d $3 - 3\sin^2\theta$ **e** $4 - 4\cos^2\theta$ **f** $\sin^3\theta + \sin\theta\cos^2\theta$

g $\cos^2\theta - 1$ **h** $\sin^2\theta - 1$ **i** $2\cos^2\theta - 2$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$ **k** $\dfrac{1 - \cos^2\theta}{\sin\theta}$ **l** $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

As with ordinary algebraic expressions we can **expand** trigonometric products.

Sometimes simplication of these expansions is possible.



Example 22

Expand and simplify if possible: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned}&(\cos\theta - \sin\theta)^2\\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta && \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\}\\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta\\ &= 1 - 2\cos\theta\sin\theta\end{aligned}$$

3 Expand and simplify if possible:

a $(1+\sin\theta)^2$ **b** $(\sin\alpha - 2)^2$ **c** $(\cos\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$ **e** $(\sin\beta - \cos\beta)^2$ **f** $-(2-\cos\alpha)^2$

Factorisation of trigonometric expressions is also possible.

Example 23

Factorise: **a** $\cos^2\alpha - \sin^2\alpha$ **b** $\sin^2\theta - 3\sin\theta + 2$

a $$\begin{aligned}&\cos^2\alpha - \sin^2\alpha\\ &= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha) && \{a^2 - b^2 = (a+b)(a-b)\}\end{aligned}$$

b $$\begin{aligned}&\sin^2\theta - 3\sin\theta + 2\\ &= (\sin\theta - 2)(\sin\theta - 1) && \{x^2 - 3x + 2 = (x-2)(x-1)\}\end{aligned}$$

4 Factorise:

a $1-\sin^2\theta$ **b** $\sin^2\alpha - \cos^2\alpha$ **c** $\cos^2\alpha - 1$

d $2\sin^2\beta - \sin\beta$ **e** $2\cos\phi + 3\cos^2\phi$ **f** $3\sin^2\theta - 6\sin\theta$

g $\sin^2\theta + 5\sin\theta + 6$ **h** $2\cos^2\theta + 7\cos\theta + 3$ **i** $6\cos^2\alpha - \cos\alpha - 1$

Example 24

Simplify: **a** $\dfrac{2-2\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

a $$\begin{aligned}&\frac{2-2\cos^2\theta}{1+\cos\theta}\\ &= \frac{2(1-\cos^2\theta)}{1+\cos\theta}\\ &= \frac{2\cancel{(1+\cos\theta)}(1-\cos\theta)}{\cancel{(1+\cos\theta)}}\\ &= 2(1-\cos\theta)\end{aligned}$$

b $$\begin{aligned}&\frac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\cancel{(\cos\theta - \sin\theta)}}{(\cos\theta + \sin\theta)\cancel{(\cos\theta - \sin\theta)}}\\ &= \frac{1}{\cos\theta + \sin\theta}\end{aligned}$$

5 Simplify:

a $\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$ **b** $\dfrac{\cos^2\beta-1}{\cos\beta+1}$ **c** $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}$

d $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$ **e** $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{3-3\sin^2\theta}{6\cos\theta}$

6 Show that:

a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2$ simplifies to 2

b $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2$ simplifies to 13

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)$ simplifies to $\dfrac{\sin^2\theta}{\cos\theta}$

d $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)$ simplifies to $\cos^2\theta$

e $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}$ simplifies to $\dfrac{2}{\sin\theta}$

f $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}$ simplifies to $\dfrac{2\cos\theta}{\sin\theta}$

g $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}$ simplifies to $\dfrac{2}{\cos^2\theta}$

GRAPHING PACKAGE

Use a graphing package to check these, by graphing each function on the same set of axes.

J DOUBLE ANGLE FORMULAE

INVESTIGATION 5 DOUBLE ANGLE FORMULAE

What to do:

1 Copy and complete using angles of your choice as well:

A	$\sin 2A$	$2\sin A$	$2\sin A\cos A$	$\cos 2A$	$2\cos A$	$\cos^2 A-\sin^2 A$
0.631						
57.81^o						
-3.697						

2 Write down any discoveries from your table of values in **1**.

The **double angle** formulae are:

$$\mathbf{\sin 2A = 2\sin A\cos A}$$

GRAPHING PACKAGE

$$\mathbf{\cos 2A = \begin{cases}\cos^2 A-\sin^2 A\\ 2\cos^2 A-1\\ 1-2\sin^2 A\end{cases}}$$

GRAPHING PACKAGE

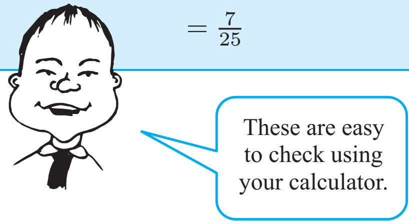
Example 25

Given that $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$ find:

a $\sin 2\alpha$ **b** $\cos 2\alpha$

a $\sin 2\alpha$
$= 2\sin\alpha\cos\alpha$
$= 2(\frac{3}{5})(-\frac{4}{5})$
$= -\frac{24}{25}$

b $\cos 2\alpha$
$= \cos^2\alpha - \sin^2\alpha$
$= (-\frac{4}{5})^2 - (\frac{3}{5})^2$
$= \frac{16}{25} - \frac{9}{25}$
$= \frac{7}{25}$



EXERCISE 13J

1 If $\sin A = \frac{4}{5}$ and $\cos A = \frac{3}{5}$ find the values of:
a $\sin 2A$ **b** $\cos 2A$

2 If $\cos A = \frac{1}{3}$, find the value of $\cos 2A$.

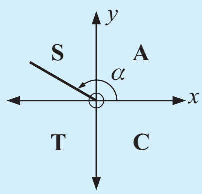
3 If $\sin\phi = -\frac{2}{3}$, find the value of $\cos 2\phi$.

Example 26

If $\sin\alpha = \frac{5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$, find the value of $\sin 2\alpha$ without using a calculator.

First we need to find $\cos\alpha$ where α is in quad 2 $\therefore$ $\cos\alpha$ is negative.

Now $\cos^2\alpha + \sin^2\alpha = 1$
$\therefore$ $\cos^2\alpha + \frac{25}{169} = 1$
$\therefore$ $\cos^2\alpha = \frac{144}{169}$
$\therefore$ $\cos\alpha = \pm\frac{12}{13}$
$\therefore$ $\cos\alpha = -\frac{12}{13}$



But $\sin 2\alpha = 2\sin\alpha\cos\alpha$
$= 2(\frac{5}{13})(-\frac{12}{13})$
$= -\frac{120}{169}$

4 **a** If $\sin\alpha = -\frac{2}{3}$ where $\pi < \alpha < \frac{3\pi}{2}$ find the value of $\cos\alpha$ and hence the value of $\sin 2\alpha$.

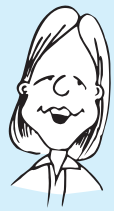
b If $\cos\beta = \frac{2}{5}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find the value of $\sin\beta$ and hence the value of $\sin 2\beta$.

Example 27

If α is acute and $\cos 2\alpha = \frac{3}{4}$ find the values of **a** $\cos\alpha$ **b** $\sin\alpha$.

a
$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\therefore \quad \tfrac{3}{4} = 2\cos^2\alpha - 1$$
$$\therefore \quad \cos^2\alpha = \tfrac{7}{8}$$
$$\therefore \quad \cos\alpha = \pm\frac{\sqrt{7}}{2\sqrt{2}}$$
$$\therefore \quad \cos\alpha = \frac{\sqrt{7}}{2\sqrt{2}} \qquad \{\text{as } \alpha \text{ is acute, } \cos\alpha \text{ is positive}\}$$

b
$$\cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\therefore \quad \tfrac{3}{4} = 1 - 2\sin^2\alpha$$
$$\therefore \quad 2\sin^2\alpha = \tfrac{1}{4}$$
$$\therefore \quad \sin^2\alpha = \tfrac{1}{8}$$
$$\therefore \quad \sin\alpha = \pm\frac{1}{2\sqrt{2}}$$
$$\therefore \quad \sin\alpha = \frac{1}{2\sqrt{2}} \qquad \{\text{as } \alpha \text{ is acute, } \sin\alpha \text{ is positive}\}$$



$\sin^2\alpha + \cos^2\alpha = 1$ could also be used in **b**.

5 If α is acute and $\cos 2\alpha = -\frac{7}{9}$, find without a calculator, the values of:

a $\cos\alpha$ **b** $\sin\alpha$

Example 28

Use an appropriate 'double angle formula' to simplify:

a $3\sin\theta\cos\theta$ **b** $4\cos^2 2B - 2$

a
$$3\sin\theta\cos\theta$$
$$= \tfrac{3}{2}(2\sin\theta\cos\theta)$$
$$= \tfrac{3}{2}\sin 2\theta$$

b
$$4\cos^2 2B - 2$$
$$= 2(2\cos^2 2B - 1)$$
$$= 2\cos 2(2B)$$
$$= 2\cos 4B$$

6 Use an appropriate 'double angle' formula to simplify:

a $2\sin\alpha\cos\alpha$	**b** $4\cos\alpha\sin\alpha$	**c** $\sin\alpha\cos\alpha$
d $2\cos^2\beta - 1$	**e** $1 - 2\cos^2\phi$	**f** $1 - 2\sin^2 N$
g $2\sin^2 M - 1$	**h** $\cos^2\alpha - \sin^2\alpha$	**i** $\sin^2\alpha - \cos^2\alpha$
j $2\sin 2A\cos 2A$	**k** $2\cos 3\alpha\sin 3\alpha$	**l** $2\cos^2 4\theta - 1$
m $1 - 2\cos^2 3\beta$	**n** $1 - 2\sin^2 5\alpha$	**o** $2\sin^2 3D - 1$
p $\cos^2 2A - \sin^2 2A$	**q** $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$	**r** $2\sin^2 3P - 2\cos^2 3P$

7 Show that:

a $(\sin\theta+\cos\theta)^2$ simplifies to $1+\sin 2\theta$

b $\cos^4\theta-\sin^4\theta$ simplifies to $\cos 2\theta$

GRAPHING PACKAGE

8 Solve for x where $0 \leqslant x \leqslant 2\pi$:

a $\sin 2x+\sin x=0$

b $\sin 2x-2\cos x=0$

c $\cos 2x-\cos x=0$

d $\cos 2x+3\cos x=1$

e $\cos 2x+5\sin x=0$

f $\sin 2x+3\sin x=0$

K THE TANGENT FUNCTION

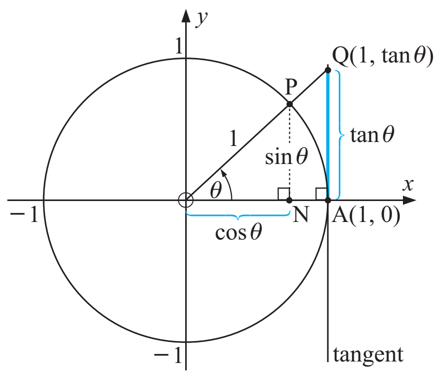
Consider the unit circle diagram given.

P$(\cos\theta, \sin\theta)$ is a point which is free to move around the circle.

In the first quadrant we extend OP to meet the tangent at A(1, 0) so that it meets this tangent at Q.

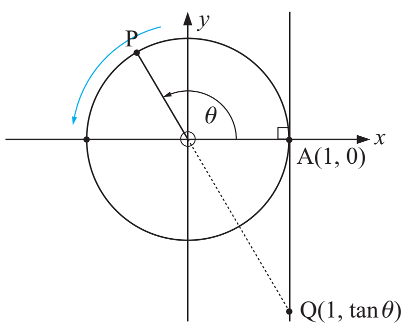
As P moves, so does Q.

Q's position relative to A is defined as the **tangent function**.



Now Δ's ONP and OAQ are equiangular and therefore similar.

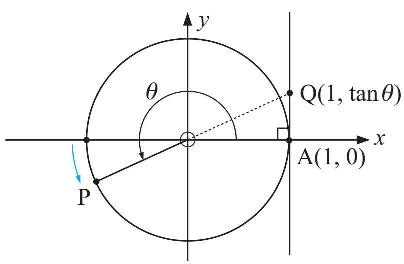
Consequently, $\dfrac{AQ}{OA}=\dfrac{NP}{ON}$ i.e., $\dfrac{AQ}{1}=\dfrac{\sin\theta}{\cos\theta}$ which suggests that $\boxed{\tan\theta=\dfrac{\sin\theta}{\cos\theta}}$.



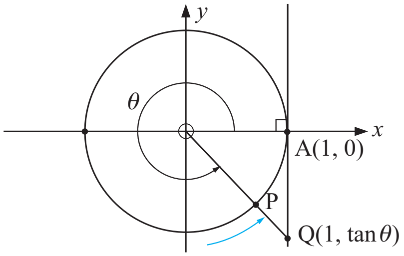
The question arises: "if P does not lie in the first quadrant, how is $\tan\theta$ defined"?

For θ obtuse, since $\sin\theta$ is positive and $\cos\theta$ is negative,

$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ is negative and PO is extended to meet the tangent at A at Q$(1, \tan\theta)$.



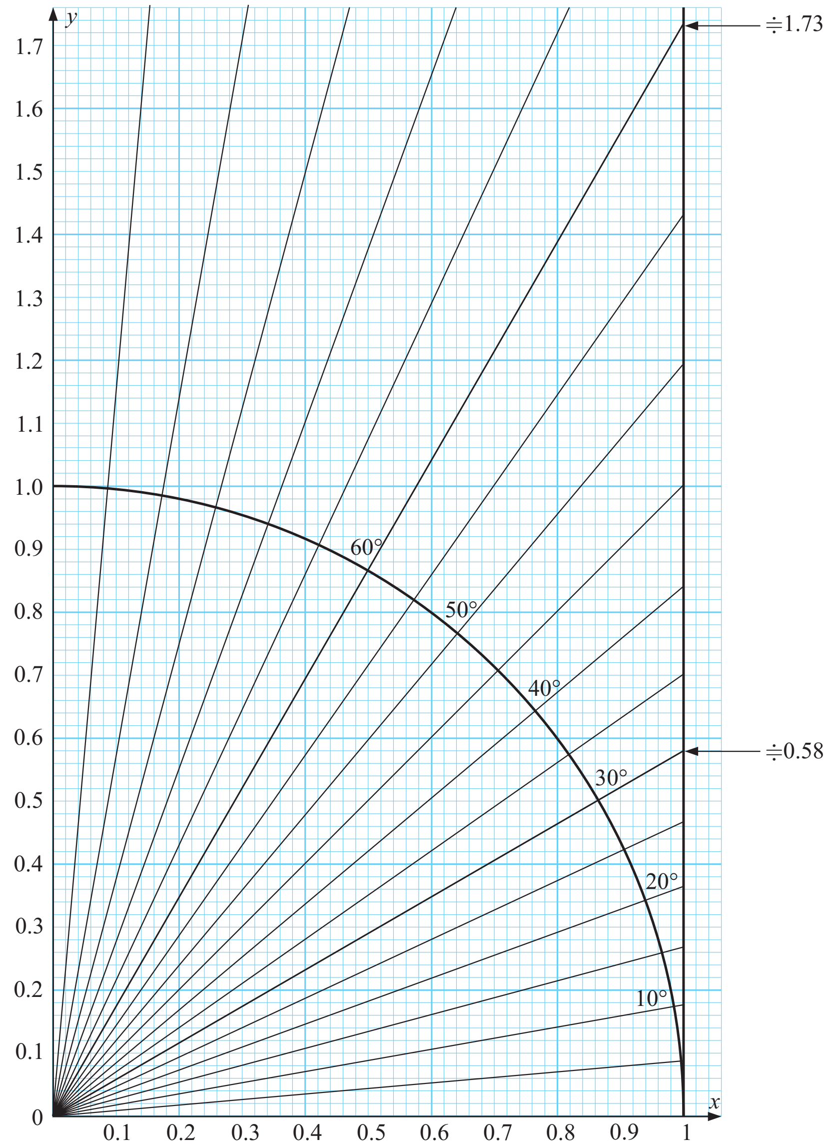
For θ in quadrant 3, $\sin\theta$ and $\cos\theta$ are both negative and so $\tan\theta$ is positive and this is clearly demonstrated as Q returns above the x-axis.



For θ in quadrant 4, $\sin\theta$ is negative and $\cos\theta$ is positive. So, $\tan\theta$ is negative.

DISCUSSION

- What is $\tan\theta$ when P is at (0, 1)?
- What is $\tan\theta$ when P is at (0, -1)?



EXERCISE 13K.1

1 Use the unit circle diagram to find the value of:

a $\tan 0^o$	**b** $\tan 15^o$	**c** $\tan 20^o$	**d** $\tan 25^o$
e $\tan 35^o$	**f** $\tan 45^o$	**g** $\tan 50^o$	**h** $\tan 55^o$

2 Use your calculator to check your answers to question **1**.

3 Explain why $\tan 45^o = 1$ exactly.

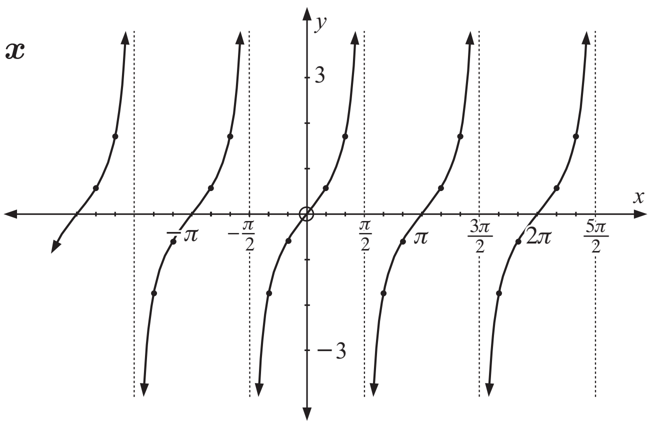
4 Why have you not been asked to find $\tan 85^o$ using the unit circle diagram? Find $\tan 85^o$ using your calculator.

Now click on the icon to see the graph of $y = \tan\theta$ demonstrated from its unit circle definition.

DEMO

THE GRAPH OF $y = \tan x$

The graph of $y = \tan x$ is



DISCUSSION

- Is the tangent function periodic? If so, what is its period?
- For what values of x does the graph not exist? What physical characteristics are shown near these values? Explain why these values must occur when $\cos x = 0$.
- Discuss how to find the x-intercepts of $y = \tan x$.
- What must $\tan\,(x - \pi)$ simplify to?
- How many solutions can the equation $\tan x = 2$ have?

EXERCISE 13K.2

1 **a** Use a transformation approach to *sketch* the graphs of the following functions for $0 \leqslant x \leqslant 3\pi$:

i $y = \tan(x - \frac{\pi}{2})$ **ii** $y = -\tan x$ **iii** $y = \tan 2x$

b Use technology to check your answers to **a**.

Look in particular for:
- asymptotes
- x-axis intercepts.

GRAPHING PACKAGE

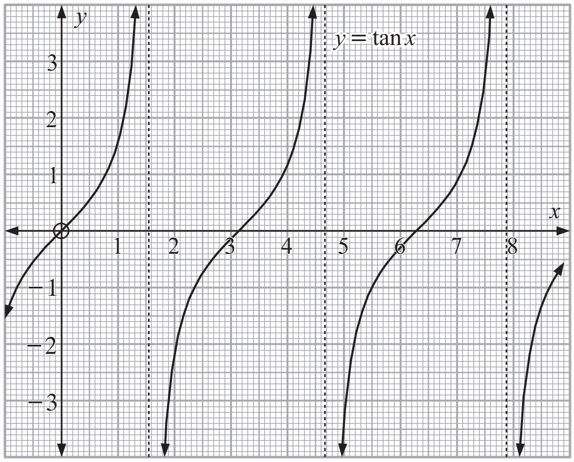
2 Use the graphing package to graph, on the same set of axes:

GRAPHING PACKAGE

- **a** $y = \tan x$ and $y = \tan(x - 1)$
- **b** $y = \tan x$ and $y = -\tan x$
- **c** $y = \tan x$ and $y = \tan\left(\frac{x}{2}\right)$

Describe the transformation which moves the first curve to the second in each case.

3 The graph of $y = \tan x$ is illustrated.



- **a** Use the graph to find estimates of:
  - **i** $\tan 1$ **ii** $\tan 2.3$
- **b** Check your answers from a calculator.
- **c** Find, correct to 1 decimal place, the solutions of:
  - **i** $\tan x = 2$ for $0 \leqslant x \leqslant 8$
  - **ii** $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$

4 What is the period of:

- **a** $y = \tan x$
- **b** $y = \tan 2x$
- **c** $y = \tan nx$?

L TANGENT EQUATIONS

In question **3** of the previous exercise we solved tangent equations graphically. Unfortunately the solutions by this method are not very accurate.

Consider solving these similar looking equations
$$\left\{\begin{aligned}\tan x &= 2.61\\ \tan(x-2) &= 2.61\\ \tan 2x &= 2.61\end{aligned}\right.$$

ALGEBRAIC SOLUTION

Since the tangent function is periodic with period π we see that $\tan(x + \pi) = \tan x$ for all values of x. This means that equal tan values are π units apart.

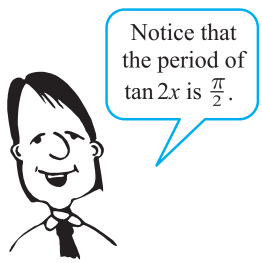
Notice all equations are of the form $\tan X = 2.61$.

If $\tan X = 2.61$
$\therefore$ $X = \tan^{-1}(2.61)$
$\therefore$ $X \doteqdot 1.205$

So, if $\tan x = 2.61$, then
$x = 1.205 + k\pi$, (k any integer)

If $\tan(x - 2) = 2.61$
then $x - 2 \doteqdot 1.205 + k\pi$
$\therefore$ $x \doteqdot 3.205 + k\pi$

If $\tan 2x = 2.61$
then $2x \doteqdot 1.205 + k\pi$
$\therefore$ $x \doteqdot \dfrac{1.205}{2} + \dfrac{k\pi}{2}$
$\therefore$ $x \doteqdot 0.602 + \dfrac{k\pi}{2}$



EXERCISE 13L.1

1 If $\tan X = 2$, find *all* solutions for X. Hence, solve the equations:

a $\tan 2x = 2$ **b** $\tan\left(\frac{x}{3}\right) = 2$ **c** $\tan(x + 1.2) = 2$

2 If $\tan X = -3$, find *all* solutions for X. Hence, solve the equations:

a $\tan(x - 2) = -3$ **b** $\tan 3x = -3$ **c** $\tan\left(\frac{x}{2}\right) = -3$

3 Find the exact solutions of $\tan X = \sqrt{3}$ in terms of π only. Hence solve the equations:

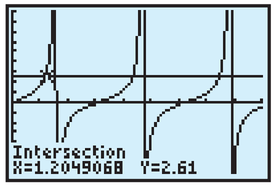
a $\tan\left(x - \frac{\pi}{6}\right) = \sqrt{3}$ **b** $\tan 4x = \sqrt{3}$ **c** $\tan^2 x = 3$

SOLUTION FROM TECHNOLOGY

Consider once again the equation $\tan x = 2.61$.

Graphing $y = \tan x$ and $y = 2.61$ on the same set of axes and finding the x-values where they intersect leads us to the solutions.

GRAPHICS CALCULATOR



Graph $Y_1 = \tan X$ and $Y_2 = 2.61$.
Use built-in functions to find the first positive point of intersection.

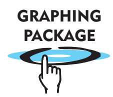
It is $X \doteqdot 1.205$. So, the solutions are $x = 1.205 + k\pi$ as the period of $y = \tan x$ is π.

Note:
- To solve $\tan(x - 2) = 2.61$ use $Y_1 = \tan(X - 2)$.
- To solve $\tan 2x = 2.61$ use $Y_1 = \tan 2X$.

GRAPHING PACKAGE

Graph $y = \tan x$ and $y = 2.61$ on the same set of axes and find the first positive point of intersection, etc.

What to do: Repeat question **1** from **Exercise 13L.1** using your technology.

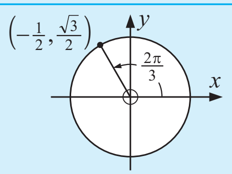


TANGENT CALCULATIONS AND SIMPLIFICATIONS

As $\tan x = \dfrac{\sin x}{\cos x}$, we can use the unit circle to find the exact value(s) of $\tan x$.

Example 29

Use a unit circle diagram to find the exact value of $\tan\left(\frac{2\pi}{3}\right)$.



We see that $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\therefore \quad \tan\left(\frac{2\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

EXERCISE 13L.2

1 Use a unit circle diagram to find the exact value of:

a $\tan 0$ **b** $\tan\left(\frac{\pi}{4}\right)$ **c** $\tan\left(\frac{\pi}{6}\right)$ **d** $\tan\left(\frac{\pi}{3}\right)$ **e** $\tan\left(\frac{\pi}{2}\right)$

f $\tan\left(\frac{3\pi}{4}\right)$ **g** $\tan\left(\frac{5\pi}{3}\right)$ **h** $\tan\left(\frac{3\pi}{2}\right)$ **i** $\tan\left(-\frac{\pi}{3}\right)$ **j** $\tan\left(\frac{-3\pi}{4}\right)$

2 Use a unit circle diagram to find all angles between 0 and 2π which have:

a a tangent of 1 **b** a tangent of -1 **c** a tangent of $\sqrt{3}$

d a tangent of 0 **e** a tangent of $\frac{1}{\sqrt{3}}$ **f** a tangent of $-\sqrt{3}$

Often expressions containing $\tan x$ can be simplified by replacing $\tan x$ by $\dfrac{\sin x}{\cos x}$.

Example 30

Simplify: **a** $\cos x \tan x + 2\sin x$ **b** $\dfrac{\tan x}{\sin x}$

a
$$\begin{aligned}&\cos x \tan x + 2\sin x\\ &= \cos x\left(\frac{\sin x}{\cos x}\right) + 2\sin x\\ &= \sin x + 2\sin x\\ &= 3\sin x\end{aligned}$$

b
$$\begin{aligned}&\frac{\tan x}{\sin x}\\ &= \frac{\sin x}{\cos x} \times \frac{1}{\sin x}\\ &= \frac{1}{\cos x}\end{aligned}$$

3 Simplify:

a $3\tan x - \tan x$ **b** $\tan x - 4\tan x$ **c** $\tan x \cos x$

d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

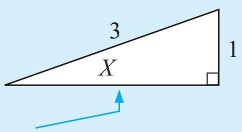
Given the exact values of $\sin x$ or $\cos x$ we can determine $\tan x$ without a calculator.

Example 31

If $\sin x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$, find the value of $\tan x$, without finding x.

Consider $\sin X = \frac{1}{3}$. $\{X$ is the *working angle* and is acute$\}$

So

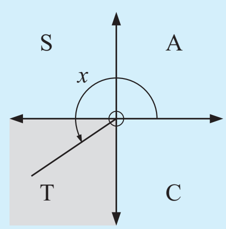


This side is $\sqrt{8}$ {Pythagoras}

$\therefore\ \tan X = \frac{1}{\sqrt{8}}$

$\therefore\ \tan x = \frac{1}{\sqrt{8}}$

{as we know that x lies in quad. 3, when tan is > 0}



4 **a** If $\sin x = \frac{1}{3}$ and $\frac{\pi}{2} < x < \pi$, find $\tan x$ in radical (surd) form.

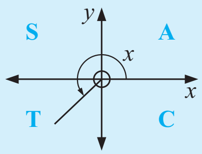
b If $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\tan x$ in radical (surd) form.

c If $\sin x = -\frac{1}{\sqrt{3}}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ in radical (surd) form.

d If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find $\tan x$ in radical (surd) form.

Example 32

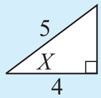
If $\tan x = \frac{3}{4}$ and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\cos x$.



x is in quadrant 3

$\therefore$ $\sin x < 0$ and $\cos x < 0$.

Consider $\tan X = \frac{3}{4}$



this side is 3 {Pythagoras}

$\therefore$ $\sin X = \frac{3}{5}$ and $\cos X = \frac{4}{5}$

and so $\sin x = -\frac{3}{5}$ and $\cos x = -\frac{4}{5}$

5 Find $\sin x$ and $\cos x$ given that:

a $\tan x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$

b $\tan x = -\frac{4}{3}$ and $\frac{\pi}{2} < x < \pi$

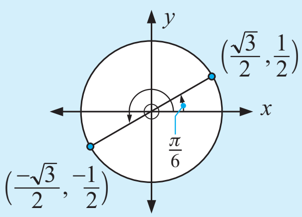
c $\tan x = \frac{\sqrt{5}}{3}$ and $\pi < x < \frac{3\pi}{2}$

d $\tan x = -\frac{12}{5}$ and $\frac{3\pi}{2} < x < 2\pi$

M OTHER EQUATIONS INVOLVING $\tan x$

Example 33

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0 \leqslant x \leqslant 2\pi$.



$$\sqrt{3}\sin x = \cos x$$

$$\therefore \quad \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}} \quad \{\text{dividing both sides by } \sqrt{3}\cos x\}$$

$$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

EXERCISE 13M

1 **a** Use your graphics calculator to sketch the graphs of $y = \sin x$ and $y = \cos x$ on the same set of axes on the domain $0 \leqslant x \leqslant 2\pi$.

b Find the x values of the points of intersection of the two graphs.

c Confirm that these values are the solutions of $\sin x = \cos x$ on $0 \leqslant x \leqslant 2\pi$.

2 Find the exact solutions to these equations for $0 \leqslant x \leqslant 2\pi$.

a $\sin x = -\cos x$ **b** $\sin(3x) = \cos(3x)$ **c** $\sin(2x) = \sqrt{3}\cos(2x)$

3 Check your answers to question **2** using a graphics calculator. Find the points of intersection of appropriate graphs.

4 Use the $\tan x = \dfrac{\sin x}{\cos x}$ identity to solve on $0 \leqslant x \leqslant 10$,

a $\sin x = 5\cos x$ **b** $4\sin x + 3\cos x = 0$

5 Check your answers to question **4** using a graphics calculator and appropriate graphs.

REVIEW SET 13A

1 Convert the following to radians in terms of π:

a 120^o **b** 225^o **c** 150^o **d** 540^o

2 Convert to radians correct to 4 significant figures:

a 71^o **b** 124.6^o **c** -142^o **d** -25.3^o

3 Convert the following radian measure to degrees:

a $\frac{2\pi}{5}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{9}$ **d** $\frac{11\pi}{6}$

4 Convert the following radian measure to degrees (to 2 decimal places):

a 3 **b** 1.46 **c** 0.435 **d** -5.271

5 Use your calculator to determine the coordinates of the point on the unit circle corresponding to an angle of:

a 320^o **b** 163^o

6 Illustrate the regions where $\sin\theta$ and $\cos\theta$ have the same sign.

7 Use a unit circle diagram to find exact values for $\sin\theta$ and $\cos\theta$ for θ equal to:

a 120^o **b** 480^o

8 Use a unit circle to find exact values for $\sin\theta$ and $\cos\theta$ for θ equal to:

a 150^o **b** 300^o

9 Explain how to use the unit circle to find θ when $\cos\theta = -\sin\theta$.

REVIEW SET 13B

1 Use a unit circle diagram to find:

a $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$ **b** $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$

2 If $\cos\theta = \frac{3}{4}$ find the possible values of $\sin\theta$.

3 If $\cos\theta = -\frac{3}{4}$, $\frac{\pi}{2} < \theta < \pi$ find $\sin\theta$.

4 Without a calculator, evaluate: **a** $\sin^2 120^o$ **b** $\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right)$

5 Without a calculator evaluate:

a $2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$ **b** $\sin^2\left(\frac{\pi}{4}\right) - 1$ **c** $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

6 Use a unit circle diagram to find all angles between 0^o and 360^o which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a sine of $\frac{1}{\sqrt{2}}$

7 Find θ in radians if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

8 Without using technology draw the graph of $y = 4\sin x$ for $0 \leqslant x \leqslant 2\pi$.

9 Without using technology draw the graph of $y = \sin 3x$ for $0 \leqslant x \leqslant 2\pi$.

REVIEW SET 13C

1 State the period of:

a $y = 4\sin\left(\frac{x}{3}\right)$ **b** $y = -2\sin 4x$

2 Without using technology draw a sketch graph of $y = \sin\left(x - \frac{\pi}{3}\right) + 2$.

3 The table below gives the mean monthly maximum temperature (^{o}C) for Perth Airport in Western Australia.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

a A sine function of the form $T \doteqdot A\sin B(t - C) + D$ is used to model the data. Find good estimates of the constants A, B, C and D without using technology. Use Jan $\equiv$ 1, Feb $\equiv$ 2, etc.

b Check your answer to **a** using your technology. How well does your model fit?

4 Use technology to solve for $0 \leqslant x \leqslant 8$:

a $\sin x = 0.382$ **b** $\sin\left(\frac{x}{2}\right) = -0.458$

5 Use technology to solve for $0 \leqslant x \leqslant 8$:

a $\sin(x - 2.4) = 0.754$ **b** $\sin\left(x + \frac{\pi}{3}\right) = 0.6049$

6 Solve algebraically in terms of π:

a $2\sin x = -1$ for $0 \leqslant x \leqslant 4\pi$ **b** $\sqrt{2}\sin x - 1 = 0$ for $-2\pi \leqslant x \leqslant 2\pi$

7 Solve algebraically in terms of π:

a $2\sin 3x + \sqrt{3} = 0$ for $0 \leqslant x \leqslant 2\pi$

b $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 0$ for $0 \leqslant x \leqslant 3\pi$

8 Solve algebraically, giving answers in terms of π:

a $\sin^2 x - \sin x - 2 = 0$ **b** $4\sin^2 x = 1$

9 The population estimate, in thousands, of a species of water beetle where $0 \leqslant t \leqslant 8$ and t is the number of weeks after the initial population estimate was made, is given by $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right)$.

a What was the initial population?

b What were the smallest and largest population sizes?

c During what time interval(s) did the population size exceed 6000?

REVIEW SET 13D

1 **a** On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \cos x - 3$.

b On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{4}\right)$.

c On the same set of axes, sketch the graphs of $y = \cos x$ and $y = 3\cos 2x$.

d On the same set of axes, sketch the graphs of $y = \cos x$ and $y = 2\cos\left(x - \frac{\pi}{3}\right) + 3$.

2 In an industrial city, the amount of pollution in the air becomes greater during the working week when factories are operating, and lessens over the weekend. The number of milligrams of pollutants in a cubic metre of air is given by

$$P(t) = 40 + 12\sin\frac{2\pi}{7}\left(t - \frac{37}{12}\right)$$

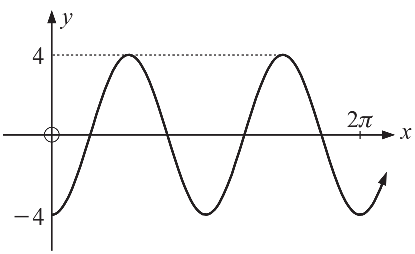
where t is the number of days after midnight on Saturday night.

a What was the minimum level of pollution?

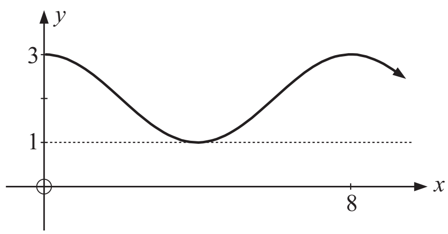
b At what time during the week does this minimum level occur?

3 For the following graphs, find the cosine function representing them:

a



b



4 Use technology to solve:

a $\cos x = 0.4379$ for $0 \leqslant x \leqslant 10$

b $\cos(x - 2.4) = -0.6014$ for $0 \leqslant x \leqslant 6$.

5 Use technology to solve:

a $\cos 4x = 0.3$ for all x **b** $4\cos 2x + 1 = 0$ for $0 \leqslant x \leqslant 5$.

6 Find the exact solutions of:

a $\cos x = -\frac{1}{\sqrt{2}}$ for $0 \leqslant x \leqslant 4\pi$

b $\cos\left(x + \frac{2\pi}{3}\right) = \frac{1}{2}$ for $-2\pi \leqslant x \leqslant 2\pi$.

7 Find the exact solutions of:

a $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)-1=0$ for $0 \leqslant x \leqslant 4\pi$.

b $2\cos 2x - 1 = 0$ for all x.

8 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$ **c** $3\cos\theta - \cos\theta$

d $5 - 5\sin^2\theta$ **e** $\dfrac{\sin^2\theta - 1}{\cos\theta}$

9 Expand and simplify if possible:

a $(2\sin\alpha - 1)^2$ **b** $(\cos\alpha - \sin\alpha)^2$

REVIEW SET 13E

1 Simplify:

a $\dfrac{1-\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$ **c** $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

2 **a** Show that $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta}$ simplifies to $\dfrac{2}{\cos\theta}$.

b Show that $\left(1+\dfrac{1}{\cos\theta}\right)(\cos\theta - \cos^2\theta)$ simplifies to $\sin^2\theta$.

3 If $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$ find the values of:

a $\sin 2A$ **b** $\cos 2A$

4 If $\sin\alpha = -\frac{3}{4}$ where $\pi < \alpha < \frac{3\pi}{2}$ find the value of $\cos\alpha$ and hence the value of $\sin 2\alpha$.

5 If $\cos x = -\frac{3}{4}$ and $\pi < x < \frac{3\pi}{2}$ find the exact value of $\sin\left(\frac{x}{2}\right)$.

6 **a** Solve algebraically:

i $\tan x = 4$ **ii** $\tan\left(\frac{x}{4}\right) = 4$ **iii** $\tan(x - 1.5) = 4$

b Find the exact solutions in terms of π only for:

i $\tan\left(x+\frac{\pi}{6}\right) = -\sqrt{3}$ **ii** $\tan 2x = -\sqrt{3}$ **iii** $\tan^2 x - 3 = 0$

c Use technology to solve $3\tan(x-1.2) = -2$.

7 If $\tan\theta = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$, find $\sin\theta$ and $\cos\theta$ without using a calculator.

8 Show that $\dfrac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1}$ simplifies to $\tan\alpha$.

Chapter 14

Matrices

Contents:

A Introduction
B Addition and subtraction of matrices
C Multiples of matrices
D Matrix algebra for addition
E Matrix multiplication
F Using technology
G Some properties of matrix multiplication
H The inverse of a 2×2 matrix
I Solving a pair of linear equations
J The 3×3 determinant
K The inverse of a 3×3 matrix
L 3×3 systems with unique solutions

Investigation: Using matrices in cryptography

Review set 14A
Review set 14B
Review set 14C
Review set 14D
Review set 14E

A INTRODUCTION

You have been using matrices for many years without realising it.

For example:

July 2001						
M	T	W	T	F	S	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

	Won	*Lost*	*Drew*	*Points*
Arsenal	24	2	4	76
Liverpool	23	3	4	73
Chelsea	21	4	5	68
Leeds	20	5	5	65
⋮				

Ingredients	*Amount*
sugar	1 tspn
flour	1 cup
milk	200 mL
salt	1 pinch

In general:

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

Consider these two items of information:

Shopping list	
Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory			
	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4

It is usual to put square or round brackets around a matrix.
We could write the shopping list and furniture inventory as:

$$\begin{array}{c|c|} & \text{number} \\ \text{B} & 2 \\ \text{J} & 1 \\ \text{E} & 6 \\ \text{C} & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|ccc|} & \text{C} & \text{T} & \text{B} \\ \text{F} & 6 & 1 & 2 \\ \text{U} & 9 & 2 & 3 \\ \text{H} & 10 & 3 & 4 \end{array}$$

and if we can remember what makes up the rows and columns, we could write them simply as:

$$\begin{bmatrix} 2 \\ 1 \\ 6 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{bmatrix}$$

In $\begin{bmatrix} 2 \\ 1 \\ 6 \\ 1 \end{bmatrix}$ we have 4 rows and 1 column and we say that this is a 4×1 **column matrix** or **column vector**.

$$\begin{bmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & \boxed{3} & 4 \end{bmatrix}$$ has 3 rows and 3 columns and is called a 3×3 **square matrix.**

This **element**, 3, is in row 3, column 2.

$$\begin{bmatrix} 3 & 0 & -1 & 2 \end{bmatrix}$$ has 1 row and 4 columns and is called a 1×4 **row matrix** or **row vector**.

Note:
- An $\underset{\text{rows}}{m} \times \underset{\text{columns}}{n}$ matrix has m rows and n columns.
- $m \times n$ specifies the **order** of a matrix.

USES OF MATRICES

Following are a few of many uses for the mathematics of matrices:

- **Solving of systems of equations** in business, physics, engineering, etc.
- **Linear programming** where, for example, we may wish to optimise a linear expression subject to linear constraints. For example, optimising profits of a business.
- **Business inventories** involving stock control, cost, revenue and profit calculations. Matrices form the basis of business computer software.
- **Markov chains**, for predicting long term probabilities such as in weather.
- **Strategies in games** where we wish to maximise our chance of winning.
- **Economic modelling** where the input from various suppliers is needed to help a business be successful.
- **Graph (network) theory** which is used in truck and airline route determination to minimise distance travelled and therefore minimise costs.
- **Assignment problems** where we have to direct resources in industrial situations in the most cost effective way.
- **Forestry and fisheries management** where we need to select an appropriate sustainable harvesting policy.
- **Cubic spline interpolation** which is used to construct fonts used in desktop publishing.
  Each font is stored in matrix form in the memory of a computer.
- **Computer graphics**, **flight simulation**, **Computer Aided Tomography** (CAT scanning) and **Magnetic Resonance Imaging** (MRI), **Fractals**, **Chaos**, **Genetics**, **Cryptography** (coding, code breaking, computer confidentiality), etc.

A matrix can be used to represent numbers of items to be purchased, prices of items to be purchased, numbers of people involved in the construction of a building, etc.

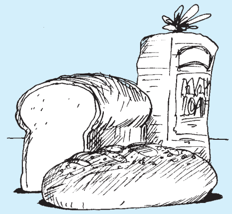
Example 1

Lisa goes shopping at store A to buy 2 loaves of bread at $2.65 each, 3 litres of milk at $1.55 per litre, a 500 g tub of butter at $2.35.

Represent the quantities purchased in a row matrix and the costs in a column matrix.

The quantities matrix is $\begin{array}{c} \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{bread} & \text{milk} & \text{butter} \end{array} \end{array}$

The costs matrix is $\begin{array}{cl} \begin{bmatrix} 2.65 \\ 1.55 \\ 2.35 \end{bmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \end{array}$



Note: If Lisa goes to a different supermarket (store B) and finds that the prices for the same items are $2.25 for bread, $1.50 for milk, and $2.20 for butter, then the costs matrix to show prices from both stores is:

$$\begin{array}{cl} \begin{bmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{bmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{store A} & \text{store B} \end{array} & \end{array}$$

EXERCISE 14A

1 Write down the order of:

a $\begin{bmatrix} 5 & 1 & 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{bmatrix}$

2 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs and 1 carton of cream. The cost of each grocery item is $1.95, $2.35, $0.15 and $0.95 respectively.

a Construct a row matrix showing quantities.

b Construct a column matrix showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.15) + (1 \times 0.95)$?

3 Big Bart's Baked Beans factory produces cans of baked beans in 3 sizes; 200 g, 300 g and 500 g. In February they produced respectively:

1000, 1500 and 1250 cans of each in week 1; 1500, 1000 and 1000 of each in week 2
800, 2300 and 1300 cans of each in week 3; and 1200 cans of each in week 4.

Construct a matrix to show February's production levels.

4 Over a long weekend holiday period, a baker produced the following food items. On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls and 40 dozen buns. On Saturday 25 dozen pies, 65 dozen pasties, 30 dozen buns and 44 dozen rolls were made. On Sunday 40 dozen pasties, 40 dozen rolls, 35 dozen of each of pies and buns were made. On Monday the totals were 40 dozen pasties, 50 dozen buns and 35 dozen of each of pies and rolls. Represent this information as a matrix.

B ADDITION AND SUBTRACTION OF MATRICES

Before attempting to add and subtract matrices it is necessary to define what we mean by **matrix equality**.

EQUALITY

Two matrices are **equal** if they have exactly the same shape (order) and elements in corresponding positions are equal.

For example, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ then $a = w$, $b = x$, $c = y$ and $d = z$.

Notice that $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

ADDITION

Thao has three stores (A, B and C). Her stock levels for dresses, skirts and blouses are given by the matrix:

$$\begin{array}{c c} & \text{Store} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{array}{c} \text{dresses} \\ \text{skirts} \\ \text{blouses} \end{array} & \begin{bmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{bmatrix} \end{array}$$

Some newly ordered stock has just arrived. For each store 20 dresses, 30 skirts and 50 blouses must be added to stock levels.

Her stock order is given by the matrix $\begin{bmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{bmatrix}$

Clearly the new levels are shown as: $\begin{bmatrix} 23+20 & 41+20 & 68+20 \\ 28+30 & 39+30 & 79+30 \\ 46+50 & 17+50 & 62+50 \end{bmatrix}$

or $\begin{bmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{bmatrix} + \begin{bmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{bmatrix} = \begin{bmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{bmatrix}$

So, to **add** two matrices they must be of the **same order** and then we simply **add corresponding elements**.

Example 2

If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{C}$

a
$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{bmatrix}$$

b $\mathbf{A}+\mathbf{C}$ cannot be found as $\mathbf{A}$ and $\mathbf{C}$ are not the same sized matrices

i.e., they have different orders.

SUBTRACTION

If Thao's stock levels were $\begin{bmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{bmatrix}$ and her sales matrix for the week is

$\begin{bmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{bmatrix}$ what are the current stock levels?

It is obvious that we subtract corresponding elements.

That is $\begin{bmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{bmatrix} - \begin{bmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{bmatrix} = \begin{bmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{bmatrix}$

So, to **subtract** matrices they must be of the **same order** and then we simply subtract corresponding elements.

Example 3

If $\mathbf{A} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{bmatrix}$ find $\mathbf{A}-\mathbf{B}$.

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{bmatrix}$$

EXERCISE 14B

1 If $\mathbf{A}=\begin{bmatrix}3 & 4\\ 5 & 2\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix}-3 & 7\\ -4 & -2\end{bmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $\mathbf{A}+\mathbf{B}+\mathbf{C}$ **c** $\mathbf{B}+\mathbf{C}$ **d** $\mathbf{C}+\mathbf{B}-\mathbf{A}$

2 If $\mathbf{P}=\begin{bmatrix}3 & 5 & -11\\ 10 & 2 & 6\\ -2 & -1 & 7\end{bmatrix}$ and $\mathbf{Q}=\begin{bmatrix}17 & -4 & 3\\ -2 & 8 & -8\\ 3 & -4 & 11\end{bmatrix}$, find:

a $\mathbf{P}+\mathbf{Q}$ **b** $\mathbf{P}-\mathbf{Q}$ **c** $\mathbf{Q}-\mathbf{P}$

3 A restaurant served 85 men, 92 women and 52 children on Friday night. On Saturday night they served 102 men, 137 women and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women and children served over the Friday-Saturday period.

4 On Monday David bought shares in five companies and on Friday he sold them. The details are:

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

a Write down David's
 i cost price column matrix
 ii selling price column matrix.

b What matrix operation is needed to find David's profit/loss matrix?

c Find David's profit/loss matrix.

5 In November, Lou E Gee sold 23 fridges, 17 stoves and 31 microwave ovens and his partner Rose A Lee sold 19 fridges, 29 stoves and 24 microwave ovens.

In December Lou's sales were: 18 fridges, 7 stoves and 36 microwaves while Rose's sales were: 25 fridges, 13 stoves and 19 microwaves.

a Write their sales for November as a 3×2 matrix.

b Write their sales for December as a 3×2 matrix.

c Write their total sales for November and December as a 3×2 matrix.

6 Find x and y if:

a $\begin{bmatrix}x & x^2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}y & 4\\ 3 & y+1\end{bmatrix}$ **b** $\begin{bmatrix}x & y\\ y & x\end{bmatrix}=\begin{bmatrix}-y & x\\ x & -y\end{bmatrix}$

7 **a** If $\mathbf{A}=\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}$ find $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$.

b Explain why $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

8 **a** For $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$ find $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

b Prove that, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are any 2×2 matrices then $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(**Hint:** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, say.)

C MULTIPLES OF MATRICES

In the pantry there are 6 cans of peaches, 4 cans of apricots and 8 cans of pears.

This information could be represented by the column vector $\mathbf{C} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}$.

Doubling these cans in the pantry we would have $\begin{bmatrix} 12 \\ 8 \\ 16 \end{bmatrix}$ which is $\mathbf{C} + \mathbf{C}$.

Now if we let $\mathbf{C} + \mathbf{C}$ be $2\mathbf{C}$ we notice that:

to get $2\mathbf{C}$ from $\mathbf{C}$ we simply multiply all matrix elements by 2.

Likewise, trebling the fruit cans in the pantry is

$$3\mathbf{C} = \mathbf{C} + \mathbf{C} + \mathbf{C} = \begin{bmatrix} 3 \times 6 \\ 3 \times 4 \\ 3 \times 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ 24 \end{bmatrix}$$

and halving them would be

$$\tfrac{1}{2}\mathbf{C} = \begin{bmatrix} \frac{1}{2} \times 6 \\ \frac{1}{2} \times 4 \\ \frac{1}{2} \times 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

In general,

> if a scalar t is multiplied by a matrix $\mathbf{A}$ the result is matrix $t\mathbf{A}$ obtained by multiplying every element of $\mathbf{A}$ by t.

Example 4

If $\mathbf{A}$ is $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ find **a** $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$

a $3\mathbf{A} = 3\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 & 15 \\ 6 & 0 & 3 \end{bmatrix}$

b $\frac{1}{2}\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & 1 & 2\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{bmatrix}$

EXERCISE 14C

1 If $\mathbf{B}=\begin{bmatrix}6 & 12\\ 24 & 6\end{bmatrix}$ find: **a** $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A}=\begin{bmatrix}2 & 3 & 5\\ 1 & 6 & 4\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$ find:

a $\mathbf{A}+\mathbf{B}$ **b** $\mathbf{A}-\mathbf{B}$ **c** $2\mathbf{A}+\mathbf{B}$ **d** $3\mathbf{A}-\mathbf{B}$

3 Frank's order for hardware items is shown in matrix form as

$$\mathbf{H}=\begin{bmatrix}6\\ 12\\ 60\\ 30\end{bmatrix}\begin{array}{l}\longleftarrow \text{hammers}\\ \longleftarrow \text{screwdriver sets}\\ \longleftarrow \text{packets of nails}\\ \longleftarrow \text{packets of screws}\end{array}$$

Find the matrix if:

a Frank doubles his order
b Frank halves his order
c Frank increases his order by 50%.

4 Isabelle sells clothing made by four different companies which we will call A, B, C and D. Her usual monthly order is:

$$\begin{array}{l} \\ \text{skirt}\\ \text{dress}\\ \text{evening}\\ \text{suit}\end{array}\begin{array}{c}\begin{array}{cccc}\text{A} & \text{B} & \text{C} & \text{D}\end{array}\\ \begin{bmatrix}30 & 40 & 40 & 60\\ 50 & 40 & 30 & 75\\ 40 & 40 & 50 & 50\\ 10 & 20 & 20 & 15\end{bmatrix}\end{array}$$

Find her order, to the nearest whole number, if:

a she increases her total order by 15%
b she decreases her total order by 15%.

5 During weekdays a video store finds that its average hirings are: 75 movies (VHS), 27 movies (DVD) and 102 video/computer games. On the weekends the average figures are: 43 DVD movies, 136 VHS movies and 129 games.

a Represent the data using *two* column matrices.
b Find the sum of the matrices in **a**.
c What does the sum matrix of **b** represent?

$$\begin{bmatrix}\quad\\ \quad\\ \quad\end{bmatrix}\begin{array}{l}\longleftarrow \text{VHS}\\ \longleftarrow \text{DVD}\\ \longleftarrow \text{games}\end{array}$$

6 A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds and 1 wardrobe.

If $\mathbf{F}=\begin{bmatrix}1\\ 4\\ 2\\ 1\end{bmatrix}$ is the matrix representing the furniture in one flat,

what, in terms of $\mathbf{F}$, is the matrix representing the furniture in **all** flats?

ZERO MATRICES

For real numbers, it is true that $a+0=0+a=a$ for all values of a.

The question arises: "Is there a matrix $\mathbf{O}$ in which $\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$ for any matrix $\mathbf{A}$?"

Simple examples like: $\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}$ suggest that $\mathbf{O}$ consists of all zeros.

A **zero matrix** is a matrix in which all elements are zero.

For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

the 2×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Zero matrices have the property that:

If $\mathbf{A}$ is a matrix of any order and $\mathbf{O}$ is the corresponding **zero matrix**, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$ is actually $-1\mathbf{A}$.

So, if $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -1 \times 3 & -1 \times -1 \\ -1 \times 2 & -1 \times 4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & -4 \end{bmatrix}$

Thus $-\mathbf{A}$ is obtained from $\mathbf{A}$ by simply reversing the signs of each element of $\mathbf{A}$.

Notice that the addition of a matrix and its negative always produces a zero matrix. For example,

$$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, in general, $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$.

D MATRIX ALGEBRA FOR ADDITION

Compare our discoveries about matrices so far with ordinary algebra. We will assume the matrices have the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a + b$ is also a real number.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A} + \mathbf{B}$ is also a matrix.
• $a + b = b + a$	• $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
• $(a + b) + c = a + (b + c)$	• $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
• $a + 0 = 0 + a = a$	• $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$
• $a + (-a) = (-a) + a = 0$	• $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$
• a half of a is $\dfrac{a}{2}$	• a half of $\mathbf{A}$ is $\frac{1}{2}\mathbf{A}$ $\left(\text{not } \dfrac{\mathbf{A}}{2}\right)$ (Dividing a matrix by a real number has no meaning in matrix algebra.)

Example 5

Explain why it is true that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$ **b** if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$

a
$$\begin{aligned} \text{if} \quad \mathbf{X}+\mathbf{A}&=\mathbf{B} \\ \text{then} \quad \mathbf{X}+\mathbf{A}+(-\mathbf{A})&=\mathbf{B}+(-\mathbf{A}) \\ \therefore \quad \mathbf{X}+\mathbf{O}&=\mathbf{B}-\mathbf{A} \\ \text{i.e.,} \quad \mathbf{X}&=\mathbf{B}-\mathbf{A} \end{aligned}$$

b
$$\begin{aligned} \text{if} \quad 3\mathbf{X}&=\mathbf{A} \\ \text{then} \quad \tfrac{1}{3}(3\mathbf{X})&=\tfrac{1}{3}\mathbf{A} \\ \therefore \quad 1\mathbf{X}&=\tfrac{1}{3}\mathbf{A} \\ \therefore \quad \mathbf{X}&=\tfrac{1}{3}\mathbf{A} \end{aligned}$$

Notice that the rules for addition (and subtraction) of matrices are identical to those of real numbers but we must be careful with scalar multiplication in matrix equations.

EXERCISE 14D

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}$ **c** $\mathbf{C}-2\mathbf{C}$
d $-\mathbf{B}+\mathbf{B}$ **e** $2(\mathbf{A}+\mathbf{B})$ **f** $-(\mathbf{A}+\mathbf{B})$
g $-(2\mathbf{A}-\mathbf{C})$ **h** $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$ **i** $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find **X** in terms of **A**, **B** and **C** if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$ **b** $\mathbf{B}+\mathbf{X}=\mathbf{C}$ **c** $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$
d $2\mathbf{X}=\mathbf{A}$ **e** $3\mathbf{X}=\mathbf{B}$ **f** $\mathbf{A}-\mathbf{X}=\mathbf{B}$
g $\frac{1}{2}\mathbf{X}=\mathbf{C}$ **h** $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$ **i** $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 **a** If $\mathbf{M}=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, find $\mathbf{X}$ if $\frac{1}{3}\mathbf{X}=\mathbf{M}$.

b If $\mathbf{N}=\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, find $\mathbf{X}$ if $4\mathbf{X}=\mathbf{N}$.

c If $\mathbf{A}=\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, and $\mathbf{B}=\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, find **X** if $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$.

E MATRIX MULTIPLICATION

Suppose you go to a shop and purchase 3 soft drink cans, 4 chocolate bars and 2 icecreams

and the prices are

soft drink cans	chocolate bars	ice creams
\$1.30	\$0.90	\$1.20

Each of these can be represented using matrices,

i.e., $\mathbf{A}=\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1.30 & 0.90 & 1.20 \end{bmatrix}$.

To work out the total cost, the following *product* could be found:

$$\begin{aligned}\mathbf{BA} &= \begin{bmatrix} 1.30 & 0.90 & 1.20 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \\ &= (1.30 \times 3) + (0.9 \times 4) + (1.20 \times 2) \\ &= 3.90 + 3.60 + 2.40 \\ &= 9.90 \end{aligned}$$

Thus the total cost is $9.90.

Notice that we write the **row matrix** first and the **column matrix** second

and that $$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = ap + bq + cr.$$

EXERCISE 14E.1

1 Determine:

a $\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$ **c** $\begin{bmatrix} 6 & -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$

2 Show that the sum of w, x, y and z is given by $\begin{bmatrix} w & x & y & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Represent the average of w, x, y and z in the same way.

3 Lucy buys 4 shirts, 3 skirts and 2 blouses costing $27, $35 and $39 respectively.

a Write down a quantities matrix **Q** and a price matrix **P**.

b Show how to use **P** and **Q** to determine the total cost.

4 In the interschool public speaking competition a first place is awarded 10 points, second place 6 points, third place 3 points and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds and 2 fourths.

a Write down this information in terms of points matrix **P**, and numbers matrix **N**.

b Show how to use **P** and **N** to find the total number of points awarded to the school.

Now consider more complicated matrix multiplication.

In **Example 1** Lisa needed $\begin{array}{ccc} \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \\ \text{milk} \quad \text{bread} \quad \text{butter} \end{array}$ and at store A, the costs matrix was $\begin{bmatrix} 2.65 \\ 1.55 \\ 2.35 \end{bmatrix}$.

To find the *total cost* Lisa needs to multiply the number of items by their respective cost,

i.e., $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \$12.30$

As the quantities do not change, her total cost in Store B is

$$2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \$11.20$$

To do this using matrices notice that:

$$\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{bmatrix} = \begin{bmatrix} 12.30 & 11.20 \end{bmatrix}$$

orders: 1×3 $\quad 3 \times 2$ $\quad 1 \times 2$

the same

resultant matrix

Now suppose Lisa's friend Olu needs 1 bread, 2 milk and 2 butter.

The quantities matrix for both Lisa and Olu would be

$$\begin{array}{c c} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Lisa} \\ \leftarrow \text{Olu} \end{array} \\ \begin{array}{ccc} \text{bread} & \text{milk} & \text{butter} \end{array} & \end{array}$$

Lisa's *total cost* at Store A is \$12.30 and at store B is \$11.20

Olu's *total cost* at Store A is $1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45$

Store B is $1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65$

So, using matrices we require that

row 1 × column 1 $\quad$ row 1 × column 2

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{bmatrix} = \begin{bmatrix} 12.30 & 11.20 \\ 10.45 & 9.65 \end{bmatrix}$$

row 2 × column 1 $\quad$ row 2 × column 2

2×3 $\quad 3 \times 2$ $\quad 2 \times 2$

the same

resultant matrix

We are now ready to give a formal definition of a matrix product.

MATRIX PRODUCTS

As a consequence of observing the usefulness of multiplying matrices as in the contextual examples we are now in a position to define multiplication more formally.

The **product** of an $m \times n$ matrix **A** with an $n \times p$ matrix **B**, is the $m \times p$ matrix (called **AB**) in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of **A** with the corresponding elements in the cth column of **B**.

For example,

if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then $\mathbf{AB} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$,

and if $\underset{2 \times 3}{\mathbf{C} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}}$ and $\underset{3 \times 1}{\mathbf{D} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$, then $\underset{2 \times 1}{\mathbf{CD} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix}}$

Example 6

If $\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$

find: **a** $\mathbf{AB}$ **b** $\mathbf{AC}$

a $\mathbf{A}$ is $1 \times \boxed{3}$ and $\mathbf{B}$ is $\boxed{3} \times 1$ $\quad \therefore$ $\mathbf{AB}$ is 1×1

$\checkmark$

$$\begin{aligned}\mathbf{AB} &= \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \\ &= [1 \times 2 + 3 \times 4 + 5 \times 7] \\ &= [49]\end{aligned}$$

b $\mathbf{A}$ is $1 \times \boxed{3}$ and $\mathbf{C}$ is $\boxed{3} \times 2$ $\quad \therefore$ $\mathbf{AC}$ is 1×2

$\checkmark$

$$\begin{aligned}\mathbf{AC} &= \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 1 + 3 \times 2 + 5 \times 1 & 1 \times 0 + 3 \times 3 + 5 \times 4 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 29 \end{bmatrix}\end{aligned}$$

EXERCISE 14E.2

1 Explain why $\mathbf{AB}$ cannot be found for $\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

2 If $\mathbf{A}$ is $2 \times n$ and $\mathbf{B}$ is $m \times 3$:

a Under what condition can we find $\mathbf{AB}$?

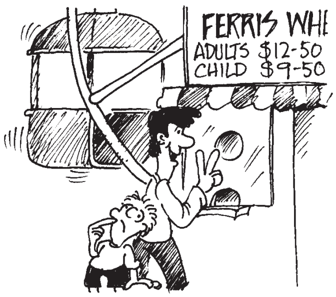
b If $\mathbf{AB}$ can be found, what is its order?

c Why can $\mathbf{BA}$ never be found?

3 **a** For $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \end{bmatrix}$, find $\mathbf{BA}$.

b For $\mathbf{A} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ find **i** $\mathbf{AB}$ **ii** $\mathbf{BA}$.

4 Find: **a** $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

5 

At the Royal Show, tickets for the Ferris wheel are \$12.50 per adult and \$9.50 per child. On the first day of the show 2375 adults and 5156 children ride this wheel. On the second day the figures are 2502 adults and 3612 children.

- **a** Write the costs matrix **C** as a 2×1 matrix and the numbers matrix **N** as a 2×2 matrix.
- **b** Find **NC** and interpret the resulting matrix.
- **c** Find the total income for the two days.

6 You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver and 2 cans of white paint and your friend wants 1 hammer, 2 screwdrivers and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$7	\$3	\$19
Store B	\$6	\$2	\$22

- **a** Write the requirements matrix **R** as a 3×2 matrix.
- **b** Write the prices matrix **P** as a 2×3 matrix.
- **c** Find **PR**.
- **d** What are your costs at store A and your friend's costs at store B?
- **e** Should you buy from store A or store B?

7 At the market, a greengrocer buys 6 boxes of apples, 7 boxes of bananas and 9 boxes of oranges. Next day, 5 boxes of apples, 8 boxes of bananas and 4 boxes of oranges are purchased. On the third day, 4 boxes of apples, 7 of bananas and 2 of oranges are purchased.

The apples cost \$18 a box, bananas \$15 a box and oranges \$13 a box over the 3-day period. Express this information in the form of two matrices and show how to find the total cost of the fruit over the 3-day period, using these matrices.

F USING TECHNOLOGY

USING A GRAPHICS CALCULATOR FOR MATRIX OPERATIONS

Click on the icon for your calculator to assist you to enter and perform operations on matrices.

USING A SPREADSHEET FOR MATRIX OPERATIONS

ADDING MATRICES

Adding matrices by hand can be tedious, particularly for matrices of higher order. A spreadsheet can significantly speed up the process.

Consider the addition: $\begin{bmatrix} 3 & 0 & 3 & 2 \\ 2 & 1 & 1 & 5 \\ 2 & 5 & 0 & 3 \\ 1 & 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 3 & 4 \\ 8 & 1 & 5 & 2 \\ 7 & 3 & 2 & 0 \\ 4 & 4 & 4 & 1 \end{bmatrix}$

SPREADSHEET

What to do:

1 Enter the two matrices on the spreadsheet.

For example:

	A	B	C	D	E	F	G	H	I	J
1										
2		3	0	3	2		2	2	3	4
3		2	1	1	5		8	1	5	2
4		2	5	0	3		7	3	2	0
5		1	4	2	8		4	4	4	1
6										
7										
8										
9										
10										
11										

2 Highlight the cells in the range E7:H10 as shown. The addition will appear in these cells.

3 With the cells still highlighted, type =B2:E5+G2:J5 To enter the formula hold down CTRL and SHIFT together and while doing this press ENTER .

SUBTRACTING MATRICES

This is identical to addition except for the formula which should now be =B2:E5-G2:J5

SCALAR MULTIPLICATION

Consider finding $3.5 \begin{bmatrix} 2 & 3 & 1 & 4 \\ 8 & 5 & 3 & 2 \\ 4 & 2 & 5 & 1 \end{bmatrix}$

	A	B	C	D	E	F	G	H
1								
2		3.5		2	3	1	4	
3				8	5	3	2	
4				4	2	5	1	
5								
6								
7								
8								
9								

What to do:

1 Enter 3.5 into B2 and the given matrix in the range D2:G4 say.

2 Highlight the cells C6:F8 and while still highlighted type =B2*D2:G4.

3 Enter the formula by holding down CTRL and SHIFT together and while doing this press ENTER .

MULTIPLYING MATRICES

Multiplying matrices by hand can be very tedious, particularly with real life matrices which could be very large. The formula = MMULT is used.

Consider finding $\begin{bmatrix} 1 & 3 & 2 & 5 & 6 \\ 3 & 0 & 2 & 1 & 2 \\ 0 & 1 & 3 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 4 & 5 & 2 & 7 \end{bmatrix}$

What to do:

1 First notice that we are multiplying a 3×5 and a 5×4 matrix.

So the result is a 3×4 matrix.

2 Enter the matrices on a spreadsheet.

For example:

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		1	3	2	5	6		2	3	1	2	
3		3	0	2	1	2		1	2	0	5	
4		0	1	3	2	4		1	0	0	1	
5								0	1	3	0	
6								4	5	2	7	
7												
8												
9												
10												
11												

3 Highlight the cells C8:F10 as shown and while these cells are highlighted type

$$=\text{MMULT}(\underbrace{\text{B2:F4}}_{\text{first matrix}},\underbrace{\text{H2:K6}}_{\text{second matrix}})$$

4 Enter the formula by holding down **CTRL** and **SHIFT** together and while doing this press **ENTER** .

EXERCISE 14F

1 Use technology to find:

a $\begin{bmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{bmatrix}$

b $\begin{bmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{bmatrix}$

c $22\begin{bmatrix} 1 & 0 & 6 & 8 & 9 \\ 2 & 7 & 4 & 5 & 0 \\ 8 & 2 & 4 & 4 & 6 \end{bmatrix}$

d $\begin{bmatrix} 2 & 6 & 0 & 7 \\ 3 & 2 & 8 & 6 \\ 1 & 4 & 0 & 2 \\ 3 & 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \\ 11 \end{bmatrix}$

Use technology to assist in solving the following problems:

2 For their holiday, Lars and Simke are planning to spend time at a popular tourist resort. They will need accommodation at one of the local motels and they are not certain how long they will stay. Their initial planning is for three nights and includes three breakfasts and two dinners. They have gathered prices from three different motels.

The Bay View has rooms at \$125 per night. A full breakfast costs \$22 per person (and therefore \$44 for them both). An evening meal for two usually costs \$75 including drinks.

By contrast, 'The Terrace' has rooms at \$150 per night, breakfast at \$40 per double and dinner costs on average \$80.

Things seem to be a little better at the Staunton Star Motel. Accommodation is \$140 per night, full breakfast (for two) is \$40, while an evening meal for two usually costs \$65.

a Write down a 'numbers' matrix as a 1×3 row matrix.

b Write down a 'prices' matrix in 3×3 form.

c Use matrix multiplication to establish total prices for each venue.

d Instead of the couple staying three nights, the alternative is to spend two nights. In that event Lars and Simke decide on having breakfast just once and one evening meal before moving on. Recalculate prices for each venue.

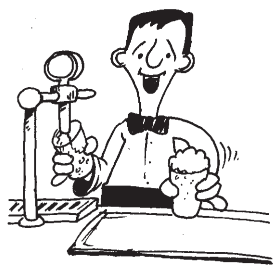
e Now remake the 'numbers' matrix (2×3) so that it includes both scenarios and recalculate the product with the 'prices' matrix.

3 A bus company runs four tours. Tour A costs \$125, Tour B costs \$315, Tour C costs \$405, and Tour D costs \$375. The numbers of clients they had over the period are shown in the table below.

$$\begin{array}{r} \\ \text{November} \\ \text{December} \\ \text{January} \\ \text{February} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Tour A} & \text{Tour B} & \text{Tour C} & \text{Tour D} \end{array} \\ \begin{bmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{bmatrix} \end{array}$$

Use the information and matrix methods to find the total income for the tour company.

4 A hotel mainly sells beer, wine, spirits and soft drinks. The number of these drinks sold during a week is shown in the table below.



	Mon	Tues	Wed	Thurs	Fri	Sat
Beer	225	195	215	240	352	321
Wine	75	62	50	92	80	97
Spirits	62	54	55	72	102	112
Soft drinks	95	60	68	85	115	146

Write the information in a suitable matrix.

The cost price per drink averages out as shown in the table below:

$$\text{Cost price (in \$)} \quad \begin{array}{c} \begin{array}{cccc} \text{Beer} & \text{Wine} & \text{Spirits} & \text{Soft drinks} \end{array} \\ \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix} \end{array}$$

The selling price for this data is:

$$\text{Selling price (in \$)} \quad \begin{array}{c} \begin{array}{cccc} \text{Beer} & \text{Wine} & \text{Spirits} & \text{Soft drinks} \end{array} \\ \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix} \end{array}$$

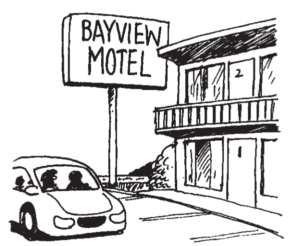
Use matrix methods to calculate the profit for the business for the week.

5 The Bay View Motel has three types of suites for guests.

Standard suites cost \$125 per night. They have 20 suites.

Deluxe suites cost \$195 per night. They have 15 suites.

Executive suites cost \$225 per night. They have 5 suites.



The rooms which are occupied also have a maintenance cost:

Standard suites cost \$85 per day to maintain.

Deluxe suites cost \$120 per day to maintain.

Executive suites cost \$130 per day to maintain.

The hotel has confirmed room bookings for the next week.

$$\begin{array}{l} \\ \text{Standard} \\ \text{Deluxe} \\ \text{Executive} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{M} & \text{T} & \text{W} & \text{Th} & \text{F} & \text{S} & \text{Su} \end{array} \\ \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

By using a process outlined below we can see that

(income from room) $\times$ (bookings/day) $-$ (maintenance cost/room) $\times$ (bookings/day)
$=$ (total expected income/day) $-$ (expected costs/day)
$=$ profit/day

a Create the matrices required to show how the profit per week can be found.

b How would the results alter if the hotel maintained (cleaned) all rooms every day? Show calculations.

c Produce a "profit per room" matrix and show how **a** could be done with a single matrix product.

G SOME PROPERTIES OF MATRIX MULTIPLICATION

In the following exercise we should discover the properties of 2×2 matrix multiplication which are like those of ordinary number multiplication, and those which are not.

EXERCISE 14G

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$ and in algebra $ab = ba$.

For matrices, is $\mathbf{AB} = \mathbf{BA}$ always?

Hint: Try $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$ say.

2 If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find $\mathbf{AO}$ and $\mathbf{OA}$.

3 For all real numbers a, b and c it is true that $a(b + c) = ab + ac$ and this is known as the **distributive law**.

a 'Make up' three 2×2 matrices **A**, **B** and **C** and verify that $\mathbf{A(B + C) = AB + AC}$.

b Now let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

and prove that in general $\mathbf{A(B + C) = AB + AC}$.

c Use the matrices you 'made up' in **a** to verify that $\mathbf{(AB)C = A(BC)}$

d As in **b** prove that $\mathbf{(AB)C = A(BC)}$

4 **a** If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ i.e., $\mathbf{AX = A}$, deduce that $w = z = 1$ and $x = y = 0$.

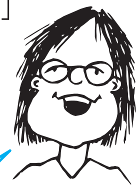
b For any real number a, it is true that $a \times 1 = 1 \times a = a$.
Is there a matrix **I**, say, such that $\mathbf{AI = IA = A}$ for all 2×2 matrices **A**?
[**Hint:** Use the results of **a** above.]

5 Suppose $\mathbf{A}^2 = \mathbf{AA}$, i.e., $\mathbf{A}$ multiplied by itself, and that $\mathbf{A}^3 = \mathbf{AAA}$.

a Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$ **b** Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$.

6 **a** If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ try to find $\mathbf{A}^2$.

b When can $\mathbf{A}^2$ be found, i.e., under what conditions can we square a matrix?



7 Show that if $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.

You should have discovered from the above exercise that:

Ordinary algebra	Matrix algebra
• If a and b are real numbers then so is ab.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix.
• $ab = ba$ for all a, b	• In general $\mathbf{AB} \neq \mathbf{BA}$.
• $a0 = 0a = 0$ for all a	• If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
• $a(b + c) = ab + ac$	• $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
• $a \times 1 = 1 \times a = a$	• If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$.
• a^n exists for all $a \geqslant 0$	• $\mathbf{A}^n$ for $n \geqslant 2$ can be determined provided that $\mathbf{A}$ is a square and n is an integer.

Example 7

Expand and simplify where possible:

a $(\mathbf{A} + 2\mathbf{I})^2$ **b** $(\mathbf{A} - \mathbf{B})^2$ {$\mathbf{I}$ is the identity matrix}

a
$$\begin{aligned}
&(\mathbf{A} + 2\mathbf{I})^2 \\
&= (\mathbf{A} + 2\mathbf{I})(\mathbf{A} + 2\mathbf{I}) && \{\mathbf{X}^2 = \mathbf{XX} \text{ by definition}\} \\
&= (\mathbf{A} + 2\mathbf{I})\mathbf{A} + (\mathbf{A} + 2\mathbf{I})2\mathbf{I} && \{\mathbf{B}(\mathbf{C} + \mathbf{D}) = \mathbf{BC} + \mathbf{BD}\} \\
&= \mathbf{A}^2 + 2\mathbf{IA} + 2\mathbf{AI} + 4\mathbf{I}^2 && \{\mathbf{B}(\mathbf{C} + \mathbf{D}) = \mathbf{BC} + \mathbf{BD} \text{ again, twice}\} \\
&= \mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{A} + 4\mathbf{I} && \{\mathbf{AI} = \mathbf{IA} = \mathbf{A} \text{ and } \mathbf{I}^2 = \mathbf{I}\} \\
&= \mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I}
\end{aligned}$$

b
$$\begin{aligned}
&(\mathbf{A} - \mathbf{B})^2 \\
&= (\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B}) && \{\mathbf{X}^2 = \mathbf{XX} \text{ by definition}\} \\
&= (\mathbf{A} - \mathbf{B})\mathbf{A} - (\mathbf{A} - \mathbf{B})\mathbf{B} && \{\mathbf{C}(\mathbf{D} - \mathbf{E}) = \mathbf{CD} - \mathbf{CE}\text{, three times}\} \\
&= \mathbf{A}^2 - \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2
\end{aligned}$$

Note: **b** cannot be simplified further as in general $\mathbf{AB} \neq \mathbf{BA}$.

8 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, explain and simplify:

a $\mathbf{A}(\mathbf{A}+\mathbf{I})$ **b** $(\mathbf{B}+2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2-2\mathbf{A}+\mathbf{I})$

d $\mathbf{A}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I})$ **e** $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$ **f** $(\mathbf{A}+\mathbf{B})^2$

g $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$ **h** $(\mathbf{A}+\mathbf{I})^2$ **i** $(3\mathbf{I}-\mathbf{B})^2$

Example 8

If $\mathbf{A}^2 = 2\mathbf{A}+3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars.

$\mathbf{A}^2 = 2\mathbf{A}+3\mathbf{I}$

$$\begin{aligned}\therefore\quad \mathbf{A}^3 &= \mathbf{A}\times\mathbf{A}^2\\ &= \mathbf{A}(2\mathbf{A}+3\mathbf{I})\\ &= 2\mathbf{A}^2+3\mathbf{A}\mathbf{I}\\ &= 2(2\mathbf{A}+3\mathbf{I})+3\mathbf{A}\mathbf{I}\\ &= 4\mathbf{A}+6\mathbf{I}+3\mathbf{A}\\ &= 7\mathbf{A}+6\mathbf{I}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\mathbf{A}^4 &= \mathbf{A}\times\mathbf{A}^3\\ &= \mathbf{A}(7\mathbf{A}+6\mathbf{I})\\ &= 7\mathbf{A}^2+6\mathbf{A}\mathbf{I}\\ &= 7(2\mathbf{A}+3\mathbf{I})+6\mathbf{A}\\ &= 14\mathbf{A}+21\mathbf{I}+6\mathbf{A}\\ &= 20\mathbf{A}+21\mathbf{I}\end{aligned}$$

9 **a** If $\mathbf{A}^2 = 2\mathbf{A}-\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars.

b If $\mathbf{B}^2 = 2\mathbf{I}-\mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$ and $\mathbf{B}^5$ in linear form.

c If $\mathbf{C}^2 = 4\mathbf{C}-3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

10 **a** If $\mathbf{A}^2 = \mathbf{I}$, simplify:

i $\mathbf{A}(\mathbf{A}+2\mathbf{I})$ **ii** $(\mathbf{A}-\mathbf{I})^2$ **iii** $\mathbf{A}(\mathbf{A}+3\mathbf{I})^2$

b If $\mathbf{A}^3 = \mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A}+\mathbf{I})^2$

c If $\mathbf{A}^2 = \mathbf{O}$, simplify:

i $\mathbf{A}(2\mathbf{A}-3\mathbf{I})$ **ii** $\mathbf{A}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-\mathbf{I})$ **iii** $\mathbf{A}(\mathbf{A}+\mathbf{I})^3$

11 The result "if $ab = 0$ then $a = 0$ or $b = 0$" for real numbers does not have an equivalent result for matrices.

a If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ find $\mathbf{AB}$.

This example provides us with evidence that
"If $\mathbf{AB} = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$" is a false statement.

b If $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ determine $\mathbf{A}^2$.

c Comment on the following argument for a 2×2 matrix $\mathbf{A}$:

It is known that $\mathbf{A}^2 = \mathbf{A}$, $\therefore\ \mathbf{A}^2 - \mathbf{A} = \mathbf{O}$

$\therefore\ \mathbf{A}(\mathbf{A}-\mathbf{I}) = \mathbf{O}$

$\therefore\ \mathbf{A} = \mathbf{O}$ or $\mathbf{A}-\mathbf{I} = \mathbf{O}$

$\therefore\ \mathbf{A} = \mathbf{O}$ or $\mathbf{I}$

d Find **all** 2×2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. [**Hint:** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.]

12 Give **one** example which shows that "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a *false* statement.

Example 9

Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,

$$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ 3a & 4a+b \end{bmatrix}$$

Thus $a + b = 7$ and $2a = 10$

$\therefore$ $a = 5$ and $b = 2$

Checking for consistency: $3a = 3(5) = 15$ ✓

$4a + b = 4(5) + (2) = 22$ ✓

13 Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to:

a $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$

14 If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$, find constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

a Hence, write $\mathbf{A}^3$ in linear form, $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

b Also write $\mathbf{A}^4$ in linear form.

H THE INVERSE OF A 2×2 MATRIX

We can solve $\left\{ \begin{array}{l} 2x + 3y = 4 \\ 5x + 4y = 17 \end{array} \right.$ algebraically to get $x = 5$, $y = -2$.

Notice that this system can be written as a matrix equation $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix}$.

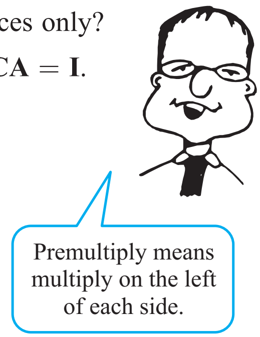
The solution $x = 5$, $y = -2$ is easily checked as

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 2(5) + 3(-2) \\ 5(5) + 4(-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix} \quad ✓$$

Notice that these matrix equations have form $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{X}$ is the unknown column matrix and $\mathbf{B}$ is the column matrix of constants.

The question arises: If $\mathbf{AX} = \mathbf{B}$, how can we find $\mathbf{X}$ using matrices only?

To answer this question, suppose there exists a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.



If we *pre-multiply* each side of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{C}$ we get

$$\begin{aligned} \mathbf{C(AX)} &= \mathbf{CB} \\ \therefore \quad \mathbf{(CA)X} &= \mathbf{CB} \\ \therefore \quad \mathbf{IX} &= \mathbf{CB} \\ \text{and so} \quad \mathbf{X} &= \mathbf{CB} \end{aligned}$$

If it exists, we will call $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$, the **multiplication inverse** of $\mathbf{A}$ and we will use the notation $\mathbf{C} = \mathbf{A}^{-1}$.

In general, the **multiplication inverse of A**, if it exists, satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$.

Notice that $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

Notice also that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\mathbf{I}$

and that $\begin{bmatrix} 5 & 11 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -11 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 37 & 0 \\ 0 & 37 \end{bmatrix} = 37\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 37\mathbf{I}$

Notice that all answers are scalar multiples of $\mathbf{I}$.

These results suggest that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = k\mathbf{I}$ for some scalar k.

On expanding this product $\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = k\mathbf{I}$

$$\therefore \quad (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k\mathbf{I}$$

$$\text{i.e.,} \quad (ad-bc)\mathbf{I} = k\mathbf{I}$$

$$\text{and so} \quad k = ad-bc.$$

Consequently, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

which suggests that if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

We also notice that $\mathbf{A}^{-1}$ exists provided $ad-bc \neq 0$, otherwise $\dfrac{1}{ad-bc}$ would be undefined. If $ad-bc \neq 0$, we say that $\mathbf{A}$ is **invertible**.

So the value of $ad-bc$ *determines* whether or not a 2×2 matrix has an inverse.

Consequently, for $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the value of $ad-bc$ is called the **determinant** of $\mathbf{A}$.

It is denoted by $|\mathbf{A}|$ or $\det\mathbf{A}$.

$\mathbf{A}$ has an **inverse** if $|\mathbf{A}| \neq 0$.

Hence, if $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1}=\dfrac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $|\mathbf{A}|=ad-bc$.

EXERCISE 14H

1 **a** Find $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$.

b Find $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$.

c Find $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}$.

2 Find $|\mathbf{A}|$ for $\mathbf{A}$ equal to:

a $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 Find $\det\mathbf{B}$ for $\mathbf{B}$ equal to:

a $\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} a & -a \\ 1 & a \end{bmatrix}$

4 Find the following determinants for $\mathbf{A}=\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$.

a $|\mathbf{A}|$ **b** $|\mathbf{A}|^2$ **c** $|2\mathbf{A}|$

5 Prove that, if $\mathbf{A}$ is any 2×2 matrix and k is a constant, then $|k\mathbf{A}|=k^2\,|\mathbf{A}|$.

6 By letting $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$

a find $|\mathbf{A}|$ and $|\mathbf{B}|$

b find $\mathbf{AB}$ and $|\mathbf{AB}|$, and hence

c show that $|\mathbf{AB}|=|\mathbf{A}|\,|\mathbf{B}|$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

7 $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

a Using the results of **5** and **6** above and the calculated values of $|\mathbf{A}|$ and $|\mathbf{B}|$, find:

i $|\mathbf{A}|$ **ii** $|2\mathbf{A}|$ **iii** $|-\mathbf{A}|$ **iv** $|-3\mathbf{B}|$ **v** $|\mathbf{AB}|$

b Check your answers without using the results of **5** and **6** above.

8 Find, if it exists, the inverse matrix of:

a $\begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

e $\begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}$ **f** $\begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix}$ **g** $\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ **h** $\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$

I SOLVING A PAIR OF LINEAR EQUATIONS

Example 10

a If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ find $|\mathbf{A}|$.

b Does $\begin{cases} 2x + y = 4 \\ 3x + 4y = -1 \end{cases}$ have a unique solution?

a $$\begin{aligned} |\mathbf{A}| &= 2(4) - 1(3) \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

b The system in matrix form is

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Now as $|\mathbf{A}| = 5 \neq 0$, $\mathbf{A}^{-1}$ exists and so we can solve for x and y

$\therefore$ the system has a unique solution.

Example 11

If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, find $\mathbf{A}^{-1}$ and hence solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$.

Now in matrix form the system is:

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix}$$

i.e., $\mathbf{AX} = \mathbf{B}$ where $|\mathbf{A}| = 8 - 15 = -7$

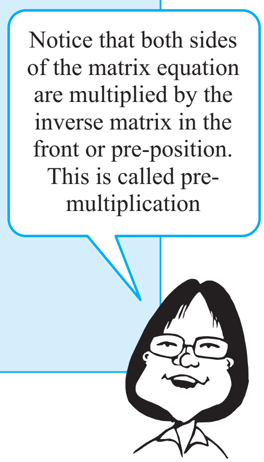
Now $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$$\therefore \quad \mathbf{IX} = \tfrac{1}{-7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 17 \end{bmatrix}$$

$$\therefore \quad \mathbf{X} = \tfrac{1}{-7}\begin{bmatrix} -35 \\ 14 \end{bmatrix}$$

$$\therefore \quad \mathbf{X} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\therefore \quad x = 5, \quad y = -2$$



EXERCISE 14I

1 Perform the following products:

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

2 Convert into matrix equations:

a $3x - y = 8$
$2x + 3y = 6$

b $4x - 3y = 11$
$3x + 2y = -5$

c $3a - b = 6$
$2a + 7b = -4$

3 Use matrix algebra to solve the system:

a $2x - y = 6$
$x + 3y = 14$

b $5x - 4y = 5$
$2x + 3y = -13$

c $x - 2y = 7$
$5x + 3y = -2$

d $3x + 5y = 4$
$2x - y = 11$

e $4x - 7y = 8$
$3x - 5y = 0$

f $7x + 11y = 18$
$11x - 7y = -11$

4 **a** Show that if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ whereas if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{BA}^{-1}$.

b Find $\mathbf{X}$ if:

i $\mathbf{X}\begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}$

ii $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

Example 12

Find $\mathbf{A}^{-1}$ when $\mathbf{A} = \begin{bmatrix} 4 & k \\ 2 & -1 \end{bmatrix}$ and state k when $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \frac{1}{-4-2k}\begin{bmatrix} -1 & -k \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2k+4} & \dfrac{k}{2k+4} \\[2ex] \dfrac{2}{2k+4} & \dfrac{-4}{2k+4} \end{bmatrix}$$

So $\mathbf{A}^{-1}$ exists provided that $2k + 4 \neq 0$, i.e., $k \neq -2$.

5 Find $\mathbf{A}^{-1}$ and state k when $\mathbf{A}^{-1}$ exists if:

a $\mathbf{A} = \begin{bmatrix} k & 1 \\ -6 & 2 \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & k \end{bmatrix}$

c $\mathbf{A} = \begin{bmatrix} k+1 & 2 \\ 1 & k \end{bmatrix}$

6 **a** If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{bmatrix}$, find $\mathbf{AB}$.

b Does your result in **a** imply that $\mathbf{A}$ and $\mathbf{B}$ are inverses? [**Hint:** Find $\mathbf{BA}$.]
The above example illustrates that only square matrices can have inverses. Why?

7 Given $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, find $\mathbf{X}$ if $\mathbf{AXB} = \mathbf{C}$.

8 **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11 \end{cases}$

i Write the equations in the form $\mathbf{AX} = \mathbf{B}$ and find $|\mathbf{A}|$.

ii Does the system have a unique solution? If so, find it.

b Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11 \end{cases}$

i Write the system in the form $\mathbf{AX} = \mathbf{B}$ and find $|\mathbf{A}|$.

ii For what value(s) of k does the system have a unique solution? Find the unique solution.

iii Find k when the system does not have a unique solution. How many solutions does it have in this case?

9 If a matrix $\mathbf{A}$ is its own inverse, then $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

then $\mathbf{A}^{-1} = \dfrac{1}{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \mathbf{A}$.

a Show that, if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.

b If $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse, show that there are exactly 4 matrices of this form.

10 **a** If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ find $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$.

b If $\mathbf{A}$ is any square matrix which has inverse $\mathbf{A}^{-1}$, simplify $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing $\mathbf{A}^{-1}$ by $\mathbf{B}$.

c What can be deduced from **b**?

11 **a** If $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$ find in simplest form:

i $\mathbf{A}^{-1}$ **ii** $\mathbf{B}^{-1}$ **iii** $(\mathbf{AB})^{-1}$

iv $(\mathbf{BA})^{-1}$ **v** $\mathbf{A}^{-1}\mathbf{B}^{-1}$ **vi** $\mathbf{B}^{-1}\mathbf{A}^{-1}$

b Choose any two invertible matrices and repeat question **a**.

c What do the results of **a** and **b** suggest?

d Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. Conclusion?

12 If k is a non-zero number, and $\mathbf{A}^{-1}$ exists, simplify $(k\mathbf{A})(\frac{1}{k}\mathbf{A}^{-1})$ and $(\frac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.
What conclusion follows from your results?

13 If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$ where $\mathbf{A}$ and $\mathbf{B}$ are invertible, find:

a $\mathbf{X}$ in terms of $\mathbf{Z}$ **b** $\mathbf{Z}$ in terms of $\mathbf{X}$.

[$\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are 2×1 and $\mathbf{A}$, $\mathbf{B}$ are 2×2]

Example 13

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^{-1}$ in linear form $r\mathbf{A} + s\mathbf{I}$, where r and s are scalars.

$$\begin{aligned}
\mathbf{A}^2 &= 2\mathbf{A} + 3\mathbf{I} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(2\mathbf{A} + 3\mathbf{I}) \qquad \{\text{premultiply both sides by } \mathbf{A}^{-1}\} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} + 3\mathbf{A}^{-1}\mathbf{I} \\
\therefore\ \mathbf{I}\mathbf{A} &= 2\mathbf{I} + 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A} - 2\mathbf{I} &= 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A}^{-1} &= \tfrac{1}{3}(\mathbf{A} - 2\mathbf{I}) \\
\text{i.e., } \mathbf{A}^{-1} &= \tfrac{1}{3}\mathbf{A} - \tfrac{2}{3}\mathbf{I}
\end{aligned}$$

14 Find $\mathbf{A}^{-1}$ in linear form given that

a $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$ **b** $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$ **c** $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$

15 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$, write $\mathbf{A}^2$ in the form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars.

Hence write $\mathbf{A}^{-1}$ in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

16 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices $\mathbf{A}$ and $\mathbf{B}$ are not necessarily invertible.

Prove that $\mathbf{A}^2 = \mathbf{A}$. [**Note:** From $\mathbf{AB} = \mathbf{A}$, you cannot deduce that $\mathbf{B} = \mathbf{I}$. Why?]

17 Under what condition is it true that "if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$"?

18 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.

19 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

J THE 3×3 DETERMINANT

The **determinant** of $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is defined as

$$|\mathbf{A}| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

The form $\begin{vmatrix} a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{vmatrix}$ may be useful.

Example 14

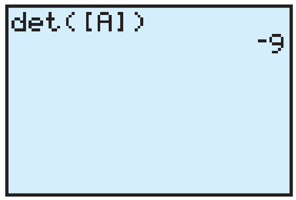
Find $|\mathbf{A}|$ for $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$

$$|\mathbf{A}| = 1\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

same same same

$$\begin{aligned} &= 1(0 - -1) + 2(3-4) + 4(-2-0) \\ &= 1 - 2 - 8 \\ &= -9 \qquad \{\text{which checks with the earlier example}\} \end{aligned}$$

Once again we observe that a 3×3 system of linear equations in matrix form $\mathbf{AX} = \mathbf{B}$ will have a **unique solution** if $|\mathbf{A}| \neq 0$.



Note: A graphics calculator or spreadsheet can be used to find the value of a determinant.

EXERCISE 14J

1 Evaluate:

a $\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix}$ **b** $\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix}$ **c** $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$

d $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **e** $\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ **f** $\begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix}$

g $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \end{vmatrix}$ **h** $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 2 & 6 & 4 \end{vmatrix}$ **i** $\begin{vmatrix} 0 & 3 & 0 \\ 1 & 2 & 5 \\ 6 & 0 & 1 \end{vmatrix}$

2 Evaluate:

a $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$ **b** $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$ **c** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

3 For what values of k does $\begin{cases} x + 2y - 3z = 5 \\ 2x - y - z = 8 \\ kx + y + 2z = 14 \end{cases}$ have a unique solution?

4 For what values of k does $\begin{cases} 2x - y - 4z = 8 \\ 3x - ky + z = 1 \\ 5x - y + kz = -2 \end{cases}$ have a unique solution?

5 Find k given that:

a $\begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7$ **b** $\begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{vmatrix} = 0$

6 Use technology to find the determinant of:

a $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{bmatrix}$

7 In MSEXCEL, = MDETERM( ) gives the determinant and = MINVERSE( ) gives the inverse matrix.

A7 = {=MINVERSE(A1:D4)}

	A	B	C	D	E	F
1	2	3	1	3		
2	4	4	3	2		determinant
3	3	1	1	0		4
4	1	2	2	1		
5						
6						
7	0.25	-0.5	0.75	0.25		
8	-1.25	3.5	-2.75	-3.25		
9	0.5	-2	1.5	2.5		
10	1.25	-2.5	1.75	2.25		

the inverse of $\begin{bmatrix} 2 & 3 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

is $\begin{bmatrix} \frac{1}{4} & -\frac{2}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{14}{4} & -\frac{11}{4} & -\frac{13}{4} \\ \frac{2}{4} & -\frac{8}{4} & \frac{6}{4} & \frac{10}{4} \\ \frac{5}{4} & -\frac{10}{4} & \frac{7}{4} & \frac{9}{4} \end{bmatrix}$

Find the determinant and inverse matrix of:

a $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 3 & 1 \\ 0 & 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$

8 If Jan bought one orange, two apples, a pear, a cabbage and a lettuce the total cost would be \$6.30. Two oranges, one apple, two pears, one cabbage and one lettuce would cost a total of \$6.70. One orange, two apples, three pears, one cabbage and one lettuce would cost a total of \$7.70. Two oranges, two apples, one pear, one cabbage and three lettuces would cost a total of \$9.80. Three oranges, three apples, five pears, two cabbages and two lettuces would cost a total of \$10.90.

a Write this information in $\mathbf{AX} = \mathbf{B}$ form where $\mathbf{A}$ is the quantities matrix, $\mathbf{X}$ is the cost per item column matrix and $\mathbf{B}$ is the total costs column matrix.

b Explain why $\mathbf{X}$ cannot be found from the given information.

c If the last lot of information is deleted and in its place "three oranges, one apple, two pears, two cabbages and one lettuce cost a total of \$9.20" is substituted, can the system be solved now, and if so, what is the solution?

Note: If a solution can be found use $\mathbf{A}^{-1}$.

K THE INVERSE OF A 3×3 MATRIX

We will use a graphics calculator to find the inverse of a 3×3 matrix.

For example, if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ what is $\mathbf{A}^{-1}$?

TI
C

On a calculator we set matrix $\mathbf{A}$ to be $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$.

We then find $\mathbf{A}^{-1}$ using appropriate keys.

You may obtain $\mathbf{A}^{-1} = \begin{bmatrix} -0.1111 & 0.8888 & -0.2222 \\ 0.1111 & 1.1111 & -0.7777 \\ 0.2222 & -0.7777 & 0.4444 \end{bmatrix}$

You will have to find how to convert each element to a fraction.

In fact, $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{9} & \frac{8}{9} & -\frac{2}{9} \\ \frac{1}{9} & \frac{10}{9} & -\frac{7}{9} \\ \frac{2}{9} & -\frac{7}{9} & \frac{4}{9} \end{bmatrix}$

EXERCISE 14K

1 Use your graphics calculator to find $\mathbf{A}^{-1}$ for:

a $\mathbf{A} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}$

2 Find $\mathbf{B}^{-1}$ for:

a $\mathbf{B} = \begin{bmatrix} 13 & 43 & -11 \\ 16 & 9 & 27 \\ -8 & 31 & -13 \end{bmatrix}$

b $\mathbf{B} = \begin{bmatrix} 1.61 & 4.32 & 6.18 \\ 0.37 & 6.02 & 9.41 \\ 7.12 & 5.31 & 2.88 \end{bmatrix}$

L 3×3 SYSTEMS WITH UNIQUE SOLUTIONS

EXERCISE 14L

1 Write as a matrix equation:

a $\begin{aligned} x - y - z &= 2 \\ x + y + 3z &= 7 \\ 9x - y - 3z &= -1 \end{aligned}$

b $\begin{aligned} 2x + y - z &= 3 \\ y + 2z &= 6 \\ x - y + z &= 13 \end{aligned}$

c $\begin{aligned} a + b - c &= 7 \\ a - b + c &= 6 \\ 2a + b - 3c &= -2 \end{aligned}$

2 Show that $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ are inverses of each other.

3 For $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}$,

calculate $\mathbf{AB}$ and hence solve the system of equations
$$\begin{aligned} 4a + 7b - 3c &= -8 \\ -a - 2b + c &= 3 \\ 6a + 12b - 5c &= -15. \end{aligned}$$

4 For $\mathbf{M} = \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$,

calculate $\mathbf{MN}$ and hence solve the system
$$\begin{aligned} 3u + 2v + 3w &= 18 \\ u - v + 2w &= 6 \\ 2u + v + 3w &= 16. \end{aligned}$$

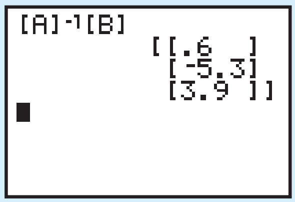
Example 15

Solve the system
$$\begin{aligned} x - y - z &= 2 \\ x + y + 3z &= 7 \\ 9x - y - 3z &= -1 \end{aligned}$$
using matrix methods and a graphics calculator.

In matrix form the system is:

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix} \quad \text{(i.e., } \mathbf{AX} = \mathbf{B}\text{)}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$



Into a calculator we enter $\mathbf{A}$ and $\mathbf{B}$ and calculate $[\mathbf{A}]^{-1}\,[\mathbf{B}]$

i.e., $x = 0.6$, $y = -5.3$, $z = 3.9$

5 Use matrix methods and technology to solve:

a $\begin{aligned} 3x + 2y - z &= 14 \\ x - y + 2z &= -8 \\ 2x + 3y - z &= 13 \end{aligned}$

b $\begin{aligned} x - y - 2z &= 4 \\ 5x + y + 2z &= -6 \\ 3x - 4y - z &= 17 \end{aligned}$

c $\begin{aligned} x + 3y - z &= 15 \\ 2x + y + z &= 7 \\ x - y - 2z &= 0 \end{aligned}$

6 Use your graphics calculator to solve:

a $\begin{aligned} x + y + z &= 6 \\ 2x + 4y + z &= 5 \\ 2x + 3y + z &= 6 \end{aligned}$

b $\begin{aligned} x + 4y + 11z &= 7 \\ x + 6y + 17z &= 9 \\ x + 4y + 8z &= 4 \end{aligned}$

c $\begin{aligned} 2x - y + 3z &= 17 \\ 2x - 2y - 5z &= 4 \\ 3x + 2y + 2z &= 10 \end{aligned}$

d $\begin{aligned} x + 2y - z &= 23 \\ x - y + 3z &= -23 \\ 7x + y - 4z &= 62 \end{aligned}$

e $\begin{aligned} 10x - y + 4z &= -9 \\ 7x + 3y - 5z &= 89 \\ 13x - 17y + 23z &= -309 \end{aligned}$

f $\begin{aligned} 1.3x + 2.7y - 3.1z &= 8.2 \\ 2.8x - 0.9y + 5.6z &= 17.3 \\ 6.1x + 1.4y - 3.2z &= -0.6 \end{aligned}$

Example 16

Rent-a-car has three different makes of vehicles, P, Q and R, for hire. These cars are located at yards A and B on either side of a city. Some cars are out (being rented). In total they have 150 cars. At yard A they have 20% of P, 40% of Q and 30% of R which is 46 cars in total. At yard B they have 40% of P, 20% of Q and 50% of R which is 54 cars in total. How many of each car type does Rent-a-car have?

Suppose Rent-a-car has x of P, y of Q and z of R.

Then as it has 150 cars in total, $x + y + z = 150$ ...... (1)

But yard A has 20% of P + 40% of Q + 30% of R and this is 46.

$$\therefore \quad \tfrac{2}{10}x + \tfrac{4}{10}y + \tfrac{3}{10}z = 46$$

$$\text{i.e.,} \quad 2x + 4y + 3z = 460 \quad \text{......} \ (2)$$

And, yard B has 40% of P + 20% of Q + 50% of R and this is 54.

$$\therefore \quad \tfrac{4}{10}x + \tfrac{2}{10}y + \tfrac{5}{10}z = 54$$

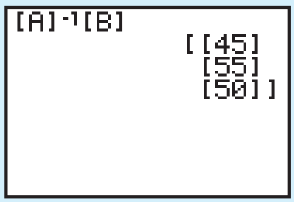
$$\text{i.e.,} \quad 4x + 2y + 5z = 540 \quad \text{......} \ (3)$$

We need to solve the system
$$\begin{aligned} x + y + z &= 150 \\ 2x + 4y + 3z &= 460 \\ 4x + 2y + 5z &= 540 \end{aligned}$$

$$\text{i.e.,} \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 150 \\ 460 \\ 540 \end{bmatrix}$$

$$\text{So} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 150 \\ 460 \\ 540 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 55 \\ 50 \end{bmatrix}$$



Thus, Rent-a-car has
45 of P, 55 of Q and 50 of R.

7 **a** Solve the system of equations
$$\begin{aligned} 2x + y + 3z &= 90 \\ 3x + 2y + z &= 81 \\ 5x + 2z &= 104. \end{aligned}$$

b Westfield School bought two cricket balls, one softball and three netballs for a total cost of \$90. Southvale School bought three cricket balls, two softballs and a netball for \$81. Eastside School bought five cricket balls and two netballs for \$104.

  i State clearly what the variables x, y and z must represent if this situation is to be described by the set of equations considered in **a**.

  ii If Northtown High School needs 4 cricket balls, 5 softballs and wishes to order as many netballs as they can afford, how many netballs will they be able to purchase if there is a total of \$315 to be spent?

8 Managers, clerks and labourers are paid according to an industry award.

Xenon employs 2 managers, 3 clerks and 8 labourers with a total salary bill of \$352 000.
Xanda employs 1 manager, 5 clerks and 4 labourers with a total salary bill of \$274 000.
Xylon employs 1 manager, 2 clerks and 11 labourers with a total salary bill of \$351 000.

- **a** If x, y and z represent the salaries (in thousands of dollars) for managers, clerks and labourers respectively, show that the above information can be represented by a system of three equations.
- **b** Solve the above system of equations.
- **c** Determine the total salary bill for Xulu company which employs 3 managers, 8 clerks and 37 labourers.

9 Herbert and Agnes had plotted three points on the graph of a quadratic function. Unfortunately, they forgot the original function and were unable to plot any more points. Given that the points were $(1, -3)$, $(3, -5)$ and $(-2, -15)$, can you help the two poor students complete the table of values below?

x	-3	-2	-1	0	1	2	3
y		-15			-3		-5

10 A mixed nut company uses cashews, macadamias and brazil nuts to make three gourmet mixes. The table along side indicates the weight in hundreds of grams of each kind of nut required to make a kilogram of mix.

	Mix A	Mix B	Mix C
Cashews	5	2	6
Macadamias	3	4	1
Brazil Nuts	2	4	3

If 1 kg of mix A costs \$12.50 to produce, 1 kg of mix B costs \$12.40 and 1 kg of mix C costs \$11.70, determine the cost per kilogram of each of the different kinds of nuts.

Hence, find the cost per kilogram to produce a mix containing 400 grams of cashews, 200 grams of macadamias and 400 grams of brazil nuts.

11 Klondike High has 76 students at Matriculation level and these students are in classes P, Q and R. There are p students in P, q in Q and r in R.

One-third of P, one-third of Q and two-fifths of R study Chemistry.

One-half of P, two-thirds of Q and one-fifth of R study Maths.

One-quarter of P, one-third of Q and three-fifths of R study Geography.

Given that 27 study Chemistry, 35 study Maths and 30 study Geography:

- **a** find a system of equations which contains this information, making sure that the coefficients of p, q and r are integers.
- **b** Solve for p, q and r.

12 Susan and James opened a new business in 1997. Their annual profit was \$160 000 in 2000, \$198 000 in 2001 and \$240 000 in 2002. Based on the information from these three years they believe that their annual profit could be predicted by the model

$$P(t) = at + b + \frac{c}{t+4} \text{ dollars}$$

where t is the number of years after 2000, i.e., $t = 0$ gives 2000 profit.

a Determine the values of a, b and c which fit the profits for 2000, 2001 and 2002.

b If the profit in 1999 was \$130 000, does this profit fit the model in **a**?

c Susan and James believe their profit will continue to grow according to this model. Predict their profit in 2003 and 2005.

INVESTIGATION — USING MATRICES IN CRYPTOGRAPHY

PORTFOLIO OPPORTUNITY

Cryptography is the study of encoding and decoding messages. Cryptography was first developed to send secret messages in written form. However, today it is used to maintain privacy when information is being transmitted via public communication services (by line or by satellite).

Messages are sent in **code** or **cipher** form. The method of converting text to ciphertext is called **enciphering** and the reverse process is called **deciphering**.

The operations of matrix addition and multiplication can be used to create codes and the coded messages are transmitted. Decoding using additive or multiplicative inverses is required by the receiver in order to read the message.

The letters of the alphabet are first assigned integer values.

Notice that Z is assigned 0.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	0

The coded form of the word SEND is therefore 19 5 14 4 which we could put in 2×2 matrix form $\begin{bmatrix} 19 & 5 \\ 14 & 4 \end{bmatrix}$.

An encoding matrix of your choice could be added to this matrix. Suppose it is $\begin{bmatrix} 2 & 7 \\ 13 & 5 \end{bmatrix}$.

The matrix to be transmitted is then $\begin{bmatrix} 19 & 5 \\ 14 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 12 \\ 27 & 9 \end{bmatrix}$

Now $\begin{bmatrix} 21 & 12 \\ 27 & 9 \end{bmatrix}$ becomes $\begin{bmatrix} 21 & 12 \\ 1 & 9 \end{bmatrix}$ as any number not in the range 0 to 25 is adjusted to be in it by adding or subtracting multiples of 26.

So, $\begin{bmatrix} 21 & 12 \\ 1 & 9 \end{bmatrix}$ is sent as 21 12 1 9.

The message SEND MONEY PLEASE could be broken into groups of four letters and each group is encoded.

SEND|MONE|YPLE|ASEE ⟵ repeat the last letter to make group of 4. This is a dummy letter.

For MONE the matrix required is $\begin{bmatrix} 13 & 15 \\ 14 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 22 \\ 27 & 10 \end{bmatrix}$ i.e., $\begin{bmatrix} 15 & 22 \\ 1 & 10 \end{bmatrix}$

For YPLE the matrix required is $\begin{bmatrix} 25 & 16 \\ 12 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 23 \\ 25 & 10 \end{bmatrix}$ i.e., $\begin{bmatrix} 1 & 23 \\ 25 & 10 \end{bmatrix}$

For ASEE the matrix required is $\begin{bmatrix} 1 & 19 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 26 \\ 18 & 10 \end{bmatrix}$ i.e., $\begin{bmatrix} 3 & 0 \\ 18 & 10 \end{bmatrix}$

So the whole message is 21 12 1 9 15 22 1 10 1 23 25 10 3 0 18 10

The decoder requires the additive inverse matrix $\begin{bmatrix} -2 & -7 \\ -13 & -5 \end{bmatrix}$ to decode the message.

What to do:

1 Use the decoder matrix to check that the original message is obtained.

2 Use the code given to decode the message:

21	12	1	9	22	15	18	25	20	22	2	21	21	1	2
25	10	12	0	20	23	1	21	20	8	1	21	10	15	2
5	23	3	6	12	4									

3 Create your own matrix addition code. Encode a short message. Supply the decoding matrix to a friend so that he/she can decode it.

4 Breaking codes where matrix multiplication is used is much more difficult.

A chosen encoder matrix is required. Suppose it is $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

The word SEND is encoded as $\begin{bmatrix} 19 & 5 \\ 14 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 43 & 67 \\ 32 & 50 \end{bmatrix}$

which is converted to $\begin{bmatrix} 17 & 15 \\ 6 & 24 \end{bmatrix}$

- **a** What is the coded form of SEND MONEY PLEASE?
- **b** What decoder matrix needs to be supplied to the receiver so that the message can be read?
- **c** Check by decoding the message.
- **d** Create your own code using matrix multiplication using a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc = 1$. Why?
- **e** What are the problems in using a 2×2 matrix when $ad - bc \neq 1$? How can these problems be overcome?

5 Research **Hill ciphers** and explain how they differ from the methods given previously.

REVIEW SET 14A

1 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$ find:

- **a** $\mathbf{A} + \mathbf{B}$
- **b** $3\mathbf{A}$
- **c** $-2\mathbf{B}$
- **d** $\mathbf{A} - \mathbf{B}$
- **e** $\mathbf{B} - 2\mathbf{A}$
- **f** $3\mathbf{A} - 2\mathbf{B}$
- **g** $\mathbf{AB}$
- **h** $\mathbf{BA}$
- **i** $\mathbf{A}^{-1}$
- **j** $\mathbf{A}^2$
- **k** $\mathbf{ABA}$
- **l** $(\mathbf{AB})^{-1}$

2 Find a, b, c and d if:

a $\begin{bmatrix} a & b-2 \\ c & d \end{bmatrix} = \begin{bmatrix} -a & 3 \\ 2-c & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 2a \\ b & -2 \end{bmatrix} + \begin{bmatrix} b & -a \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2 \\ 2 & 6 \end{bmatrix}$

3 Make $\mathbf{Y}$ the subject of:

a $\mathbf{B} - \mathbf{Y} = \mathbf{A}$ **b** $2\mathbf{Y} + \mathbf{C} = \mathbf{D}$ **c** $\mathbf{AY} = \mathbf{B}$

d $\mathbf{YB} = \mathbf{C}$ **e** $\mathbf{C} - \mathbf{AY} = \mathbf{B}$ **f** $\mathbf{AY}^{-1} = \mathbf{B}$

4 Solve using matrix methods:

a $3x - 4y = 2$, $5x + 2y = -1$ **b** $4x - y = 5$, $2x + 3y = 9$

c $\mathbf{X}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{X}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

5 If $\mathbf{A}$ is $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{B}$ is $\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$ find, if possible:

a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

6 For $\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$ find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{Q} - \mathbf{P}$ **c** $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

REVIEW SET 14B

1 What is the 2×2 matrix which when multiplied by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ gives an answer of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$? **Hint:** Let the matrix be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2 $\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Find, if possible:

a $\mathbf{AB}$ **b** $\mathbf{BA}$ **c** $\mathbf{AC}$ **d** $\mathbf{CA}$ **e** $\mathbf{CB}$

3 Find, if they exist, the inverse matrices of each of the following:

a $\begin{bmatrix} 6 & 8 \\ 5 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ **c** $\begin{bmatrix} 11 & 5 \\ -6 & -3 \end{bmatrix}$

4 A café sells two types of cola drinks. The drinks each come in three sizes: small, medium and large. At the beginning of the day the fridge was stocked with the number of units of each as shown in the matrix below. At the end of the day the stock was again counted. The results are shown below:

Start of the day

$$\begin{array}{l c} & \begin{array}{cc}\text{Brand C} & \text{Brand P}\end{array} \\ \begin{array}{l}\text{small} \longrightarrow \\ \text{medium} \longrightarrow \\ \text{large} \longrightarrow\end{array} & \begin{bmatrix} 42 & 54 \\ 36 & 27 \\ 34 & 30 \end{bmatrix}\end{array}$$

At the end of the day

$$\begin{array}{l c} & \begin{array}{cc}\text{Brand C} & \text{Brand P}\end{array} \\ \begin{array}{l}\text{small} \longrightarrow \\ \text{medium} \longrightarrow \\ \text{large} \longrightarrow\end{array} & \begin{bmatrix} 27 & 31 \\ 28 & 15 \\ 28 & 22 \end{bmatrix}\end{array}$$

The profit matrix is profit $\begin{array}{c}\begin{array}{ccc}\text{small} & \text{medium} & \text{large}\end{array} \\ \begin{bmatrix} \$0.75 & \$0.55 & \$1.20 \end{bmatrix}\end{array}$

Use matrix methods to calculate the profit made for the day from the sale of these drinks.

5 **a** If $\mathbf{A} = 2\mathbf{A}^{-1}$, show that $\mathbf{A}^2 = 2\mathbf{I}$.

b If $\mathbf{A} = 2\mathbf{A}^{-1}$, simplify $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I})$ giving your answer in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are real numbers.

6 If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, find $\mathbf{AB}$ and $\mathbf{BA}$ and hence find $\mathbf{A}^{-1}$ in terms of $\mathbf{B}$.

REVIEW SET 14C

1 When does the system $\begin{aligned} kx + 3y &= -6 \\ x + (k+2)y &= 2 \end{aligned}$ have a unique solution?

Comment on the solutions for the non-unique cases.

2 Find x if $\begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} = 0$, given that x is real.

3 If $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, evaluate if possible:

a $2\mathbf{A} - 2\mathbf{B}$ **b** $\mathbf{AC}$ **c** $\mathbf{CB}$ **d** $\mathbf{D}$, given that $\mathbf{DA} = \mathbf{B}$.

4 Find $\mathbf{X}$ if $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$.

5 Write $5\mathbf{A}^2 - 6\mathbf{A} = 3\mathbf{I}$ in the form $\mathbf{AB} = \mathbf{I}$ and hence find $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

6 **a** Under what conditions are the following true? ($\mathbf{A}$, $\mathbf{B}$ square)

i If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$

ii $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

b If $\mathbf{M} = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}\begin{bmatrix} k-1 & -2 \\ -3 & k \end{bmatrix}$ has an inverse $\mathbf{M}^{-1}$, what values can k have?

REVIEW SET 14D

1 Solve the system
$$\begin{aligned} 3x - y + 2z &= 8 \\ 2x + 3y - z &= -3 \\ x - 2y + 3z &= 9 \end{aligned}$$

2 Prove that $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$.

3 If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$, find $\mathbf{A}^3$, $\mathbf{A}^4$, $\mathbf{A}^5$, $\mathbf{A}^6$ in the form $r\mathbf{A} + s\mathbf{I}$.

4 The cost of producing x hundred bottles of correcting fluid per day is given by the function $C(x) = ax^3 + bx^2 + cx + d$ dollars where a, b, c and d are constants.

a If it costs \$80 before any bottles are produced, find d.

b It costs \$100 to produce 100 bottles, \$148 to produce 200 bottles and \$376 to produce 400 bottles per day. Determine a, b and c.

5 If $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -12 & -11 \\ -10 & -1 \end{bmatrix}$, find $\mathbf{X}$ if $\mathbf{AXB} = \mathbf{C}$.

6 Find the solution set of the following:

a
$$\begin{aligned} 2x + y - z &= 9 \\ 3x + 2y + 5z &= 19 \\ x + y - 3z &= 1 \end{aligned}$$

b
$$\begin{aligned} 2x + y - z &= 3 \\ 3x + 2y + z &= 1 \\ x - 3y &= 5 \end{aligned}$$

REVIEW SET 14E

1 Hung, Quan and Ariel bought tickets for three separate performances. The table below shows the number of tickets bought by each person.

	Opera	*Play*	*Concert*
Hung	3	2	5
Quan	2	3	1
Ariel	1	5	4

a If the total cost for Hung was \$267, for Quan \$145 and for Ariel \$230, represent this information in the form of three equations.

b Find the cost per ticket for each of the performances.

c Determine how much it would cost Phuong to purchase 4 opera, 1 play and 2 concert tickets.

2 Solve the system of equations
$$\begin{aligned} 2x + y + z &= 8 \\ 4x - 7y + 3z &= 10 \\ 3x - 2y - z &= 1 \end{aligned}$$

3 If $\mathbf{A} = \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$ find, if possible:

a $3\mathbf{A}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$ **d** $\mathbf{AC}$ **e** $\mathbf{BC}$

4 If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$ show by calculation that
$\det(\mathbf{AB}) = \det\mathbf{A}.\det\mathbf{B} = 80$.

5 A rock thrown upwards from the top of a cliff followed a path such that its distance above sea level was given by $s(t) = at^2 + bt + c$, where t is the time in seconds after the rock is released. After 1 second the rock was 63 m above sea level, after 2 seconds 72 m and after 7 seconds 27 m.

a Find a, b and c and hence an expression for $s(t)$.

b Find the height of the cliff.

c Find the time taken for the rock to reach sea level.

6 A matrix $\mathbf{A}$ has the property that $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$.
Find expressions for $\mathbf{A}^n$ for $n = 3, 4, ..., 8$ in terms of $\mathbf{A}$ and $\mathbf{I}$ (i.e., in the form $a\mathbf{A} + b\mathbf{I}$). Hence:

a deduce simple expressions for $\mathbf{A}^{6n+3}$ and $\mathbf{A}^{6n+5}$

b express $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

Chapter 15

Vectors in 2-dimensions

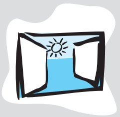
Contents:

A Vectors
B Operations with vectors
C Vectors in component form
D Vector equations
E Vectors in coordinate geometry
F Parallelism
G Unit vectors
H Angles and scalar product

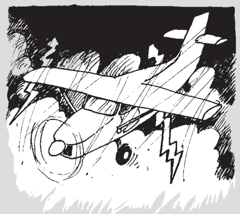
Review set 15A
Review set 15B
Review set 15C
Review set 15D

A VECTORS

OPENING PROBLEM



An aeroplane in calm conditions is flying due east. A cold wind suddenly blows in from the south west. The aeroplane, cruising at 800 km/h, is blown slightly off course by the 35 km/h wind.



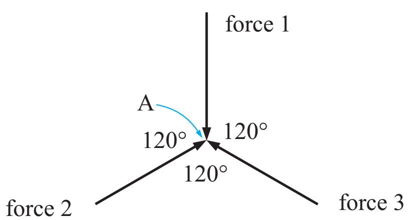
- What effect does the wind have on the speed and direction of the aeroplane?
- How can we accurately determine the new speed and direction using mathematics?
- How much of the force of the wind operates in the direction of the aeroplane, and how does this affect fuel consumption and the time of the flight?

VECTORS AND SCALARS

In order to handle the **Opening Problem** and problems similar to it we need to examine the **size** or **magnitude** of the quantities under consideration as well as the direction in which they are acting.

For example, the effect of the wind on an aeroplane would be different if the wind was blowing from behind the plane rather than against it.

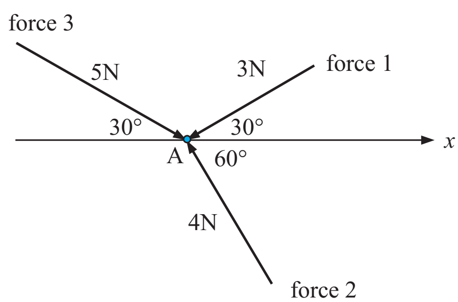
Consider the problem of forces acting at a point.



If three equal forces act on point A and they are from directions 120^o apart then clearly A would not move.

(Imagine three people pushing a fourth person with equal force and 120^o apart.)

Now suppose three forces act on the point A as shown. What is the resultant force acting on A and in what direction would A move under these three forces?



VECTORS

To handle these situations we need to consider quantities called **vectors** which have both size (magnitude) and direction.

Quantities which have only magnitude are called **scalars**.

Quantities which have both size (magnitude) and direction are called **vectors**.

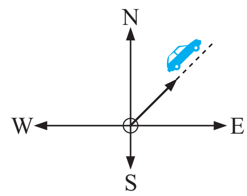
For example, velocity is a vector since it deals with speed (a scalar) in a particular direction.

Other examples of vector quantities are:

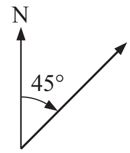
- acceleration
- force
- displacement
- momentum
- weight

DIRECTED LINE SEGMENT REPRESENTATION

Consider the following example where a car is travelling at 80 kmph in a NE direction.



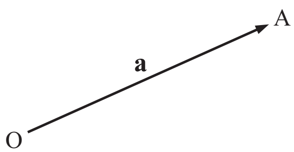
One good way of representing this is to use an arrow on a scale diagram.



Scale: 1 cm represents 40 kmph

The **length of the arrow** represents the size (magnitude) of the quantity and the **arrowhead** shows the direction of travel.

Consider the vector represented by the line segment from O to A.



- This **vector** could be represented by

  $\overrightarrow{OA}$ or $\mathbf{a}$ or $\underset{\sim}{a}$

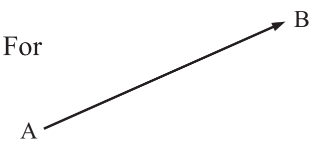
  $\mathbf{a}$ ↑ bold used in text books

  $\underset{\sim}{a}$ ↑ used by students

- The **magnitude** (**length**) could be represented by

  $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\underset{\sim}{a}|$

For

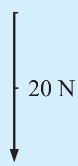


the vector which **emanates** at A and **terminates** at B, $\overrightarrow{AB}$ is the **position vector** of B relative to (from) A.

Example 1

On a scale diagram, sketch the vector which represents "a force of 20 Newtons in a southerly direction".

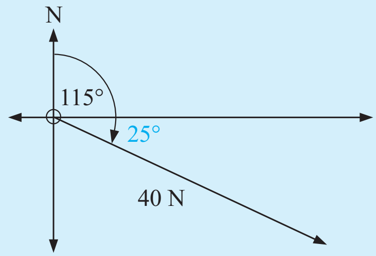
Scale: 1 cm $\equiv$ 10 Newtons



Example 2

Draw a scaled arrow diagram representing '40 Newtons on a bearing 115^o'.

Scale: 1 cm $\equiv$ 10 N



EXERCISE 15A.1

1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

- **a** 30 Newtons in a SE direction
- **b** 25 ms^{-1} in a northerly direction
- **c** a displacement of 35 m in a direction 070^o
- **d** an aeroplane taking off at an angle of 10^o to the runway with a speed of 50 ms^{-1}.

2 If $\longrightarrow$ represents a velocity of 50 ms^{-1} due east, draw a directed line segment representing a velocity of:

- **a** 100 ms^{-1} due west
- **b** 75 ms^{-1} north east.

3 Draw a scaled arrow diagram representing the following vectors:

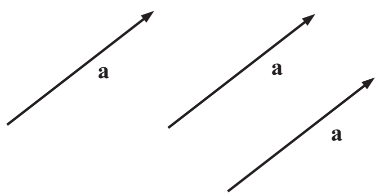
- **a** a force of 30 Newtons in the NW direction
- **b** a velocity of 40 ms^{-1} in a direction 146^o
- **c** a displacement of 25 km in the direction $\text{S}32^o\text{E}$
- **d** an aeroplane taking off at an angle of 8^o to the runway at a speed of 150 kmph.

VECTOR EQUALITY

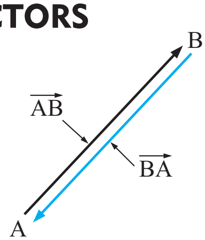
Two vectors are **equal** if they have the same magnitude and direction.

So, if arrows are used to represent vectors, then equal vectors are **parallel** and **equal in length**.

This means that equal vector arrows are translations of one another.



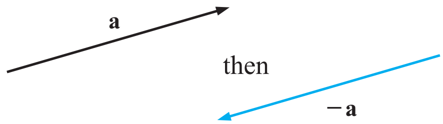
NEGATIVE VECTORS



Notice that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length but have opposite directions.

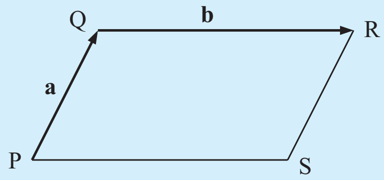
We say that $\overrightarrow{BA}$ is the negative of $\overrightarrow{AB}$ and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

Also, if



as these two vectors are parallel, equal in length, but opposite in direction.

Example 3



PQRS is a parallelogram, and $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$.

Find vector expressions for:

a $\overrightarrow{QP}$ **b** $\overrightarrow{RQ}$ **c** $\overrightarrow{SR}$ **d** $\overrightarrow{SP}$

a $\overrightarrow{QP} = -\mathbf{a}$ {the negative vector of $\overrightarrow{PQ}$}

b $\overrightarrow{RQ} = -\mathbf{b}$ {the negative vector of $\overrightarrow{QR}$}

c $\overrightarrow{SR} = \mathbf{a}$ {parallel to and the same length as $\overrightarrow{PQ}$}

d $\overrightarrow{SP} = -\mathbf{b}$ {parallel to and the same length as $\overrightarrow{RQ}$}

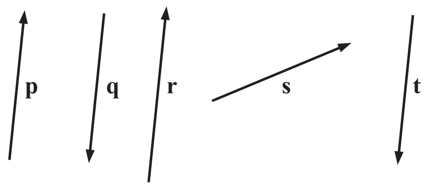
EXERCISE 15A.2

1 State the vectors which are:

a equal in magnitude **b** parallel

c in the same direction **d** equal

e negatives of one another.

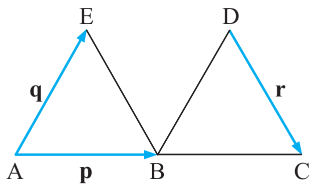


2 The figure shown consists of 2 congruent equilateral triangles. ABC lie on a straight line. $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AE} = \mathbf{q}$ and $\overrightarrow{DC} = \mathbf{r}$.

Which of the following statements is true?

a $\overrightarrow{EB} = \mathbf{r}$ **b** $|\mathbf{p}| = |\mathbf{q}|$ **c** $\overrightarrow{BC} = \mathbf{r}$

d $\overrightarrow{DB} = \mathbf{q}$ **e** $\overrightarrow{ED} = \mathbf{p}$ **f** $\mathbf{p} = \mathbf{q}$



DISCUSSION

Could we have a zero vector?

What would its length be?

What direction?

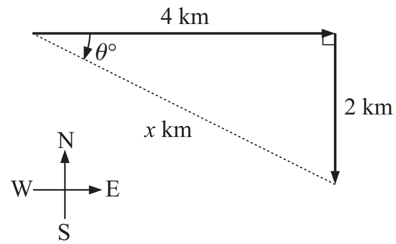
B OPERATIONS WITH VECTORS

We have already been operating with vectors without realising it.

Bearing problems are an example of this. The vectors in this case are **displacements**.

A typical problem could be, "A runner runs in an easterly direction for 4 km and then in a southerly direction for 2 km.

How far is she from her starting point and in what direction?"



Notice that trigonometry and Pythagoras' Rule are used to answer such problems as we need to find θ and x.

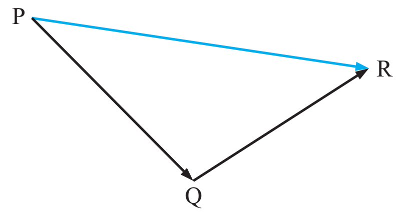
DISPLACEMENT VECTORS

Suppose we have three towns P, Q and R.

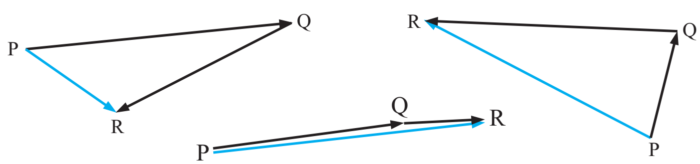
A trip from P to Q followed by a trip from Q to R is equivalent to a trip from P to R.

This can be expressed in a vector form as

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ where the $+$ sign could mean '*followed by*'.



This triangular diagram could take all sorts of shapes. For example



VECTOR ADDITION

After considering displacements in diagrams like those above, we define vector addition geometrically as:

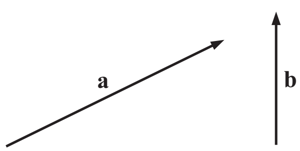
To add **a** and **b**

Step 1: first draw **a**, then

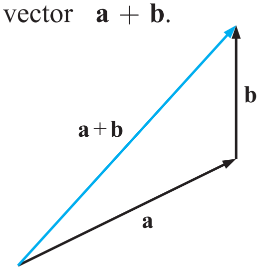
Step 2: at the arrowhead end of **a** draw **b**, and then

Step 3: join the beginning of **a** to the arrowhead end of **b** and this is vector **a** + **b**.

So given



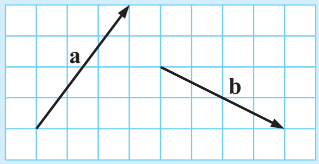
we have

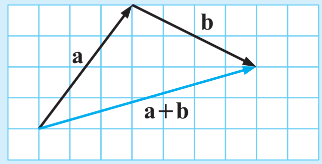


DEMO

Example 4

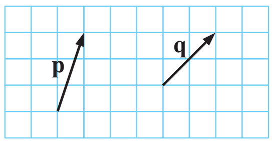
Given **a** and **b** as shown, construct **a** + **b**.

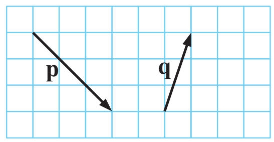


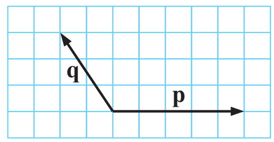


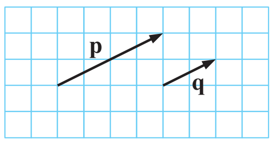
EXERCISE 15B.1

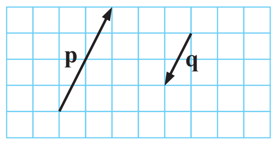
1 Copy the given vectors **p** and **q** and hence show how to find **p** + **q**:

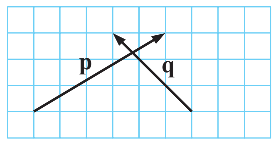
a


b


c


d


e


f


Example 5

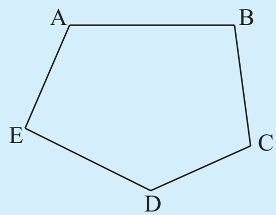
Find a single vector which is equal to:

a $\overrightarrow{BC} + \overrightarrow{CA}$

b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC}$

c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

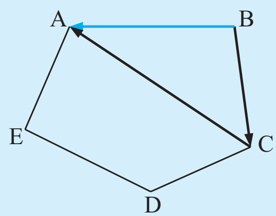


a $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$ {as shown}

b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{BC}$

c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

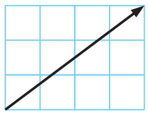
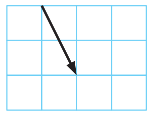


2 Find a single vector which is equal to:

a $\overrightarrow{AB} + \overrightarrow{BC}$ **b** $\overrightarrow{BC} + \overrightarrow{CD}$

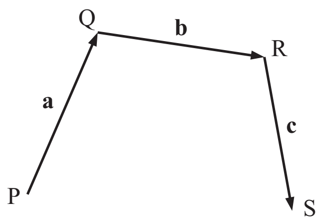
c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **d** $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

3 **a** Use vector diagrams to find **i** $\mathbf{p}+\mathbf{q}$ **ii** $\mathbf{q}+\mathbf{p}$ given that

$\mathbf{p}$ is  and $\mathbf{q}$ is 

b For any two vectors $\mathbf{p}$ and $\mathbf{q}$, is $\mathbf{p}+\mathbf{q}=\mathbf{q}+\mathbf{p}$?

4 Consider:



Notice that one way of finding $\overrightarrow{PS}$

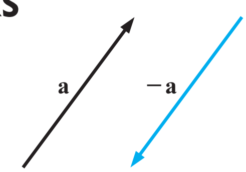
is $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$

$= (\mathbf{a}+\mathbf{b})+\mathbf{c}$.

Use the diagram to show that $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$.

Before defining vector subtraction it is necessary to look again at what we mean by **negative vectors**.

NEGATIVE VECTORS



$-\mathbf{a}$ is the **negative** of $\mathbf{a}$.

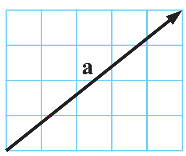
Notice that $-\mathbf{a}$ has the same magnitude as $\mathbf{a}$ but is in the opposite direction.

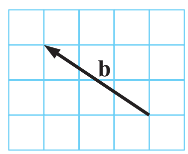
ZERO VECTOR

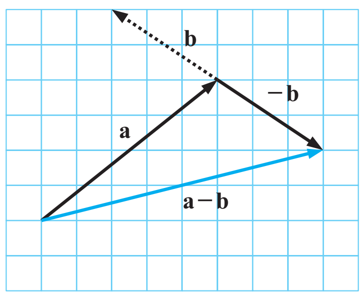
The **zero vector** is written as $\mathbf{0}$ and for any vector $\mathbf{a}$, $\mathbf{a}+(-\mathbf{a}) = (-\mathbf{a})+\mathbf{a} = \mathbf{0}$.

VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**, i.e., $\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b})$.

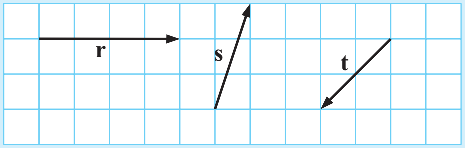
Geometrically: For 

and 

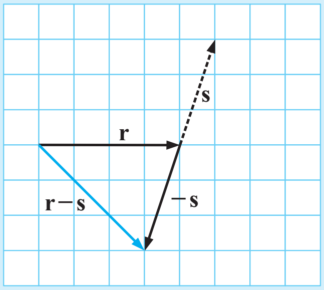
then 

Example 6

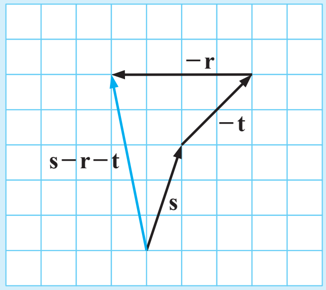
For **r**, **s** and **t** as shown find geometrically:
a $\mathbf{r}-\mathbf{s}$ **b** $\mathbf{s}-\mathbf{t}-\mathbf{r}$



a



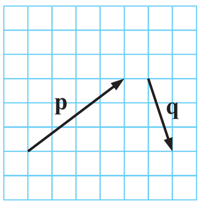
b



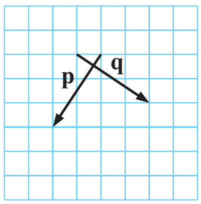
EXERCISE 15B.2

1 For the following vectors **p** and **q**, show how to construct $\mathbf{p}-\mathbf{q}$:

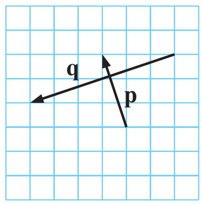
a



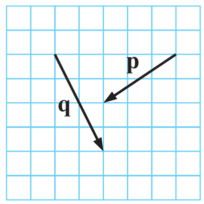
b



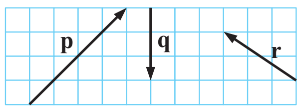
c



d



2 For the following vectors:



show how to construct:

a $\mathbf{p}+\mathbf{q}-\mathbf{r}$ **b** $\mathbf{p}-\mathbf{q}-\mathbf{r}$ **c** $\mathbf{r}-\mathbf{q}-\mathbf{p}$

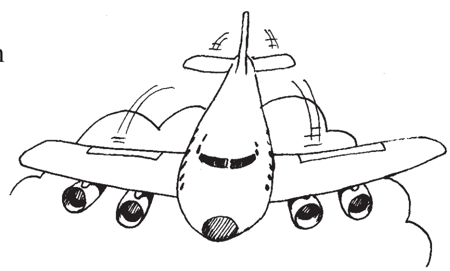
AN APPLICATION OF VECTOR SUBTRACTION

Vector subtraction is used in problem solving involving displacement, velocity and force.

Consider the following velocity application:

An aeroplane needs to fly due east from one city to another at a speed of 400 km/h. However a 50 km/h wind blows constantly from the north-east.

In what direction must the aeroplane head and at what speed must it travel?

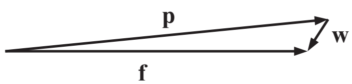


Notice that on this occasion we know:

SIMULATION

We also know that the aeroplane would have to head a little north of its final destination as the north-easterly would blow it back to its final direction.

So



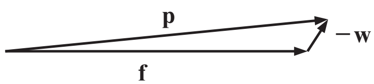
Even though the plane moves in the **f** direction it is actually lined up in the **p** direction.

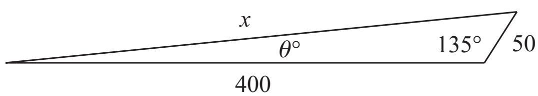
Notice that $\quad \mathbf{p}+\mathbf{w}=\mathbf{f}$ and so $\mathbf{p}+\mathbf{w}+(-\mathbf{w})=\mathbf{f}+(-\mathbf{w})$

$$\therefore\ \mathbf{p}+\mathbf{0}=\mathbf{f}-\mathbf{w}$$

$$\text{i.e., } \mathbf{p}=\mathbf{f}-\mathbf{w}$$

The solution:





By the cosine rule,

$x^2=50^2+400^2-2\times 50\times 400\cos 135^o$

gives $\quad x \doteqdot 436.8$

By the sine rule

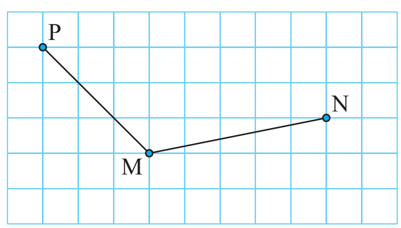
$$\frac{\sin\theta}{50}=\frac{\sin 135}{436.8} \quad \text{gives} \quad \theta \doteqdot 4.6$$

Consequently, the aeroplane must fly 4.6^o north of east at 436.8 kmph.

3 **a** Copy this diagram and on it mark the points:

- **i** X such that $\overrightarrow{MX}=\overrightarrow{MN}+\overrightarrow{MP}$
- **ii** Y such that $\overrightarrow{MY}=\overrightarrow{MN}-\overrightarrow{MP}$
- **iii** Z such that $\overrightarrow{PZ}=2\overrightarrow{PM}$

b What type of figure is MNYZ?



Example 7

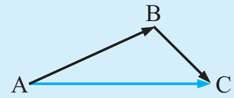
For points A, B, C and D, simplify the following vector expressions:

a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

a $\overrightarrow{AB} - \overrightarrow{CB}$

$= \overrightarrow{AB} + \overrightarrow{BC}$ $\{$as $\overrightarrow{BC} = -\overrightarrow{CB}\}$

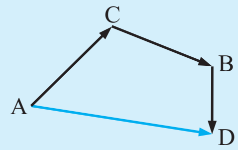
$= \overrightarrow{AC}$



b $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

$= \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

$= \overrightarrow{AD}$



4 For points A, B, C and D, simplify the following vector expressions:

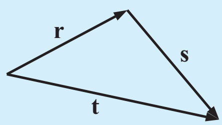
a $\overrightarrow{AC} + \overrightarrow{CB}$ **b** $\overrightarrow{AD} - \overrightarrow{BD}$ **c** $\overrightarrow{AC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **e** $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$ **f** $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

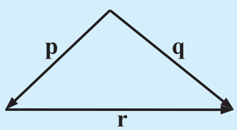
Example 8

Construct vector equations for:

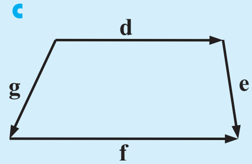
a

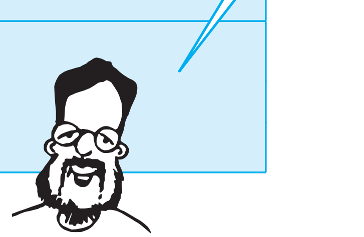


b



c





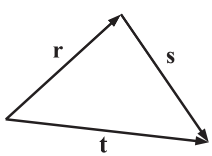
a $\mathbf{t} = \mathbf{r} + \mathbf{s}$

b $\mathbf{r} = -\mathbf{p} + \mathbf{q}$

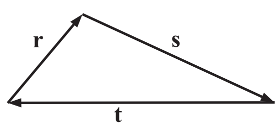
c $\mathbf{f} = -\mathbf{g} + \mathbf{d} + \mathbf{e}$

5 Construct vector equations for:

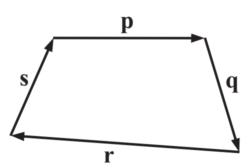
a



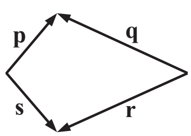
b



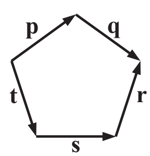
c



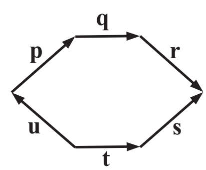
d



e



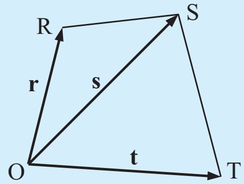
f



Example 9

For the given diagram, find, in terms of **r**, **s** and **t**:

a $\overrightarrow{RS}$ **b** $\overrightarrow{SR}$ **c** $\overrightarrow{ST}$



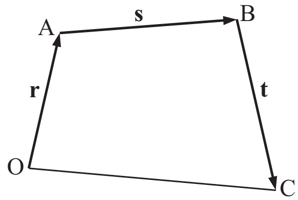
a $\overrightarrow{RS}$
$= \overrightarrow{RO} + \overrightarrow{OS}$
$= -\overrightarrow{OR} + \overrightarrow{OS}$
$= -\mathbf{r} + \mathbf{s}$
$= \mathbf{s} - \mathbf{r}$

b $\overrightarrow{SR}$
$= \overrightarrow{SO} + \overrightarrow{OR}$
$= -\overrightarrow{OS} + \overrightarrow{OR}$
$= -\mathbf{s} + \mathbf{r}$
$= \mathbf{r} - \mathbf{s}$

c $\overrightarrow{ST}$
$= \overrightarrow{SO} + \overrightarrow{OT}$
$= -\overrightarrow{OS} + \overrightarrow{OT}$
$= -\mathbf{s} + \mathbf{t}$
$= \mathbf{t} - \mathbf{s}$

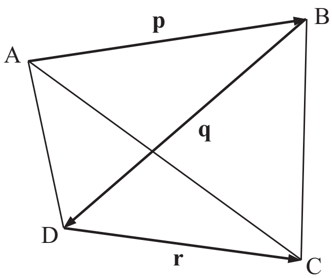
6 **a** Find, in terms of **r**, **s** and **t**:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$ **iii** $\overrightarrow{OC}$



b For $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{BD} = \mathbf{q}$ and $\overrightarrow{DC} = \mathbf{r}$
find, in terms of **p**, **q** and **r**:

i $\overrightarrow{AD}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{AC}$



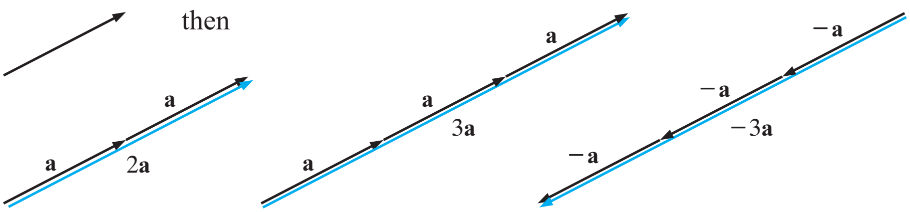
SCALAR MULTIPLICATION

Numbers such as 2 and -3 are also referred to as scalars. If **a** is a vector, what would $2\mathbf{a}$ and $-3\mathbf{a}$ mean?

By definition, $2\mathbf{a} = \mathbf{a} + \mathbf{a}$, $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$, etc

$$-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$$

So, if **a** is then



So, $2\mathbf{a}$ is in the direction of **a** but is twice as long as **a**

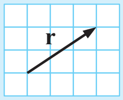
$3\mathbf{a}$ is in the direction of **a** but is three times longer than **a**

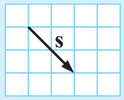
$-3\mathbf{a}$ is oppositely directed to **a** and is three times longer than **a**.

Note:

> If $\mathbf{a}$ is a vector and k is a scalar,
>
> $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.
>
> If $k > 0$, $k\mathbf{a}$ and $\mathbf{a}$ have the same direction.
>
> If $k < 0$, $k\mathbf{a}$ and $\mathbf{a}$ have opposite directions.

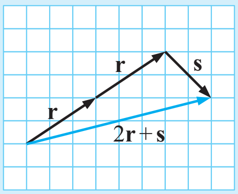
Example 10

Given vectors 

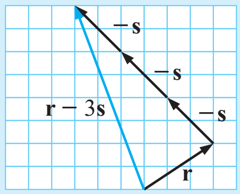
and 

show how to find **a** $2\mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} - 3\mathbf{s}$ geometrically.

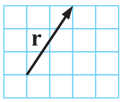
a

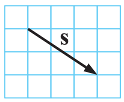


b



EXERCISE 15B.3

1 Given vectors 

and 

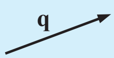
, show how to find geometrically:

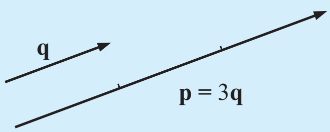
a $-\mathbf{r}$ **b** $2\mathbf{s}$ **c** $\frac{1}{2}\mathbf{r}$ **d** $-\frac{3}{2}\mathbf{s}$

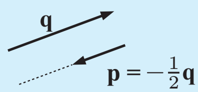
e $2\mathbf{r} - \mathbf{s}$ **f** $2\mathbf{r} + 3\mathbf{s}$ **g** $\frac{1}{2}\mathbf{r} + 2\mathbf{s}$ **h** $\frac{1}{2}(\mathbf{r} + 3\mathbf{s})$

Example 11

Draw sketches of vectors $\mathbf{p}$ and $\mathbf{q}$ if **a** $\mathbf{p} = 3\mathbf{q}$ **b** $\mathbf{p} = -\frac{1}{2}\mathbf{q}$.

Let $\mathbf{q}$ be 

a 

b 

2 Draw sketches of $\mathbf{p}$ and $\mathbf{q}$ if:

a $\mathbf{p} = \mathbf{q}$ **b** $\mathbf{p} = -\mathbf{q}$ **c** $\mathbf{p} = 2\mathbf{q}$ **d** $\mathbf{p} = \frac{1}{3}\mathbf{q}$ **e** $\mathbf{p} = -3\mathbf{q}$

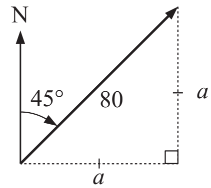
C VECTORS IN COMPONENT FORM

So far we have examined vectors from their geometric representation.

We have used arrows where:

- the **length** of the arrow represents size (magnitude)
- the **arrowhead** indicates direction.

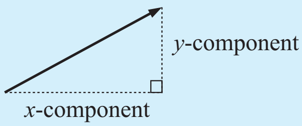
Consider a car travelling at 80 km/h in a NE direction.



The velocity vector could be represented by using the x and y-steps which are necessary to go from the start to the finish.

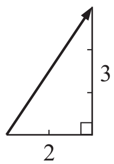
In this case the column vector $\begin{bmatrix} 56.6 \\ 56.6 \end{bmatrix}$ gives the x and y steps.

$$\begin{aligned} a^2 + a^2 &= 80^2 \\ \therefore \quad 2a^2 &= 6400 \\ \therefore \quad a^2 &= 3200 \\ \therefore \quad a &\doteqdot 56.6 \end{aligned}$$



$\begin{bmatrix} x\text{-component} \\ y\text{-component} \end{bmatrix}$ is the **component form** of a vector.

For example, given $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ we could draw



and vice versa.

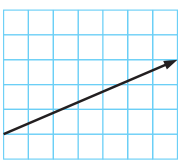
2 is the horizontal step and 3 is the vertical step.

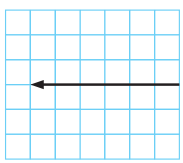
EXERCISE 15C.1

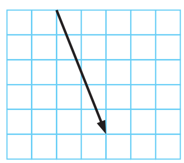
1 Draw arrow diagrams to represent the vectors:

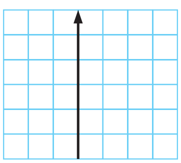
a $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **c** $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ **d** $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$

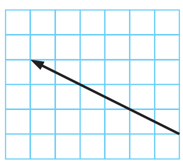
2 Write the illustrated vectors in component form:

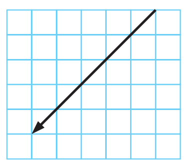
a


b


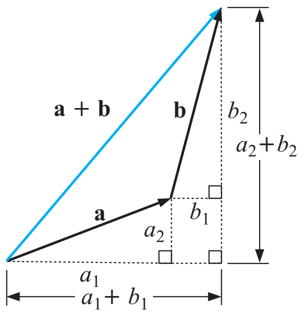
c


d


e


f


VECTOR ADDITION



Consider adding vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Notice that the

horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$ and the

vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.

So, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$.

Example 12

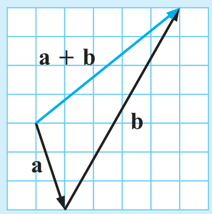
If $\mathbf{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

find $\mathbf{a} + \mathbf{b}$.

Check graphically.

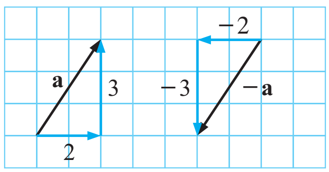
$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$
$$= \begin{bmatrix} 1+4 \\ -3+7 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Check:



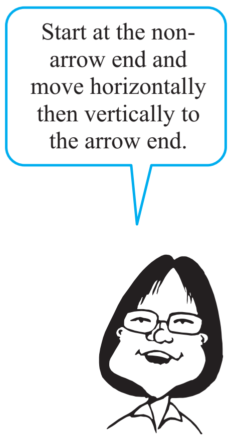
NEGATIVE VECTORS

Consider the vector $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.



Notice that $-\mathbf{a} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

In general, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ then $-\mathbf{a} = \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix}$.



ZERO VECTOR

The zero vector is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and for any vector $\mathbf{a}$, $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.

VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**, i.e., $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

Notice that, if $\mathbf{a}=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$ then

$$\begin{aligned}\mathbf{a}-\mathbf{b}&=\mathbf{a}+(-\mathbf{b})\\&=\begin{bmatrix}a_1\\a_2\end{bmatrix}+\begin{bmatrix}-b_1\\-b_2\end{bmatrix}\\&=\begin{bmatrix}a_1-b_1\\a_2-b_2\end{bmatrix}\end{aligned}$$

i.e., if $\mathbf{a}=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$ then $\mathbf{a}-\mathbf{b}=\begin{bmatrix}a_1-b_1\\a_2-b_2\end{bmatrix}$.

EXERCISE 15C.2

VECTOR RACE GAME

1 If $\mathbf{a}=\begin{bmatrix}-3\\2\end{bmatrix}$ $\mathbf{b}=\begin{bmatrix}1\\4\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}-2\\-5\end{bmatrix}$ find:

a $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{b}+\mathbf{a}$ **c** $\mathbf{b}+\mathbf{c}$ **d** $\mathbf{c}+\mathbf{b}$
e $\mathbf{a}+\mathbf{c}$ **f** $\mathbf{c}+\mathbf{a}$ **g** $\mathbf{a}+\mathbf{a}$ **h** $\mathbf{b}+\mathbf{a}+\mathbf{c}$

Example 13

Given $\mathbf{p}=\begin{bmatrix}3\\-2\end{bmatrix}$, $\mathbf{q}=\begin{bmatrix}1\\4\end{bmatrix}$ and $\mathbf{r}=\begin{bmatrix}-2\\-5\end{bmatrix}$ find: **a** $\mathbf{q}-\mathbf{p}$ **b** $\mathbf{p}-\mathbf{q}-\mathbf{r}$

a $\mathbf{q}-\mathbf{p}$

$$=\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}$$
$$=\begin{bmatrix}1-3\\4+2\end{bmatrix}$$
$$=\begin{bmatrix}-2\\6\end{bmatrix}$$

b $\mathbf{p}-\mathbf{q}-\mathbf{r}$

$$=\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}-2\\-5\end{bmatrix}$$
$$=\begin{bmatrix}3-1--2\\-2-4--5\end{bmatrix}$$
$$=\begin{bmatrix}4\\-1\end{bmatrix}$$

2 Given $\mathbf{p}=\begin{bmatrix}-4\\2\end{bmatrix}$, $\mathbf{q}=\begin{bmatrix}-1\\-5\end{bmatrix}$ and $\mathbf{r}=\begin{bmatrix}3\\-2\end{bmatrix}$ find:

a $\mathbf{p}-\mathbf{q}$ **b** $\mathbf{q}-\mathbf{r}$ **c** $\mathbf{p}+\mathbf{q}-\mathbf{r}$
d $\mathbf{p}-\mathbf{q}-\mathbf{r}$ **e** $\mathbf{q}-\mathbf{r}-\mathbf{p}$ **f** $\mathbf{r}+\mathbf{q}-\mathbf{p}$

3 **a** Given $\overrightarrow{BA}=\begin{bmatrix}2\\-3\end{bmatrix}$ and $\overrightarrow{BC}=\begin{bmatrix}-3\\1\end{bmatrix}$ find $\overrightarrow{AC}$. **Hint:** $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$
$=-\overrightarrow{BA}+\overrightarrow{BC}$.

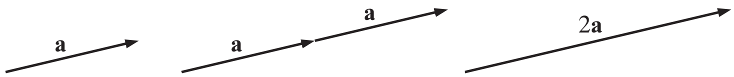
b If $\overrightarrow{AB}=\begin{bmatrix}-1\\3\end{bmatrix}$, $\overrightarrow{CA}=\begin{bmatrix}2\\-1\end{bmatrix}$, find $\overrightarrow{CB}$.

c If $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $\overrightarrow{RQ} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{RS} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find $\overrightarrow{SP}$.

SCALAR MULTIPLICATION

Recall the geometric approach for scalar multiplication.

For example:



A **scalar** is a non-vector quantity.

The word scalar is also used for a constant number.

Consider $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. $\quad \mathbf{a} + \mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{a} + \mathbf{a} + \mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

Examples like these suggest the following definition for **scalar multiplication**:

If k is a scalar, then $k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix}$.

Notice that:

- $(-1)\mathbf{a} = \begin{bmatrix} (-1)a_1 \\ (-1)a_2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = -\mathbf{a}$
- $(0)\mathbf{a} = \begin{bmatrix} (0)a_1 \\ (0)a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$

Example 14

For $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ find: **a** $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$

$= 3\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ -9 \end{bmatrix}$

b $\mathbf{p} + 2\mathbf{q}$

$= \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{bmatrix}$

$= \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

$= \frac{1}{2}\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{bmatrix}$

$= \begin{bmatrix} -4 \\ 9\frac{1}{2} \end{bmatrix}$

EXERCISE 15C.3

1 For $\mathbf{p} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$

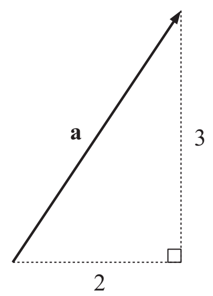
e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

2 If $\mathbf{p} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix}2\\-1\end{bmatrix}$ find by diagram:

a $\mathbf{p}+\mathbf{p}+\mathbf{q}+\mathbf{q}+\mathbf{q}$ **b** $\mathbf{p}+\mathbf{q}+\mathbf{p}+\mathbf{q}+\mathbf{q}$

c $\mathbf{q}+\mathbf{p}+\mathbf{q}+\mathbf{p}+\mathbf{q}$ Comment on the results.

LENGTH OF A VECTOR



Consider vector $\mathbf{a} = \begin{bmatrix}2\\3\end{bmatrix}$ as illustrated.

Recall that $|\mathbf{a}|$ represents the length of $\mathbf{a}$.

By Pythagoras $|\mathbf{a}|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$\therefore \quad |\mathbf{a}| = \sqrt{13}$ units

In general, if $\mathbf{a} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$, then $|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2}$.

Example 15

If $\mathbf{p} = \begin{bmatrix}3\\-5\end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix}-1\\-2\end{bmatrix}$ find: **a** $|\mathbf{p}|$ **b** $|\mathbf{q}|$ **c** $|\mathbf{p} - 2\mathbf{q}|$

a $\mathbf{p} = \begin{bmatrix}3\\-5\end{bmatrix}$ $\therefore \quad |\mathbf{p}| = \sqrt{9+25} = \sqrt{34}$ units

b $\mathbf{q} = \begin{bmatrix}-1\\-2\end{bmatrix}$ $\therefore \quad |\mathbf{q}| = \sqrt{1+4} = \sqrt{5}$ units

c $\mathbf{p} - 2\mathbf{q} = \begin{bmatrix}3\\-5\end{bmatrix} - 2\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$ $\therefore \quad |\mathbf{p} - 2\mathbf{q}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$ units

EXERCISE 15C.4

1 For $\mathbf{r} = \begin{bmatrix}2\\3\end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix}-1\\4\end{bmatrix}$ find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r}+\mathbf{s}|$ **d** $|\mathbf{r}-\mathbf{s}|$ **e** $|\mathbf{s} - 2\mathbf{r}|$

2 If $\mathbf{p} = \begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{q} = \begin{bmatrix}-2\\4\end{bmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$ **c** $|-2\mathbf{p}|$ **d** $|3\mathbf{p}|$ **e** $|-3\mathbf{p}|$

f $|\mathbf{q}|$ **g** $|4\mathbf{q}|$ **h** $|-4\mathbf{q}|$ **i** $|\frac{1}{2}\mathbf{q}|$ **j** $|-\frac{1}{2}\mathbf{q}|$

3 From your answers in **2**, you should have noticed that $|k\mathbf{a}| = |k|\,|\mathbf{a}|$

i.e., (the length of $k\mathbf{a}$) $=$ (the modulus of k) $\times$ (the length of $\mathbf{a}$).

By letting $\mathbf{a} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$, prove that $|k\mathbf{a}| = |k|\,|\mathbf{a}|$.

D VECTOR EQUATIONS

The rules for solving vector equations are similar to those for solving real number equations except that there is no such thing as dividing a vector by a scalar.

We avoid this problem by multiplying by reciprocals.

So, for example, if $2\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{2}\mathbf{a}$ and *not* $\dfrac{\mathbf{a}}{2}$.

$\dfrac{\mathbf{a}}{2}$ has no meaning in vector algebra.

Two useful rules are

- if $\mathbf{x} + \mathbf{a} = \mathbf{b}$ then $\mathbf{x} = \mathbf{b} - \mathbf{a}$
- if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$ $(k \neq 0)$

To establish these notice that:

if $\mathbf{x} + \mathbf{a} = \mathbf{b}$

then $\mathbf{x} + \mathbf{a} + (-\mathbf{a}) = \mathbf{b} + (-\mathbf{a})$

$\therefore\ \mathbf{x} + \mathbf{0} = \mathbf{b} - \mathbf{a}$

$\therefore\ \mathbf{x} = \mathbf{b} - \mathbf{a}$

and if $k\mathbf{x} = \mathbf{a}$

then $\frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{a}$

$\therefore\ 1\mathbf{x} = \frac{1}{k}\mathbf{a}$

$\therefore\ \mathbf{x} = \frac{1}{k}\mathbf{a}$

Example 16

Solve for $\mathbf{x}$: **a** $3\mathbf{x} - \mathbf{r} = \mathbf{s}$ **b** $\mathbf{c} - 2\mathbf{x} = \mathbf{d}$

a $3\mathbf{x} - \mathbf{r} = \mathbf{s}$

$\therefore\ 3\mathbf{x} = \mathbf{s} + \mathbf{r}$

$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{s} + \mathbf{r})$

b $\mathbf{c} - 2\mathbf{x} = \mathbf{d}$

$\therefore\ \mathbf{c} - \mathbf{d} = 2\mathbf{x}$

$\therefore\ \frac{1}{2}(\mathbf{c} - \mathbf{d}) = \mathbf{x}$

EXERCISE 15D

1 Solve the following vector equations for $\mathbf{x}$:

a $2\mathbf{x} = \mathbf{q}$ **b** $\frac{1}{2}\mathbf{x} = \mathbf{n}$ **c** $-3\mathbf{x} = \mathbf{p}$

d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$ **e** $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$ **f** $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

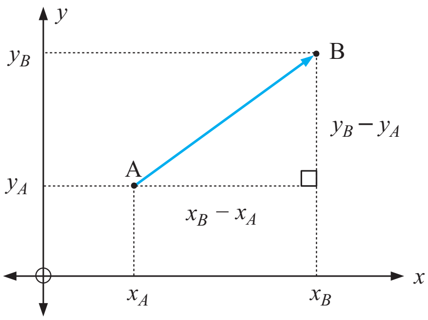
2 If $\mathbf{r} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $\mathbf{y}$ if:

a $2\mathbf{y} = \mathbf{r}$ **b** $\frac{1}{2}\mathbf{y} = \mathbf{s}$ **c** $\mathbf{r} + 2\mathbf{y} = \mathbf{s}$ **d** $3\mathbf{s} - 4\mathbf{y} = \mathbf{r}$

3 If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, and $k\mathbf{x} = \mathbf{a}$, show by equating components that $\mathbf{x} = \dfrac{1}{k}\mathbf{a}$.

E VECTORS IN COORDINATE GEOMETRY

VECTORS BETWEEN TWO POINTS



Consider points $A(x_A, y_A)$ and $B(x_B, y_B)$

In going from A to B,

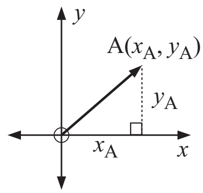
$x_B - x_A$ is the x-**step**, and

$y_B - y_A$ is the y-**step**.

Consequently

$$\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}.$$

Notice that if O is (0, 0) and A is (x_A, y_A) then $\overrightarrow{OA}$ is $\begin{bmatrix} x_A \\ y_A \end{bmatrix}$.



DISTANCE BETWEEN TWO POINTS

The **distance** between two points A and B is the length of $\overrightarrow{AB}$ (or $\overrightarrow{BA}$) and is denoted $|\overrightarrow{AB}|$.

Example 17

If P is $(-1, 2)$ and $Q(3, 1)$ find

a $\overrightarrow{PQ}$ **b** the distance from P to Q.

a $\overrightarrow{PQ}$

$= \begin{bmatrix} 3-(-1) \\ 1-2 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

b distance

$= |\overrightarrow{PQ}|$

$= \sqrt{4^2 + (-1)^2}$

$= \sqrt{17}$ units

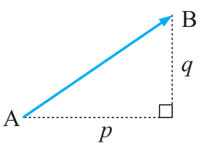
Notice that if

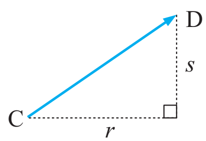
$\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$

then

$\overrightarrow{BA} = \begin{bmatrix} x_A - x_B \\ y_A - y_B \end{bmatrix}.$

VECTOR EQUALITY





Two vectors are **equal** if they have the same length and direction.

Consequently, their x-steps are equal i.e., $p = r$

and their y-steps are equal i.e., $q = s$

i.e., $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} \Leftrightarrow p = r$ and $q = s$.

(where $\Leftrightarrow$ reads "if and only if")

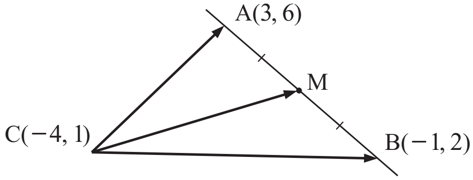
EXERCISE 15E

1 Given A(3, −2), B(2, 6) and C(−1, −4) find:

a $\overrightarrow{AB}$ and the distance from A to B **b** $\overrightarrow{BC}$ and the distance from B to C

c $\overrightarrow{CA}$ and the distance from C to A

2



Find:

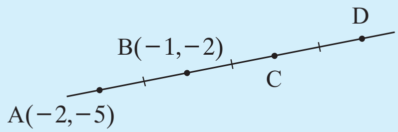
a the coordinates of M

b vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$ and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

Example 18

Find the coordinates of C and D in:



$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\therefore$ C is $(-1+1,\ -2+3)$ i.e., C(0, 1)

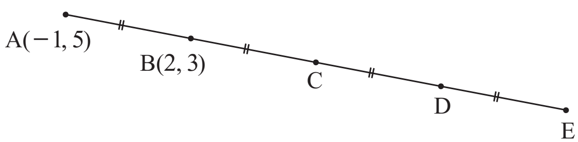
and D is $(0+1,\ 1+3)$ i.e., D(1, 4)

3 Find B if C is the centre of a circle with diameter AB:

a A is (3, −2) and C(1, 4) **b** A is (0, 5) and C(−1, −2)

c A is (−1, −4) and C(3, 0)

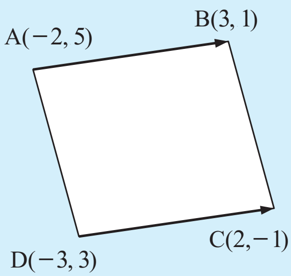
4



Find the coordinates of C, D and E using equal vectors (steps).

Example 19

Use vectors to show that ABCD is a parallelogram where A is (−2, 5), B(3, 1), C(2, −1) and D is (−3, 3).



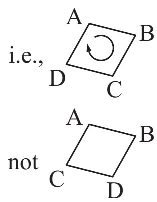
$$\overrightarrow{AB} = \begin{bmatrix} 3-(-2) \\ 1-5 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\overrightarrow{DC} = \begin{bmatrix} 2-(-3) \\ -1-3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

i.e., $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ side AB is parallel to side DC and as they are equal in length, i.e., $\sqrt{25+16}$, this is sufficient to deduce that ABCD is a parallelogram.

Given ABCD, the ordering of letters is cyclic,



5 Use vectors to find whether or not ABCD is a parallelogram:

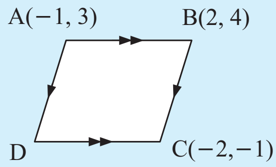
a A(3, −1), B(4, 2), C(−1, 4) and D(−2, 1)

b A(5, 0), B(−1, 2), C(4, −3) and D(10, −6)

c A(2, −3), B(1, 4), C(−2, 6) and D(−1, −1)

Example 20

Use vector methods to find the remaining vertex of:



If D is (a, b) then $\overrightarrow{CD} = \begin{bmatrix} a-(-2) \\ b-(-1) \end{bmatrix} = \begin{bmatrix} a+2 \\ b+1 \end{bmatrix}$

But $\overrightarrow{CD} = \overrightarrow{BA}$

$\therefore \begin{bmatrix} a+2 \\ b+1 \end{bmatrix} = \begin{bmatrix} -1-2 \\ 3-4 \end{bmatrix}$

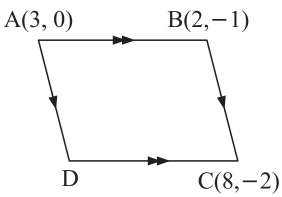
$\therefore\ a+2=-3$ and $b+1=-1$

$\therefore\ a=-5$ and $b=-2$

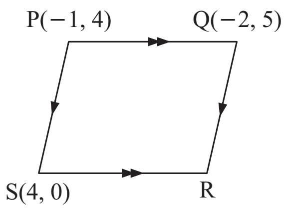
So, D is $(-5, -2)$.

6 Use vector methods to find the remaining vertex of:

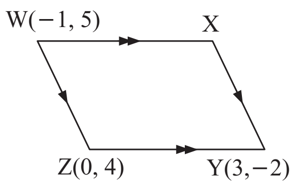
a



b



c



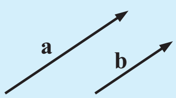
7 Find scalars r and s such that:

a $r\begin{bmatrix} 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$

b $r\begin{bmatrix} 2 \\ -3 \end{bmatrix} + s\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$

F PARALLELISM

If two vectors are **parallel** then one is a **scalar multiple** of the other.



Note:
- If $\mathbf{a} \parallel \mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then $\mathbf{a} \parallel \mathbf{b}$.

For example, if $\mathbf{a} = \begin{bmatrix}12\\6\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}4\\2\end{bmatrix}$ then $\mathbf{a}$ is parallel to $\mathbf{b}$ because $\mathbf{a} = 3\mathbf{b}$.

Notice that $|\mathbf{a}| = \sqrt{144+36} = \sqrt{180} = 6\sqrt{5}$ and

$|\mathbf{b}| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

i.e., $|\mathbf{a}| = 3\,|\mathbf{b}|$

In general,

- $k\mathbf{a}$ is a vector **parallel** to $\mathbf{a}$.
  It has the **same direction** as $\mathbf{a}$ if $k > 0$ and **opposite direction** if $k < 0$.
- $|k\mathbf{a}| = |k|\,|\mathbf{a}|$

length modulus length

EXERCISE 15F

1 Find the pairs of parallel vectors in the following:

$\mathbf{a} = \begin{bmatrix}2\\5\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}3\\-1\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-2\\5\end{bmatrix}$, $\mathbf{d} = \begin{bmatrix}-20\\50\end{bmatrix}$, $\mathbf{e} = \begin{bmatrix}-12\\4\end{bmatrix}$, $\mathbf{f} = \begin{bmatrix}10\\25\end{bmatrix}$

2 Find scalar t if the following pairs of vectors are parallel:

a $\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}9\\t\end{bmatrix}$ **b** $\begin{bmatrix}5\\-4\end{bmatrix}$ and $\begin{bmatrix}-10\\t\end{bmatrix}$ **c** $\begin{bmatrix}16\\t\end{bmatrix} = \begin{bmatrix}12\\9\end{bmatrix}$

G UNIT VECTORS

A **unit vector** is any vector which is one unit long.

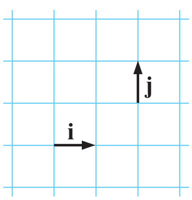
For example, if $\mathbf{a} = \begin{bmatrix}-1\\0\end{bmatrix}$, then $|\mathbf{a}| = \sqrt{(-1)^2+0^2} = \sqrt{1} = 1$ unit

and if $\mathbf{b} = \begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix}$, then

$$\begin{aligned}|\mathbf{b}| &= \sqrt{(\tfrac{\sqrt{3}}{2})^2 + (\tfrac{1}{2})^2}\\ &= \sqrt{\tfrac{3}{4}+\tfrac{1}{4}}\\ &= \sqrt{1}\\ &= 1 \text{ unit}\end{aligned}$$

So, both $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.

SPECIAL UNIT VECTORS i AND j



Notice that $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are unit vectors which are **horizontal** and **vertical** respectively and are directed positively.

We let $\mathbf{i} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix}0\\1\end{bmatrix}$.

$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the unit vector in the positive x-direction.

$\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the unit vector in the positive y-direction.

Notice that $3\mathbf{i} + 2\mathbf{j} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

In fact, any vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ can be written in **unit vector form**

as $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is the **component form** of a vector.

Example 21

Write **a** $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ in unit vector form **b** $2\mathbf{i} - 5\mathbf{j}$ in component form.

a $\begin{bmatrix} -3 \\ 4 \end{bmatrix} = -3\mathbf{i} + 4\mathbf{j}$

b $2\mathbf{i} - 5\mathbf{j} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

EXERCISE 15G

1 Which of the following are unit vectors?

a $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ **c** $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}$ **e** $\begin{bmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{bmatrix}$

2 Write in terms of $\mathbf{i}$ and $\mathbf{j}$:

a $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ **c** $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

3 Write in component form:

a $3\mathbf{i} + 5\mathbf{j}$ **b** $5\mathbf{i} - 4\mathbf{j}$ **c** $-4\mathbf{i}$ **d** $3\mathbf{j}$ **e** $\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

Example 22

Find k given that $\begin{bmatrix} -\frac{1}{3} \\ k \end{bmatrix}$ is a unit vector.

Since $\begin{bmatrix} -\frac{1}{3} \\ k \end{bmatrix}$ is a unit vector, $\sqrt{(-\frac{1}{3})^2 + k^2} = 1$

$\therefore \sqrt{\frac{1}{9} + k^2} = 1$

$\therefore \frac{1}{9} + k^2 = 1$

$\therefore k^2 = \frac{8}{9}$

$\therefore k = \pm\frac{\sqrt{8}}{3}$

4 Find k for the unit vectors:

a $\begin{bmatrix} 0 \\ k \end{bmatrix}$ **b** $\begin{bmatrix} k \\ 0 \end{bmatrix}$ **c** $\begin{bmatrix} k \\ 1 \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{1}{2} \\ k \end{bmatrix}$ **e** $\begin{bmatrix} k \\ \frac{2}{3} \end{bmatrix}$

Example 23

Find the length of $2\mathbf{i} - 5\mathbf{j}$.

As $2\mathbf{i} - 5\mathbf{j} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, its length is

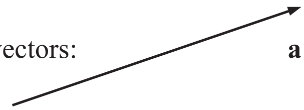
$$\sqrt{2^2 + (-5)^2}$$
$$= \sqrt{29} \text{ units}$$

5 Find the length of the vectors:

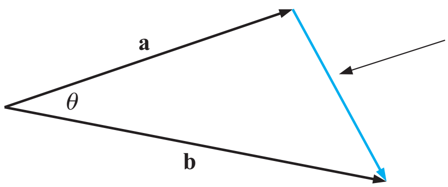
a $3\mathbf{i} + 4\mathbf{j}$ **b** $3\mathbf{i} - 4\mathbf{j}$ **c** $-2\mathbf{i} - 7\mathbf{j}$ **d** $-2.36\mathbf{i} + 5.65\mathbf{j}$

6 Find the unit vector in the direction of **a** $\mathbf{i} + 2\mathbf{j}$ **b** $2\mathbf{i} - 2\mathbf{j}$ **c** $-2\mathbf{i} - 5\mathbf{j}$

H ANGLES AND SCALAR PRODUCT

Consider vectors:  $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

We translate one of the vectors so that they both emanate from the same point.



This vector is $-\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$ and has length $|\mathbf{b} - \mathbf{a}|$.

Using the cosine rule, $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta$

But $\mathbf{b} - \mathbf{a} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}$

So $(b_1 - a_1)^2 + (b_2 - a_2)^2 = {a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta$

which simplifies to $a_1 b_1 + a_2 b_2 = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

So, $\cos\theta = \dfrac{a_1 b_1 + a_2 b_2}{|\mathbf{a}|\,|\mathbf{b}|}$ can be used to find the angle between two vectors **a** and **b**.

SCALAR PRODUCT

Two vectors are perpendicular if $\theta = 90^o$. So, $\cos\theta = 0$ and $a_1 b_1 + a_2 b_2 = 0$.

The quantity $a_1 b_1 + a_2 b_2$ is a scalar and is called the **scalar product** (or **dot product** or **inner product**) of vectors **a** and **b**.

$\mathbf{a} \bullet \mathbf{b}$ is the notation used for the scalar product.

Notice that for $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\boxed{\mathbf{a} \bullet \mathbf{b} = \underbrace{a_1b_1 + a_2b_2}_{\text{algebraic form}} = \underbrace{|\mathbf{a}|\,|\mathbf{b}| \cos\theta}_{\text{geometric form}}.}$

Example 24

Find $\mathbf{p} \bullet \mathbf{q}$ for

$\mathbf{p} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$$\begin{aligned} \mathbf{p} \bullet \mathbf{q} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ &= 2 \times 1 + 3 \times 4 \\ &= 2 + 12 \\ &= 14 \end{aligned}$$

EXERCISE 15H

1 For $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, find:

- **a** $\mathbf{q} \bullet \mathbf{p}$
- **b** $\mathbf{q} \bullet \mathbf{r}$
- **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$
- **d** $3\mathbf{r} \bullet \mathbf{q}$
- **e** $2\mathbf{p} \bullet 2\mathbf{p}$
- **f** $\mathbf{i} \bullet \mathbf{p}$
- **g** $\mathbf{q} \bullet \mathbf{j}$
- **h** $\mathbf{i} \bullet \mathbf{i}$

Example 25

Find t such that $\mathbf{a} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ t \end{bmatrix}$ are perpendicular.

Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a} \bullet \mathbf{b} = 0$

$$\begin{aligned} \therefore \quad \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ t \end{bmatrix} &= 0 \\ \therefore \quad (-1)(2) + 5t &= 0 \\ \therefore \quad -2 + 5t &= 0 \\ \therefore \quad 5t &= 2 \\ \text{and so} \quad t &= \tfrac{2}{5} \end{aligned}$$

2 Find t given that these vectors are perpendicular:

- **a** $\mathbf{p} = \begin{bmatrix} 3 \\ t \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- **b** $\mathbf{m} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} 10 \\ t \end{bmatrix}$
- **c** $\mathbf{a} = \begin{bmatrix} t \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} t \\ 6 \end{bmatrix}$
- **d** $\mathbf{r} = \begin{bmatrix} t \\ t+2 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$
- **e** $\mathbf{a} = \begin{bmatrix} t \\ t+2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2-3t \\ t \end{bmatrix}$

Example 26

Use scalar product to check if triangle ABC is right angled and if so find the right angle. A is (2, 1), B(6, −1) and C(5, −3).

$\overrightarrow{AB} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $\overrightarrow{BC} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

and we notice that $\overrightarrow{AB} \bullet \overrightarrow{BC} = 4(-1) + (-2)(-2) = -4 + 4 = 0$

$\therefore$ $\overrightarrow{AB} \perp \overrightarrow{BC}$. So triangle ABC is right angled at B.

3 Use scalar product to check if ΔABC is right angled and if so, find the right angle.

a A(−2, 1), B(−2, 5) and C(3, 1)
b A(4, 7), B(1, 2) and C(−1, 6)
c A(2, −2), B(5, 7) and C(−1, −1)
d A(10, 1), B(5, 2) and C(7, 4)

Example 27

Find the form of all vectors which are perpendicular to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 3 \end{bmatrix} = -12 + 12 = 0.$$

So, $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is one such vector

$\therefore$ required vectors have form $k\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ where $k \neq 0$.

4 Find the form of all vectors which are perpendicular to:

a $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$
b $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$
c $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$
d $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$
e $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Notice that, as $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$:

if θ is acute, $\cos\theta > 0$ and $\therefore$ $\mathbf{a} \bullet \mathbf{b} > \mathbf{0}$
if θ is obtuse, $\cos\theta < 0$ and $\therefore$ $\mathbf{a} \bullet \mathbf{b} < \mathbf{0}$.

Example 28

$\mathbf{a}$ and $\mathbf{b}$ have lengths 5 units and $\sqrt{7}$ units respectively and the angle between them is 110^o. Find $\mathbf{a} \bullet \mathbf{b}$.

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \quad \{\text{geometric form}\} \\ &= 5 \times \sqrt{7} \times \cos 110^o \\ &\doteqdot -4.525 \end{aligned}$$

5 Find $\mathbf{p} \bullet \mathbf{q}$ for:

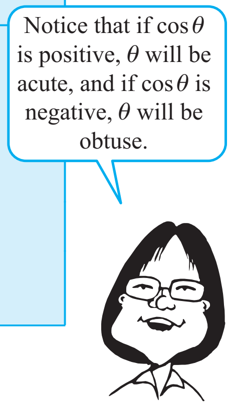
a $|\mathbf{p}| = 2$, $|\mathbf{q}| = 5$ and $\theta = 60^o$ **b** $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$ and $\theta = 120^o$

Example 29

Find the angle between $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

Since $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, then

$$(3)(1) + (-2)(7) = \sqrt{9+4}\sqrt{1+49}\cos\theta$$
$$\therefore \quad -11 = \sqrt{13}\sqrt{50}\cos\theta$$
$$\therefore \quad \cos\theta = \frac{-11}{\sqrt{650}}$$
$$\therefore \quad \theta = \cos^{-1}\left(\frac{-11}{\sqrt{650}}\right) \doteqdot 115.6^o$$



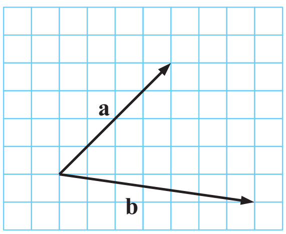
6 For each of the following find:

i the vectors **a** and **b** in component form

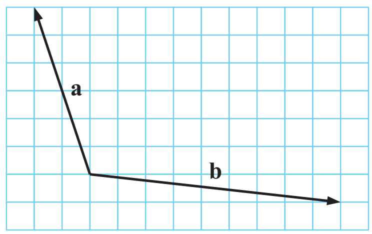
ii the angle between the two vectors.

Check your answers by using a protractor.

a



b



7 **a** Accurately draw diagrams to show $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$.

b Use vector methods to calculate the size of $\measuredangle$BAC.

c Measure $\measuredangle$BAC using a protractor.

8 Find the measure of the angle between the vectors **r** and **s** for:

a $\mathbf{r} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\mathbf{r} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

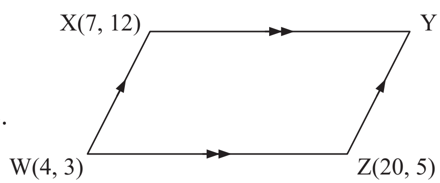
c $\mathbf{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **d** $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{i} + \mathbf{j}$

9 Given the parallelogram shown:

a find the coordinates of Y.

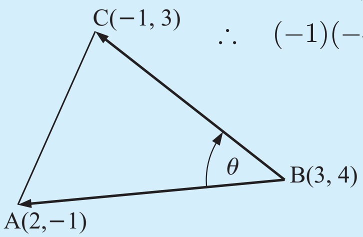
b Use $\overrightarrow{WX}$ and $\overrightarrow{WZ}$ to find the size of $\measuredangle$XWZ.

c Find the area of WXYZ.



Example 30

Find the measure of angle ABC for A(2, −1), B(3, 4) and C(−1, 3).



We draw vectors **away** from B, i.e., $\overrightarrow{BA}$ and $\overrightarrow{BC}$

$\overrightarrow{BA} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$ and $\overrightarrow{BC} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$.

$$\text{Now} \quad \overrightarrow{BA} \bullet \overrightarrow{BC} = |\overrightarrow{BA}|\ |\overrightarrow{BC}| \cos\theta$$

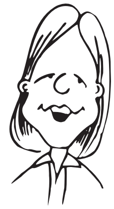
$$\therefore \quad (-1)(-4) + (-5)(-1) = \sqrt{1+25}\sqrt{16+1}\cos\theta$$

$$\therefore \quad 9 = \sqrt{26}\sqrt{17}\cos\theta$$

$$\therefore \quad \cos\theta = \frac{9}{\sqrt{442}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{9}{\sqrt{442}}\right)$$

$$\therefore \quad \theta \doteqdot 64.65^o$$



Notice that vectors used must both be away from B (or towards B). If this is not done you will be finding the exterior angle at B.

10 Find the measure of angle ABC for:

a A(3, 1), B(1, 2) and C(4, −1) **b** A(5, 0), B(−1, 3) and C(2, 8)

11 Find the measure of all angles of triangle ABC for:

a A(3, −1), B(−2, 4) and C(1, 0) **b** A(1, 4), B(3, −1) and C(−1, 2)

Example 31

Find the measure of the angle between the lines $2x + y = 5$ and $3x - 2y = 8$.

$2x + y = 5$ has slope $-\frac{2}{1}$ and $\therefore$ has direction vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{a}$, say.

$3x - 2y = 8$ has slope $\frac{3}{2}$ and $\therefore$ has direction vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{b}$, say.

If the angle between the lines is θ, then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}$$

$$= \frac{(1 \times 2) + (-2 \times 3)}{\sqrt{1+4}\sqrt{4+9}}$$

$$= \frac{-4}{\sqrt{5}\sqrt{13}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-4}{\sqrt{65}}\right) \doteqdot 119.7^o$$

$\therefore$ the angle is 119.7^o or 60.3^o.

If a line has slope $\frac{b}{a}$ it has direction vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

12 Find the measure of the angle between the lines:

a $x - y = 3$ and $3x + 2y = 11$ **b** $y = x + 2$ and $y = 1 - 3x$

c $y + x = 7$ and $x - 3y + 2 = 0$ **d** $y = 2 - x$ and $x - 2y = 7$

REVIEW SET 15A

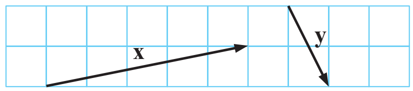
1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

a an aeroplane taking off at an angle of 8^o to the runway with a speed of 60 ms^{-1}

b a displacement of 45 m in a direction 060^o.

2 Copy the given vectors and find geometrically:

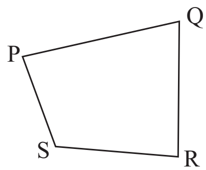
a $\mathbf{x} + \mathbf{y}$ **b** $\mathbf{y} - 2\mathbf{x}$



3 Find a single vector which is equal to:

a $\overrightarrow{PR} + \overrightarrow{RQ}$

b $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR}$



4 Dino walks for 9 km in a direction 246^o and then for 6 km in a direction 096^o. Find his displacement from his starting point.

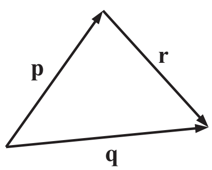
5 Simplify **a** $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$.

6 What geometrical facts can be deduced from the equations:

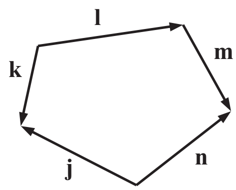
a $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD}$ **b** $\overrightarrow{AB} = 2\overrightarrow{AC}$?

7 Construct vector equations for:

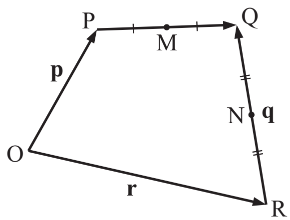
a



b



8



In the figure alongside $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$ and $\overrightarrow{RQ} = \mathbf{q}$.

If M and N are midpoints of the sides as shown, find in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$:

a $\overrightarrow{OQ}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{ON}$ **d** $\overrightarrow{MN}$

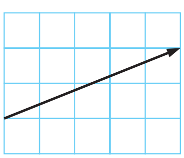
REVIEW SET 15B

1 Draw arrow diagrams to represent:

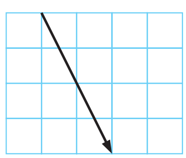
a $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ **c** $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$

2 Write the illustrated vectors in component form:

a



b



3 If $\mathbf{p} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, and $\mathbf{r} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ find:

a $2\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} - 3\mathbf{r}$ **c** $\mathbf{p} - \mathbf{q} + \mathbf{r}$

4 Given $\mathbf{p} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ find:

a $\mathbf{p} + \mathbf{q} - \mathbf{r}$ **b** $2\mathbf{q} - 3\mathbf{r}$ **c** $\mathbf{r} + 2\mathbf{p} - \mathbf{q}$

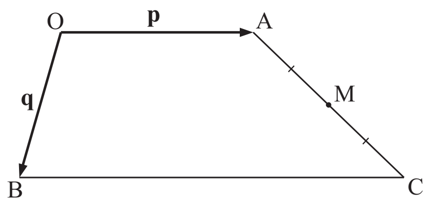
5 If $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\overrightarrow{CA} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find CB.

6 If $\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $\overrightarrow{RQ} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\overrightarrow{RS} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\overrightarrow{SP}$.

7 If $\mathbf{r} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|2\mathbf{s} - \mathbf{r}|$

8



BC is parallel to OA and is twice its length. Find in terms of **p** and **q** vector expressions for **a** $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$.

REVIEW SET 15C

1 If $\mathbf{p} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find $\mathbf{x}$ if:

a $\mathbf{p} - 3\mathbf{x} = \mathbf{0}$ **b** $2\mathbf{q} - \mathbf{x} = \mathbf{r}$

2 For P$(-3, 5)$, Q$(2, -1)$, and R$(1, 3)$ find:

a $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$ **c** $|\overrightarrow{PR}|$

3 Use vectors to show that WYZX is a parallelogram if X is $(-2, 5)$, Y$(3, 4)$, W$(-3, -1)$, and Z$(4, 10)$.

4 Find scalars r and s such that $r\begin{bmatrix} -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 13 \\ -24 \end{bmatrix}$.

5 AB and CD are diameters of a circle centre O. If $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{OB} = \mathbf{r}$, find:

a $\overrightarrow{DB}$ in terms of **q** and **r** **b** $\overrightarrow{AC}$ in terms of **q** and **r**.

What can be deduced about DB and AC?

6 If $\mathbf{p} = \mathbf{j} - 2\mathbf{i}$ find the:

a unit vector in the direction of $-\mathbf{p}$ **b** magnitude and direction of **p**.

7 If $\mathbf{x} = -\mathbf{i} + 3\mathbf{j}$ and $\mathbf{y} = -\mathbf{i} - 2\mathbf{j}$ find:

a $3\mathbf{x} - \mathbf{y}$ **b** $|\mathbf{x}|$ **c** a unit vector in the direction of $\mathbf{y} - \mathbf{x}$.

8 Find k if the following are unit vectors: **a** $\begin{bmatrix} \frac{4}{7} \\ \frac{1}{k} \end{bmatrix}$ **b** $\begin{bmatrix} k \\ k \end{bmatrix}$

REVIEW SET 15D

1 If $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, and $\mathbf{r} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r})$

2 Using $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ verify that $\mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$.

3 Determine the value of t if $\begin{bmatrix} 3 \\ 3 - 2t \end{bmatrix}$ and $\begin{bmatrix} t^2 + t \\ -2 \end{bmatrix}$ are perpendicular.

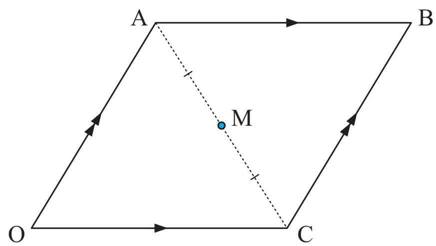
4 Given A(2, 3), B(−1, 4) and C(3, k), find k if $\measuredangle$BAC is a right angle.

5 Find all vectors which are perpendicular to the vector $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$.

6 Find the measure of all angles of triangle KLM for K(−2, 1), L(3, 2) and M(1, −3).

7 Determine the angle between the two lines with equations $4x - 5y = 11$ and $2x + 3y = 7$.

8 **a** **Do not** assume any diagonal properties of parallelograms. OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{p}$ and $\overrightarrow{OC} = \mathbf{q}$. M is the midpoint of AC.

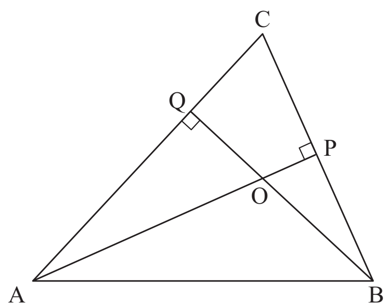


i Find in terms of $\mathbf{p}$ and $\mathbf{q}$:

(1) $\overrightarrow{OB}$ **(2)** $\overrightarrow{OM}$

ii Show using **i** only that O, M and B are collinear and M is the midpoint of OB.

b AP and BQ are altitudes of triangle ABC. Let $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OB} = \mathbf{q}$ and $\overrightarrow{OC} = \mathbf{r}$.



i Find vector expressions for $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.

ii Deduce that $\mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r}$.

iii Hence prove that OC is perpendicular to AB.

Chapter 16

Vectors in 3-dimensions

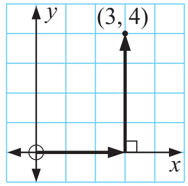
Contents:

A 3-dimensional coordinates
B 3-dimensional vectors
C Algebraic operations with 3-D vectors
D Parallelism
E Unit vectors
F Collinear points and ratio of division – extension
G The scalar product of 3-D vectors

Review set 16A
Review set 16B

INTRODUCTION

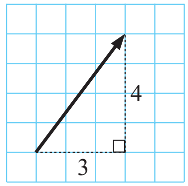
In **Chapter 15**, we considered 2-dimensional coordinate and vector geometry.



Points were specified in terms of ordered pairs such as (3, 4).

(3, 4) is found by starting at the origin O(0, 0), moving 3 units along the x-axis in the positive direction and then 4 units vertically.

Vectors were also specified using a pair of components.



$\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a **vector** which is represented by a directed line segment or arrow and has x-step 3 and y-step 4 as shown.

> **Vectors** are used to specify quantities which have size (magnitude) and direction.

Such quantities include:

- velocity
- force
- acceleration
- displacement
- work
- moments
- momentum
- weight

Vectors are used in navigation, physics, engineering and a host of other applied sciences.

In 3-dimensional geometry we extend the number plane to space where there is a third dimension, depth.
Consequently, points are specified as an ordered number triple, for example, (3, 4, 7) and vectors likewise, for example, $\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$.

It should be noted that vectors become a bridge between geometry and algebra.
For example, if **a** is parallel to **b** then $\mathbf{a} = k\mathbf{b}$ for some scalar k.

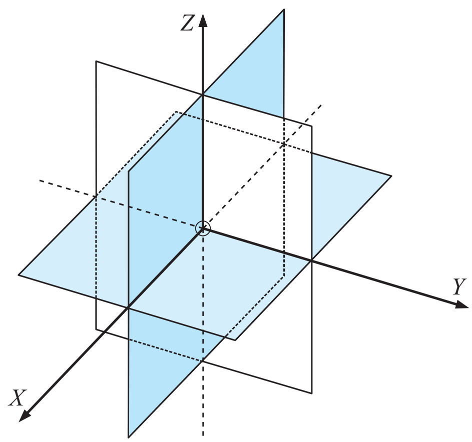
A 3-DIMENSIONAL COORDINATES

To specify points in **space** (or **3-dimensional space**) we need a point of reference, O, called the **origin**.

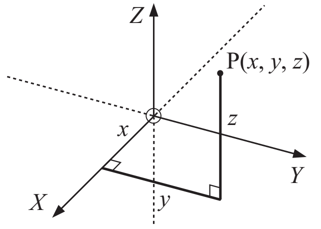
Through O we draw 3 **mutually perpendicular** lines and call them the X, Y and Z-axes. The X-axis is considered to come directly out of the page.

In the diagram alongside the **coordinate planes** divide space into 8 regions, each pair of planes intersecting on the axes.

The **positive direction** of each axis is a solid line whereas the **negative direction** is 'dashed'.



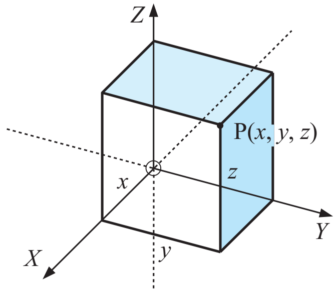
Any point P, in space can be specified by an **ordered triple** of numbers $(x,\ y,\ z)$ where x, y and z are the **steps** in the X, Y and Z directions from the origin O, to P.



The **position vector** of P is $\overrightarrow{\mathrm{OP}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

To help us visualise the 3-D position of a point on our 2-D paper, it is useful to complete a rectangular prism (or box) with the origin O as one vertex, the axes as sides adjacent to it, and P is at the vertex opposite O.

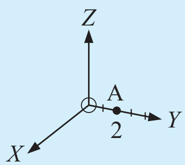
3-D POINT PLOTTER



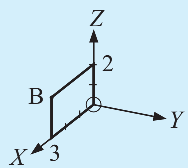
Example 1

Illustrate the points: **a** A(0, 2, 0) **b** B(3, 0, 2) **c** C(−1, 2, 3)

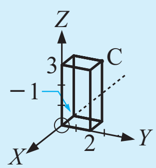
a



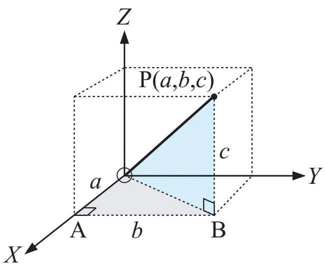
b



c



DISTANCE AND MIDPOINTS



Triangle OAB is right angled at A

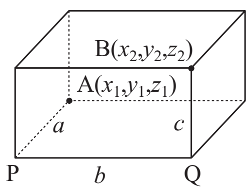
$\therefore\ \mathrm{OB}^2 = a^2 + b^2$ ...... (1) {Pythagoras}

Triangle OBP is right angled at B

$\therefore\ \mathrm{OP}^2 = \mathrm{OB}^2 + c^2$ {Pythagoras}

$\therefore\ \mathrm{OP}^2 = a^2 + b^2 + c^2$ {from (1)}

$\therefore\ \mathrm{OP} = \sqrt{a^2 + b^2 + c^2}$



Now for two general points $\mathrm{A}(x_1,\ y_1,\ z_1)$ and $\mathrm{B}(x_2,\ y_2,\ z_2)$

a, the x-step from A to B $= x_2 - x_1 = \Delta x$
b, the y-step from A to B $= y_2 - y_1 = \Delta y$
c, the z-step from A to B $= z_2 - z_1 = \Delta z$

$$\therefore\ \mathrm{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\text{or}\quad \mathrm{AB} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

$$\therefore\ \mathrm{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A simple extension from 2-D to 3-D geometry also gives the

$$\text{midpoint of AB} = \left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right).$$

Example 2

If A$(-1, 2, 4)$ and B$(1, 0, -1)$ are two points in space, find:

a the distance from A to B **b** the coordinates of the midpoint of AB.

a $\text{AB} = \sqrt{(1--1)^2+(0-2)^2+(-1-4)^2}$
$= \sqrt{4+4+25}$
$= \sqrt{33}$ units

b midpoint is $\left(\frac{-1+1}{2}, \frac{2+0}{2}, \frac{4+(-1)}{2}\right)$ i.e., $(0, 1, \frac{3}{2})$

EXERCISE 16A

1 Illustrate P and find its distance to the origin O if P is:

a $(0, 0, -3)$ **b** $(0, -1, 2)$ **c** $(3, 1, 4)$ **d** $(-1, -2, 3)$

2 For each of the following:

i find the distance AB **ii** find the midpoint of AB.

a A$(-1, 2, 3)$ and B$(0, -1, 1)$ **b** A$(0, 0, 0)$ and B$(2, -1, 3)$
c A$(3, -1, -1)$ and B$(-1, 0, 1)$ **d** A$(2, 0, -3)$ and B$(0, 1, 0)$

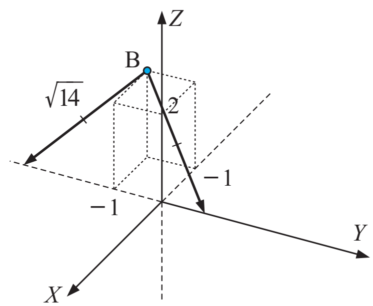
3 Show that P$(0, 4, 4)$, Q$(2, 6, 5)$ and R$(1, 4, 3)$ are vertices of an isosceles triangle.

4 Determine the nature of triangle ABC using distances for:

a A$(2, -1, 7)$, B$(3, 1, 4)$ and C$(5, 4, 5)$
b A$(0, 0, 3)$, B$(2, 8, 1)$ and C$(-9, 6, 18)$
c A$(5, 6, -2)$, B$(6, 12, 9)$ and C$(2, 4, 2)$.
d A$(1, 0, -3)$, B$(2, 2, 0)$ and C$(4, 6, 6)$.

5 A sphere has centre C$(-1, 2, 4)$ and diameter AB where A is $(-2, 1, 3)$. Find the coordinates of B and the radius of the circle.

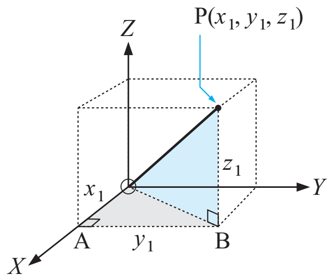
6 **a** State the coordinates of any point on the Y-axis.
b Find the coordinates of two points on the Y-axis which are $\sqrt{14}$ units from B$(-1, -1, 2)$.



B 3-DIMENSIONAL VECTORS

Consider a point $P(x_1, y_1, z_1)$.

The x, y and z-steps from the origin to P are x_1, y_1 and z_1 respectively.



So $\quad \overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is the vector which emanates from O and terminates at P.

In general, if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in space then:

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \begin{array}{l} \leftarrow x\text{-step} \\ \leftarrow y\text{-step} \\ \leftarrow z\text{-step} \end{array} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

$\overrightarrow{AB}$ is called 'vector AB' or 'the **position vector of B relative to A** (or from A)'

Example 3

If A is $(3, -1, 2)$ and B is $(1, 0, -2)$ find: **a** $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$

a $\overrightarrow{OA} = \begin{bmatrix} 3-0 \\ -1-0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ **b** $\overrightarrow{AB} = \begin{bmatrix} 1-3 \\ 0-(-1) \\ -2-2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$

Note: $\overrightarrow{OA}$ has length $|\overrightarrow{OA}|$ or simply OA, $\overrightarrow{AB}$ has length $|\overrightarrow{AB}|$ or simply AB.

Example 4

If P is $(-3, 1, 2)$ and Q is $(1, -1, 3)$, find $|\overrightarrow{PQ}|$.

$\overrightarrow{PQ} = \begin{bmatrix} 1-(-3) \\ -1-1 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ $\quad \therefore \quad |\overrightarrow{PQ}| = \sqrt{4^2 + (-2)^2 + 1^2}$

$= \sqrt{21}$ units

EXERCISE 16B.1

1 Consider the point $T(3, -1, 4)$.

a Draw a diagram to locate the position of T in space.

b Find $\overrightarrow{OT}$. **c** How far is it from O to T?

2 Given $A(-3, 1, 2)$ and $B(1, 0, -1)$ find:

a $\overrightarrow{AB}$ and $\overrightarrow{BA}$ **b** the length of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

3 Given $A(3, 1, 0)$ and $B(-1, 1, 2)$ find $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{AB}$.

Example 5

If A is $(-1, 3, 2)$ and $B(2, 1, -4)$ find:

a the position vector of A from B **b** the distance between A and B.

a The position vector of A from B is $\overrightarrow{BA} = \begin{bmatrix} -1-2 \\ 3-1 \\ 2-(-4) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}$

b $\text{AB} = |\overrightarrow{BA}|$

$= \sqrt{9+4+36}$

$= 7$ units

4 Given $M(4, -2, -1)$ and $N(-1, 2, 0)$ find:

a the position vector of M from N **b** the position vector of N from M

c the distance between M and N.

5 For $A(-1, 2, 5)$, $B(2, 0, 3)$ and $C(-3, 1, 0)$ find the position vector of:

a A from O and the distance of A from O

b C from A and the distance of C from A

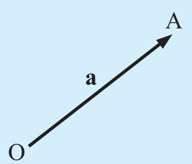
c B from C and the distance of B from C.

6 Find the distance from $Q(3, 1, -2)$ to:

a the Y-axis **b** the origin **c** the YOZ plane.

GEOMETRIC REPRESENTATION

As for 2-D vectors, 3-D vectors are represented by **directed line segments** (called **arrows**).
Consider the vector represented by the line segment from O to A.



- This **vector** could be represented by

  $\overrightarrow{OA}$ or **a** or $\underset{\sim}{a}$.

  bold used in text books

  used by students

- The **magnitude (length)** could be represented by $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\underset{\sim}{a}|$.

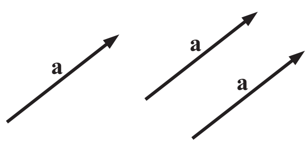
  If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ then $|\mathbf{a}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$.

VECTOR EQUALITY

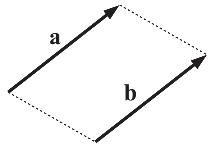
Two vectors are **equal** if they have the same magnitude and direction.

So, if arrows are used to represent vectors, then equal vectors are **parallel** and **equal in length**.

This means that equal vector arrows are translations of one another, but in space.



If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then $\mathbf{a} = \mathbf{b} \ \Leftrightarrow \ a_1 = b_1, \ a_2 = b_2, \ a_3 = b_3$.



$\mathbf{a} = \mathbf{b}$ implies that vector **a** is parallel to vector **b**.

Consequently, **a** and **b** are opposite sides of a parallelogram, and certainly lie in the same plane.

DISCUSSION

- Do any three points in space define a plane? What about four points? Illustrate.
- What simple test(s) on four points in space enables us to deduce that the points are vertices of a parallelogram? Consider using vectors and not using vectors.

Example 6

Find a, b, and c if $\begin{bmatrix} a-3 \\ b-2 \\ c-1 \end{bmatrix} = \begin{bmatrix} 1-a \\ -b \\ -3-c \end{bmatrix}$.

Equating components we get

$$\begin{array}{llll} & a-3 = 1-a & b-2 = -b & c-1 = -3-c \\ \therefore & 2a = 4 & 2b = 2 & 2c = -2 \\ \therefore & a = 2 & b = 1 & c = -1 \end{array}$$

EXERCISE 16B.2

1 Find a, b and c if:

a $\begin{bmatrix} a-4 \\ b-3 \\ c+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ **b** $\begin{bmatrix} a-5 \\ b-2 \\ c+3 \end{bmatrix} = \begin{bmatrix} 3-a \\ 2-b \\ 5-c \end{bmatrix}$

2 Find scalars a, b and c if:

a $2\begin{bmatrix} 1 \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} = \begin{bmatrix} b \\ a^2 \\ a+b \end{bmatrix}$

c $a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

3 A$(-1, 3, 4)$, B$(2, 5, -1)$, C$(-1, 2, -2)$ and D(r, s, t) are four points in space. Find r, s and t if:

a $\overrightarrow{AC} = \overrightarrow{BD}$ **b** $\overrightarrow{AB} = \overrightarrow{DC}$

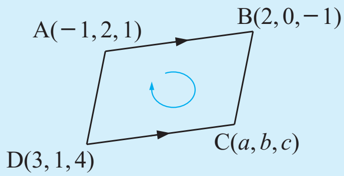
4 A quadrilateral has vertices A$(1, 2, 3)$, B$(3, -3, 2)$, C$(7, -4, 5)$ and D$(5, 1, 6)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$. **b** What can be deduced about the quadrilateral ABCD?

Example 7

ABCD is a parallelogram. A is $(-1,\ 2,\ 1)$, B is $(2,\ 0,\ -1)$ and D is $(3,\ 1,\ 4)$. Find the coordinates of C.

First we draw an axis free sketch:



Let C be $(a,\ b,\ c)$.

Now as AB is parallel to DC and has the same length then $\overrightarrow{DC} = \overrightarrow{AB}$,

i.e., $$\begin{bmatrix} a-3 \\ b-1 \\ c-4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

$\therefore\quad a-3=3,\quad b-1=-2,\quad c-4=-2$

$\therefore\quad a=6,\quad \therefore\quad b=-1,\quad \therefore\quad c=2$ So, C is $(6,\ -1,\ 2)$.

Check: midpoint of DB is
$$\left(\frac{3+2}{2},\ \frac{1+0}{2},\ \frac{4+-1}{2}\right)$$
i.e., $\left(\frac{5}{2},\ \frac{1}{2},\ \frac{3}{2}\right)$

midpoint of AC is
$$\left(\frac{-1+6}{2},\ \frac{2+-1}{2},\ \frac{1+2}{2}\right)$$
i.e., $\left(\frac{5}{2},\ \frac{1}{2},\ \frac{3}{2}\right)$

5 PQRS is a parallelogram. P is $(-1,\ 2,\ 3)$, $Q(1,\ -2,\ 5)$ and $R(0,\ 4,\ -1)$.

- **a** Use vectors to find the coordinates of S.
- **b** Use midpoints of diagonals to check your answer.

C ALGEBRAIC OPERATIONS WITH 3-D VECTORS

Similar operations can be performed with 3-D vectors to those performed in **Chapter 15** with 2-D vectors. These are:

If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ then $\mathbf{a}+\mathbf{b} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$, $\mathbf{a}-\mathbf{b} = \begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{bmatrix}$

and $k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$ for some scalar k

SOME PROPERTIES OF VECTORS

- $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$
  $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$
  $\mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a} = \mathbf{a}$
  $\mathbf{a}+(-\mathbf{a}) = (-\mathbf{a})+\mathbf{a} = \mathbf{0}$
- $\underbrace{|k\mathbf{a}|}_{\text{length of } k\mathbf{a}} = \underbrace{|k|}_{\text{modulus of } k}\ \underbrace{|\mathbf{a}|}_{\text{length of } \mathbf{a}}$ where $k\mathbf{a}$ is parallel to $\mathbf{a}$

Example 8

If $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, find: **a** $\mathbf{p} + 2\mathbf{q}$ **b** $2\mathbf{p} - 3\mathbf{q}$

a $\mathbf{p} + 2\mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+8 \\ 2-2 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$

b $2\mathbf{p} - 3\mathbf{q} = 2\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} - 3\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2-12 \\ 4+3 \\ -6-6 \end{bmatrix} = \begin{bmatrix} -14 \\ 7 \\ -12 \end{bmatrix}$

EXERCISE 16C

1 For $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{b} + 2\mathbf{c}$ **d** $\mathbf{a} - 3\mathbf{c}$

e $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **f** $\mathbf{c} - \frac{1}{2}\mathbf{a}$ **g** $\mathbf{a} - \mathbf{b} - \mathbf{c}$ **h** $2\mathbf{b} - \mathbf{c} + \mathbf{a}$

Example 9

If $\mathbf{a} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, find $|\mathbf{a}|$.

$$\begin{aligned} |\mathbf{a}| &= \sqrt{(-1)^2 + 3^2 + 2^2} \\ &= \sqrt{1+9+4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

2 If $\mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$ find:

a $|\mathbf{a}|$ **b** $|\mathbf{b}|$ **c** $|\mathbf{b} + \mathbf{c}|$ **d** $|\mathbf{a} - \mathbf{c}|$ **e** $|\mathbf{a}|\mathbf{b}$ **f** $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$

Example 10

If $\mathbf{a} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$ find x if: **a** $2\mathbf{x} = \mathbf{b}$ **b** $\mathbf{b} - 2\mathbf{x} = \mathbf{a}$

a
$$\begin{aligned} 2\mathbf{x} &= \mathbf{b} \\ \therefore\ \mathbf{x} &= \tfrac{1}{2}\mathbf{b} \\ \therefore\ \mathbf{x} &= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \end{aligned}$$

b
$$\begin{aligned} \mathbf{b} - 2\mathbf{x} &= \mathbf{a} \\ \therefore\ \mathbf{x} &= \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \quad \{\text{as } 2\mathbf{x} = \mathbf{b} - \mathbf{a}\} \\ \therefore\ \mathbf{x} &= \tfrac{1}{2}\left(\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \end{aligned}$$

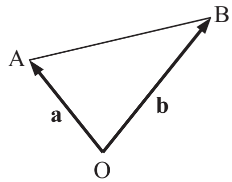
3 Solve for $\mathbf{x}$:

a $2\mathbf{x} + \mathbf{a} = \mathbf{b}$ **b** $\mathbf{b} - 3\mathbf{x} = 2\mathbf{a}$ **c** $\mathbf{a} + 2\mathbf{x} = \mathbf{b} - \mathbf{x}$

4 If $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ find $\mathbf{x}$ if:

a $2\mathbf{a} + \mathbf{x} = \mathbf{b}$ **b** $3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$ **c** $2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$

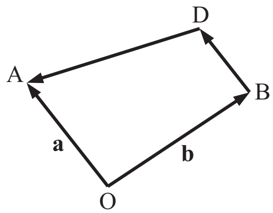
Notice that if $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ where O is the origin then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$.



5 If $\overrightarrow{OA} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ find $\overrightarrow{AB}$ and hence find the distance from A to B.

6 The position vectors of A, B, C and D from O are $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ respectively. Deduce that $\overrightarrow{BD} = 2\overrightarrow{AC}$.

7



In the given figure BD is parallel to OA and half its length. Find in terms of $\mathbf{a}$ and $\mathbf{b}$ vector expressions for:

a $\overrightarrow{BD}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

d $\overrightarrow{OD}$ **e** $\overrightarrow{AD}$ **f** $\overrightarrow{DA}$

Example 11

If $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ find $\overrightarrow{BC}$.

$$\begin{aligned} \overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\ &= -\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} \end{aligned}$$

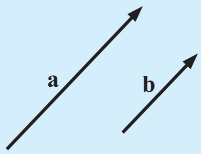
8 If $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\overrightarrow{BD} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ find:

a $\overrightarrow{AD}$ **b** $\overrightarrow{CB}$ **c** $\overrightarrow{CD}$

D PARALLELISM

PARALLELISM

If two vectors are **parallel**, then one is a scalar multiple of the other and vice versa.



Note:
- If **a** is parallel to **b**, then there exists a scalar, k say, such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - ▶ **a** is parallel to **b**, and
  - ▶ $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

Notice that $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}$ is parallel to $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 4 \\ 12 \\ -8 \end{bmatrix}$ as $\mathbf{a} = 2\mathbf{b}$ and $\mathbf{a} = \frac{1}{2}\mathbf{c}$.

Also $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}$ is parallel to $\mathbf{d} = \begin{bmatrix} -3 \\ -9 \\ 6 \end{bmatrix}$ as $\mathbf{a} = -\frac{3}{2}\mathbf{d}$.

Example 12

Find r and s given that $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ r \end{bmatrix}$ is parallel to $\mathbf{b} = \begin{bmatrix} s \\ 2 \\ -3 \end{bmatrix}$.

Since **a** and **b** are parallel, then $\mathbf{a} = k\mathbf{b}$ for some scalar k

$$\therefore \quad \begin{bmatrix} 2 \\ -1 \\ r \end{bmatrix} = k\begin{bmatrix} s \\ 2 \\ -3 \end{bmatrix}$$

$$\therefore \quad 2 = ks, \quad -1 = 2k \quad \text{and} \quad r = -3k$$

Consequently, $k = -\frac{1}{2}$ and $\therefore\ 2 = -\frac{1}{2}s$ and $r = -3\left(\frac{-1}{2}\right)$

$$\therefore \quad r = \frac{3}{2} \text{ and } s = -4$$

EXERCISE 16D

1 $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -6 \\ r \\ s \end{bmatrix}$ are parallel. Find r and s.

2 Find scalars a and b, given that $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} a \\ 2 \\ b \end{bmatrix}$ are parallel.

3 **a** Find a vector of length 1 unit which is parallel to $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.
(**Hint:** Let the vector be $k\mathbf{a}$.)

b Find a vector of length 2 units which is parallel to $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

4 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$ **d** $\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AC}$

5 The position vectors of P, Q, R and S from O are $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ respectively.

a Deduce that PR and QS are parallel.

b What is the relationship between the lengths of PR and QS?

6 Prove that $|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$ using a geometrical argument.

[**Hint:** Consider **a** **a** not parallel to **b** **b** **a** and **b** parallel **c** any other cases.]

E UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

For example,

- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a unit vector as its length is $\sqrt{1^2+0^2+0^2} = 1$
- $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is a unit vector as its length is $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$

$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are special unit vectors in the direction of the X, Y and Z-axes respectively.

Notice that $\underbrace{\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\text{component form}} \Leftrightarrow \mathbf{a} = \underbrace{a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}_{\text{unit vector form}}$.

Thus, $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$ can be written as $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and vice versa.

EXERCISE 16E

1 Express the following vectors in component form and find their length:

a $\mathbf{i} - \mathbf{j} + \mathbf{k}$ **b** $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ **c** $\mathbf{i} - 5\mathbf{k}$ **d** $\frac{1}{2}(\mathbf{j} + \mathbf{k})$

2 Express in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

a $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$ **c** $\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

3 Find the length of:

a $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ **c** $\mathbf{i} - \mathbf{j} + \mathbf{k}$ **d** $2\mathbf{i} - \mathbf{j}$

4 For $\mathbf{a} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} - \mathbf{a}$ **c** $2\mathbf{a} + 5\mathbf{b}$ **d** $3\mathbf{a} - 2\mathbf{b}$

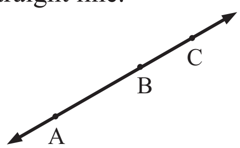
5 Find the unit vector in the direction of **a** $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ **b** $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

F COLLINEAR POINTS AND RATIO OF DIVISION – EXTENSION

Three or more points are said to be **collinear** if they lie on the same straight line.

Notice that,

A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.



Example 13

Prove that A$(-1, 2, 3)$, B$(4, 0, -1)$ and C$(14, -4, -9)$ are collinear and hence find the ratio in which B divides CA.

$$\overrightarrow{AB} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \qquad \overrightarrow{BC} = \begin{bmatrix} 10 \\ -4 \\ -8 \end{bmatrix} = 2\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \qquad \therefore \ \overrightarrow{BC} = 2\overrightarrow{AB}$$

$\therefore$ BC is parallel to AB and since B is common to both, A, B and C are collinear.

To find the ratio in which B divides CA, we find

$$\overrightarrow{CB} : \overrightarrow{BA} = -2\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} : -\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} = 2 : 1$$

$\therefore$ B divides CA internally in the ratio $2 : 1$.

EXERCISE 16F

1 **a** Prove that A$(-2, 1, 4)$, B$(4, 3, 0)$ and C$(19, 8, -10)$ are collinear and hence find the ratio in which A divides CB.

b Prove that P$(2, 1, 1)$, Q$(5, -5, -2)$ and R$(-1, 7, 4)$ are collinear and hence find the ratio in which Q divides PR.

2 **a** A(2, −3, 4), B(11, −9, 7) and C(−13, a, b) are collinear. Find a and b.

b K(1, −1, 0), L(4, −3, 7) and M(a, 2, b) are collinear. Find a and b.

G THE SCALAR PRODUCT OF 3-D VECTORS

If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the **scalar product** of **a** and **b** (also known as the **dot product**) is defined as $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

This definition is simply an extension of the 2-dimensional definition, adding the Z-component.

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

Dot product has the same algebraic properties for 3-D vectors as it has for its 2-D counterparts.

- $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
- $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ and
  $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

These properties are proven in general by using vectors such as $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, etc.

GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

- If θ is the angle between vectors **a** and **b** then: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$
- For non-zero vectors **a** and **b**: $\mathbf{a} \bullet \mathbf{b} = 0 \quad \Leftrightarrow \quad$ **a** and **b** are perpendicular.

The proofs of these results are identical for those in 2-dimensions.

Example 14

If $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** the angle between **p** and **q**

a
$$\begin{aligned} &\mathbf{p} \bullet \mathbf{q} \\ &= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= 2(-1) + 3(0) + (-1)2 \\ &= -2 + 0 - 2 \\ &= -4 \end{aligned}$$

b
$$\begin{aligned} \mathbf{p} \bullet \mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}| \cos\theta \\ \therefore\ -4 &= \sqrt{4+9+1}\sqrt{1+0+4}\cos\theta \\ \therefore\ -4 &= \sqrt{14}\sqrt{5}\cos\theta \\ \therefore\ -4 &= \sqrt{70}\cos\theta \\ \therefore\ \cos\theta &= -\tfrac{4}{\sqrt{70}} \\ \therefore\ \theta &= \cos^{-1}\left(-\tfrac{4}{\sqrt{70}}\right) \doteqdot 118.56^o \end{aligned}$$

EXERCISE 16G

1 For $\mathbf{a}=\begin{bmatrix}2\\1\\3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ find:

a $\mathbf{a}\bullet\mathbf{b}$ **b** $\mathbf{b}\bullet\mathbf{a}$ **c** $|\mathbf{a}|^2$

d $\mathbf{a}\bullet\mathbf{a}$ **e** $\mathbf{a}\bullet(\mathbf{b}+\mathbf{c})$ **f** $\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}$

2 Find:

a $(\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(2\mathbf{j}+\mathbf{k})$ **b** $\mathbf{i}\bullet\mathbf{i}$ **c** $\mathbf{i}\bullet\mathbf{j}$

3 Using $\mathbf{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ prove that $\mathbf{a}\bullet(\mathbf{b}+\mathbf{c})=\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}$.

Hence, prove that $(\mathbf{a}+\mathbf{b})\bullet(\mathbf{c}+\mathbf{d})=\mathbf{a}\bullet\mathbf{c}+\mathbf{a}\bullet\mathbf{d}+\mathbf{b}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{d}$.

4 Find t if $\begin{bmatrix}3\\-1\\t\end{bmatrix}$ and $\begin{bmatrix}2t\\-3\\-4\end{bmatrix}$ are perpendicular.

5 Show that $\mathbf{a}=\begin{bmatrix}3\\1\\2\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix}1\\5\\-4\end{bmatrix}$ are mutually perpendicular.

6 **a** Show that $\begin{bmatrix}1\\1\\5\end{bmatrix}$ and $\begin{bmatrix}2\\3\\-1\end{bmatrix}$ are perpendicular.

b Find t if $\begin{bmatrix}3\\t\\-2\end{bmatrix}$ is perpendicular to $\begin{bmatrix}1-t\\-3\\4\end{bmatrix}$.

7 Consider triangle ABC in which A is (5, 1, 2), B(6, -1, 0) and C(3, 2, 0). Using scalar product only, show that the triangle is right angled.

8 A(2, 4, 2), B(-1, 2, 3), C(-3, 3, 6) and D(0, 5, 5) are vertices of a quadrilateral.

a Prove that ABCD is a parallelogram.

b Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$. What can be said about ABCD?

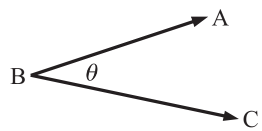
c Find $\overrightarrow{AC}\bullet\overrightarrow{BD}$. What property of figure ABCD has been found to be valid?

9 For $\mathbf{a}=-\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ find:

a $\mathbf{a}\bullet\mathbf{b}$ **b** the angle between $\mathbf{a}$ and $\mathbf{b}$

10 Find the angle ABC of triangle ABC for A(3, 0, 1), B(-3, 1, 2) and C(-2, 1, -1).

Reminder: To find the angle at B, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are used.



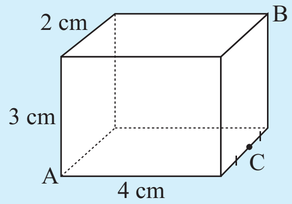
What angle is found if $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used?

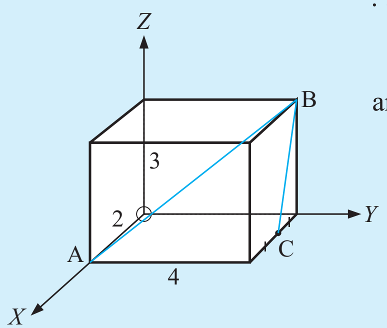
11 Find the measure of angle PQR for:

- **a** P(1, 2, 3), Q(0, 2, -1) and R(2, -1, 2)
- **b** P(2, 2, 1), Q(1, 2, 4) and R(3, 1, -1)
- **c** P(4, 3, 0), Q(0, 3, -2) and R(-3, 0, 1)
- **d** P(6, 2, 1), Q(4, 3, 1) and R(3, 1, -3)

Example 15

Use vector methods to determine the measure of angle ABC.





Placing the coordinate axes as illustrated,

A is (2, 0, 0), B is (0, 4, 3) and C is (1, 4, 0)

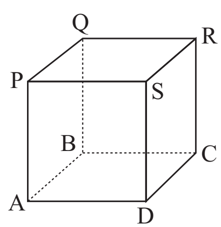
$$\therefore \quad \overrightarrow{BA} \text{ is } \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \text{ and } \overrightarrow{BC} \text{ is } \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$\begin{aligned} \text{and } \cos \measuredangle ABC &= \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}|\ |\overrightarrow{BC}|} \\ &= \frac{2(1) + (-4)(0) + (-3)(-3)}{\sqrt{4+16+9}.\sqrt{1+0+9}} \\ &= \frac{2+0+9}{\sqrt{29}.\sqrt{10}} \\ &= \frac{11}{\sqrt{290}} \end{aligned}$$

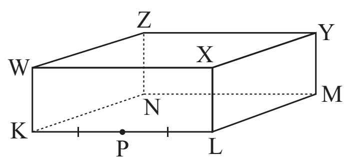
$$\therefore \quad \measuredangle ABC = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \doteqdot 49.76^o$$

12 For the cube alongside with sides of length 2 cm, find using vector methods:

- **a** the measure of angle ABS
- **b** the measure of angle RBP
- **c** the measure of angle PBS.



13



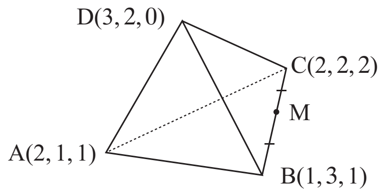
KL, LM and LX are 8, 5 and 3 units long respectively.
P is the midpoint of KL.
Find, using vector methods:

- **a** the measure of angle YNX
- **b** the measure of angle YNP.

14 For the tetrahedron ABCD:

a find the coordinates of M

b find the measure of angle DMA.



15 **a** Find t if $2\mathbf{i}+t\mathbf{j}+(t-2)\mathbf{k}$ and $t\mathbf{i}+3\mathbf{j}+t\mathbf{k}$ are perpendicular.

b Find r, s and t if $\mathbf{a}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}2\\2\\r\end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix}s\\t\\1\end{bmatrix}$ are mutually perpendicular.

16 Find the angle made by:

a the X-axis and the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$

b a line parallel to the Y-axis and the vector $\begin{bmatrix}-1\\1\\3\end{bmatrix}$.

17 Explain why $\mathbf{a}\bullet\mathbf{b}\bullet\mathbf{c}$ is meaningless.

REVIEW SET 16A

1 Given P(2, −5, 6) and Q(−1, 7, 9), find:

a the position vector of Q from P

b the distance from P to Q

c the distance from P to the x-axis.

2 For $\mathbf{m}=\begin{bmatrix}6\\-3\\1\end{bmatrix}$, $\mathbf{n}=\begin{bmatrix}2\\3\\-4\end{bmatrix}$ and $\mathbf{p}=\begin{bmatrix}-1\\3\\6\end{bmatrix}$, find:

a $\mathbf{m}-\mathbf{n}+\mathbf{p}$

b $2\mathbf{n}-3\mathbf{p}$

c $|\mathbf{m}+\mathbf{p}|$

3 If $\overrightarrow{AB}=\begin{bmatrix}2\\-7\\4\end{bmatrix}$ and $\overrightarrow{AC}=\begin{bmatrix}-6\\1\\-3\end{bmatrix}$, find $\overrightarrow{CB}$.

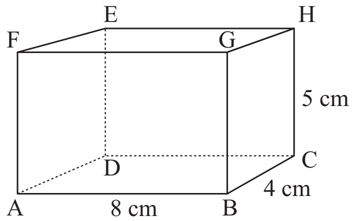
4 Find m and n if $\begin{bmatrix}3\\m\\n\end{bmatrix}$ and $\begin{bmatrix}-12\\-20\\2\end{bmatrix}$ are parallel vectors.

5 Prove that P(−6, 8, 2), Q(4, 6, 8) and R(19, 3, 17) are collinear. Hence find the ratio in which Q divides PR.

6 Find t if $\begin{bmatrix}-4\\t+2\\t\end{bmatrix}$ and $\begin{bmatrix}t\\1+t\\-3\end{bmatrix}$ are perpendicular vectors.

7 Determine the angle between $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

8 Find the measure of angle GAC in the rectangular box alongside. Use vector methods.



REVIEW SET 16B

1 Illustrate on a set of axes the points **a** A$(-1, 2, 1)$ **b** B$(2, -3, 4)$.

2 For P$(2, 3, -1)$ and Q$(-4, 4, 2)$ find:

a $\overrightarrow{PQ}$ **b** the distance between P and Q **c** the midpoint of PQ.

3 For $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ find:

a $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q} - \mathbf{r}$ **c** the angle between $\mathbf{p}$ and $\mathbf{r}$.

4 Determine all angles of the triangle with vertices K$(3, 1, 4)$, L$(-2, 1, 3)$ and M$(4, 1, 3)$.

5 Find t if $\begin{bmatrix} -4 \\ t \\ 1-t \end{bmatrix}$ is perpendicular to $\begin{bmatrix} t \\ t \\ -6-t \end{bmatrix}$.

6 If $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ verify that:

a $\mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$ **b** $\mathbf{x} \bullet (\mathbf{y} - \mathbf{z}) = \mathbf{x} \bullet \mathbf{y} - \mathbf{x} \bullet \mathbf{z}$.

7 Find the angle between $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

8 If A$(4, 2, -1)$, B$(-1, 5, 2)$, C$(3, -3, c)$ and triangle ABC is right angled at B, find possible values of c.

Chapter 17

Lines in the plane and in space

Contents:

A Vector and parametric form of a line in 2–dimensional geometry
B The velocity vector of a moving object
C Constant velocity problems
Investigation: The two yachts problem
D The closest distance
E Geometric applications of $\mathbf{r} = \mathbf{a} + t\mathbf{b}$
F Lines in space
G Line classification

Review set 17A
Review set 17B

INTRODUCTION

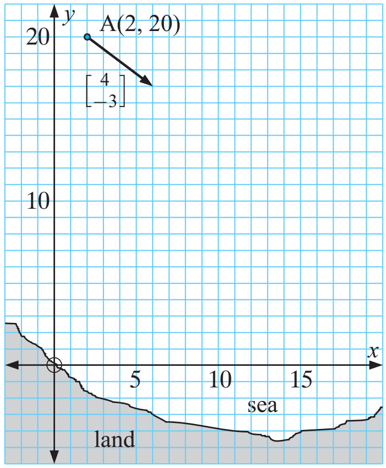
Suppose the vector $\begin{bmatrix}1\\0\end{bmatrix}$ represents a displacement of 1 km due East and

$\begin{bmatrix}0\\1\end{bmatrix}$ represents a displacement of 1 km due North.

The diagram shows the path of a yacht relative to a yacht club which is situated at (0, 0). At 12:00 noon the yacht is at the point A(2, 20).

The yacht is travelling in the direction $\begin{bmatrix}4\\-3\end{bmatrix}$.

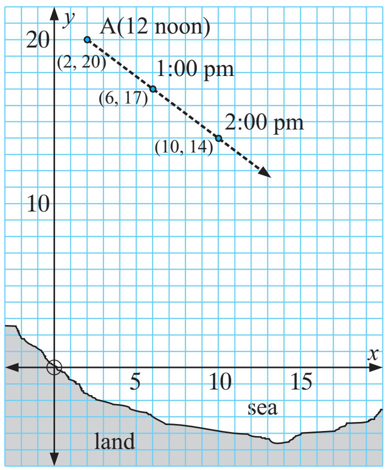
It has a constant speed of 5 km/h.



The following diagram shows the position of the yacht at 1:00 pm and 2:00 pm.

Since $\left|\begin{bmatrix}4\\-3\end{bmatrix}\right| = 5$ and the speed of the yacht is 5 km/h then $\begin{bmatrix}4\\-3\end{bmatrix}$ not only gives the direction of travel, but it also gives the distance travelled in one hour.

So, $\begin{bmatrix}4\\-3\end{bmatrix}$ is called the **velocity vector** of the yacht.



In order to define the position of the yacht at any time t hours after 12 noon, we can use the **parametric equations** $x = 2 + 4t$ and $y = 20 - 3t$ where t is called the **parameter**.

Note:
- if $t = 0$, $x = 2$ and $y = 20$, i.e., the yacht is at (2, 20)
  if $t = 1$, $x = 6$ and $y = 17$, i.e., the yacht is at (6, 17)
  if $t = 2$, $x = 10$ and $y = 14$, i.e., the yacht is at (10, 14)
- t hours after 12 noon, the position is $(2 + 4t, 20 - 3t)$
- $(2 + 4t, 20 - 3t)$ describes the position of the yacht at any time t hours.

The **cartesian equation** of the yacht's path can be found.

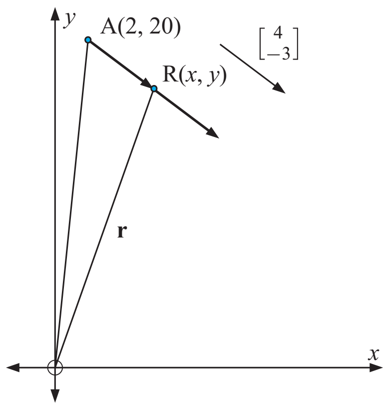
$$\text{As}\quad x = 2 + 4t, \qquad 4t = x - 2 \quad \text{and so} \quad t = \frac{x-2}{4}$$

Substituting into the second equation, $y = 20 - 3t$ we get

$$\begin{aligned} y &= 20 - 3\left(\frac{x-2}{4}\right) \\ \therefore\quad 4y &= 80 - 3(x-2) \\ \text{i.e.,}\quad 4y &= 80 - 3x + 6 \\ \therefore\quad 3x + 4y &= 86 \end{aligned}$$

The **vector equation** of the yacht's path can also be found.

Suppose the yacht is at R, t hours after 12:00 noon.



$$\text{Then}\quad \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

$$\therefore\quad \mathbf{r} = \begin{bmatrix} 2 \\ 20 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

and if R is at (x, y), then

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 20 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and this is called the **vector equation** of the yacht's path.

Note: The parametric equations are easily found from the vector equation,

i.e., if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 20 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ then $x = 2 + 4t$ and $y = 20 - 3t$.

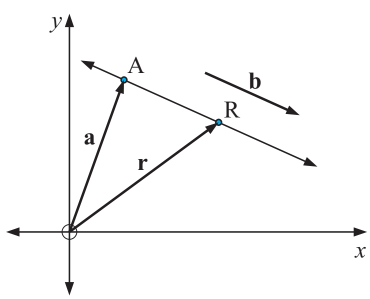
Consequently, the coordinates of the yacht's position are expressed in terms of t, the time since 12:00 noon.

For example, at 3:00 pm, $t = 3$ and so $x = 14$ and $y = 11$,
i.e., the yacht is at (14, 11).

A VECTOR AND PARAMETRIC FORM OF A LINE IN 2-DIMENSIONAL GEOMETRY

Suppose a line passes through a fixed point A (where $\overrightarrow{OA} = \mathbf{a}$) and has direction defined by vector $\mathbf{b}$.

Let R move on the line so that $\overrightarrow{OR} = \mathbf{r}$.



$$\text{We see that}\quad \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

$$\therefore\quad \boxed{\mathbf{r} = \mathbf{a} + t\mathbf{b}} \qquad \{\overrightarrow{AR} \parallel \mathbf{b}, \text{ so } \overrightarrow{AR} = t\mathbf{b}\}$$

a and **r** are the position vectors of A and R respectively.

So $\mathbf{r} = \mathbf{a} + t\mathbf{b}, \quad t \in \mathbf{R}$ is the vector equation of the line.

If $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then

- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is the **vector equation** of the line.
- $x = a_1 + b_1 t, \quad y = a_2 + b_2 t$ are the **parametric equations** of the line.

Example 1

Find **a** the vector equation **b** the parametric equations

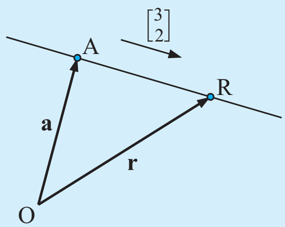
of the line passing through the point (1, 5) with direction $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

a $\mathbf{a} = \overrightarrow{OA} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

But $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad t \in \mathcal{R}$



b From **a**, $x = 1 + 3t$ and $y = 5 + 2t, \quad t \in \mathcal{R}$

EXERCISE 17A

1 Find **i** the vector equation **ii** the parametric equations of the lines:

a passing through $(3, -4)$ and in the direction $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

b passing through $(5, 2)$ and in the direction $\begin{bmatrix} -8 \\ 2 \end{bmatrix}$

c cutting the x-axis at -6 and travelling in the direction $3\mathbf{i} + 7\mathbf{j}$

d travelling in the direction $-2\mathbf{i} + \mathbf{j}$ and passing through $(-1, 11)$

2 Find the parametric equations of the line passing through $(-1, 4)$ with direction vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and parameter λ. Find the points on the line when $\lambda = 0, 1, 3, -1, -4$.

3 **a** Does $(3, -2)$ lie on the line with parametric equations $x = t + 2, \quad y = 1 - 3t$? Does $(0, 6)$ lie on this line?

b $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t, \quad y = 1 + t$. Find k.

Example 2

A particle at $P(x(t), y(t))$ moves such that $x(t) = 2 - 3t$ and $y(t) = 2t + 4$, $t \geqslant 0$, t is in seconds. Distance units are metres.

a Find the initial position of P.

b Illustrate the motion showing points where $t = 0, 1, 2$ and 3.

c Find the speed of P.

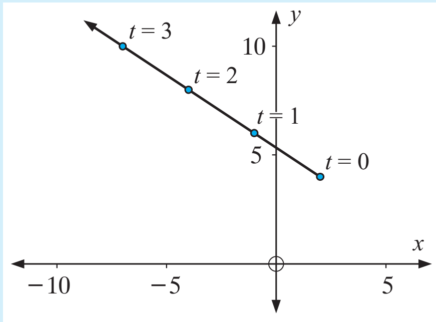
a $x(0) = 2, \quad y(0) = 4$

$\therefore$ the initial position of P is $(2, 4)$

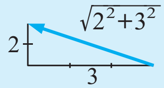
b $x(1) = -1, \quad y(1) = 6$

$x(2) = -4, \quad y(2) = 8$

$x(3) = -7, \quad y(3) = 10$



c Every second P moves with x-step -3 and y-step 2 a distance of $\sqrt{13}$ m.



$\therefore$ the speed is constant and is $\sqrt{13}$ m/s.

4 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$, t in seconds. Distance units are centimetres.

a Find the initial position of P.

b Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.

c Find the speed of P.

B THE VELOCITY VECTOR OF A MOVING OBJECT

In **Example 2**, the particle moves $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ every second.

Hence, $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is called the **velocity vector** of the particle

and as $\left|\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$,

the **velocity** of the particle is $\sqrt{13}$ metres per second in the direction $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

The **speed** of the particle is $\sqrt{13}$ metres per second.

In general, if $\begin{bmatrix} a \\ b \end{bmatrix}$ is the velocity vector of a moving object, then it is travelling at a speed of $\left|\begin{bmatrix} a \\ b \end{bmatrix}\right| = \sqrt{a^2 + b^2}$ in the direction of $\begin{bmatrix} a \\ b \end{bmatrix}$.

Example 3

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} + t\begin{bmatrix} 6 \\ -8 \end{bmatrix}$ is the vector equation of the path of an object where $t \geqslant 0$, t in seconds. Distance units are metres.

a Find the object's initial position. **b** Find the velocity vector of the object.
c Find the object's speed.

a At $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ $\therefore$ the object is at (7, 5).

b The velocity vector is $\begin{bmatrix} 6 \\ -8 \end{bmatrix}$ because the object moves $\begin{bmatrix} 6 \\ -8 \end{bmatrix}$ every second.

c The speed is $\left|\begin{bmatrix} 6 \\ -8 \end{bmatrix}\right| = \sqrt{36+64} = \sqrt{100} = 10 \text{ ms}^{-1}$.

EXERCISE 17B

1 Each of the following vector equations represents the path of a moving object. t is measured in seconds and $t \geqslant 0$. Distance units are metres. Find for the object the:

i initial position **ii** velocity vector **iii** speed.

a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} + t\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ **c** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix} + t\begin{bmatrix} -6 \\ -4 \end{bmatrix}$

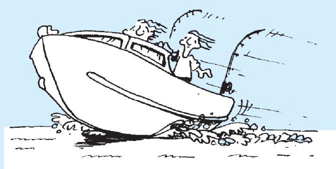
2 Given the following parametric equations for the path of a moving object (where t is measured in hours, $t \geqslant 0$ and distance is in kilometres), find:

i the velocity vector **ii** the speed of the object

a $x = 5 + 8t$ and $y = -5 + 4t$ **b** $x = 6t$ and $y = 3 + 2t$
c $x(t) = -12 + 7t$ and $y(t) = 15 + 24t$

Example 4

Find the velocity vector of a speed boat moving parallel to $\begin{bmatrix} -5 \\ 12 \end{bmatrix}$ with a speed of 65 km/h.



$$\left|\begin{bmatrix} -5 \\ 12 \end{bmatrix}\right| = \sqrt{(-5)^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$$

The speed of the boat is $65 = 5 \times 13$ kmph

$\therefore$ the velocity vector is $5\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -25 \\ 60 \end{bmatrix}$.

Note: $\left|\begin{bmatrix} -25 \\ 60 \end{bmatrix}\right| = \sqrt{(-25)^2 + 60^2} = \sqrt{4225} = 65$

3 Find the velocity vector of a speed boat:

a moving parallel to $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ with a speed of 150 km/h

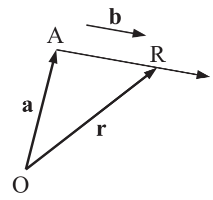
b moving parallel to $\begin{bmatrix} 24 \\ 7 \end{bmatrix}$ with a speed of 12.5 kmh^{-1}

c moving parallel to $2\mathbf{i} + \mathbf{j}$ with a speed of 50 kmh^{-1}

d moving parallel to $-3\mathbf{i} + 4\mathbf{j}$ with a speed of 100 km/h^{-1}.

C CONSTANT VELOCITY PROBLEMS

Suppose a body (or object) moves with constant velocity $\mathbf{b}$. If the body is initially at A (when time, $t = 0$) and at time t it is at R, then



$$\overrightarrow{AR} = t\mathbf{b} \qquad \{\text{time} \times \frac{\text{distance}}{\text{time}} = \text{distance}\}$$

Now $\mathbf{r} = \overrightarrow{OA} + \overrightarrow{AR}$

$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$

Thus if a body has initial position vector $\mathbf{a}$, and moves with constant velocity $\mathbf{b}$, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

Example 5

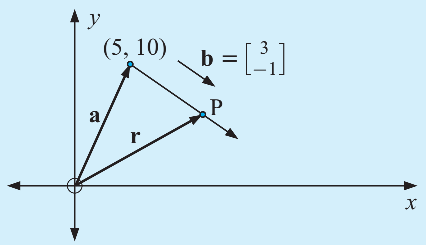
An object is initially at (5, 10) and moves with velocity vector $3\mathbf{i} - \mathbf{j}$.
Find:

a the position of the object at any time t (t in minutes)

b the position at $t = 3$

c the time when the object is due East of (0, 0).

a



$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad t \in \mathcal{R}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 + 3t \\ 10 - t \end{bmatrix}$$

$\therefore$ P is at $(5 + 3t, 10 - t)$

b At $t = 3$, $5 + 3t = 14$ and $10 - t = 7$, $\therefore$ it is at (14, 7).

c When the object is East of (0, 0) y is zero,

$\therefore \quad 10 - t = 0$

$\therefore \quad t = 10$

i.e., 10 minutes from the start.

EXERCISE 17C

1 A remote controlled toy car is initially at the point $(-3, -2)$ and moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. Distance units are centimetres and time is in seconds. Find:

a the position vector of the car at any time, $t \geqslant 0$

b the position of the car at $t = 2.5$

c the time when the car is **i** due North **ii** due West of the observation point.

d Plot the car's path at $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, ...$

2 For the following remote controlled toy cars, write vector equations for their positions in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ ($\mathbf{a}$ is the initial position vector, $\mathbf{b}$ is the velocity vector, $\mathbf{r}$ is the position vector at any time t sec, distances in cm):

a the car is initially at $(8, -10)$ and travelling at 5 cms^{-1} in the direction $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

b the car is initially at $(-2, 6)$ and travelling at constant velocity reaches $(18, 21)$ in 10 seconds

c the car is initially at $(-5, 0)$ and travelling parallel to the vector $2\mathbf{i} + \mathbf{j}$ at $\sqrt{5}$ cms^{-1}

d the car is travelling at 15 cms^{-1} in the direction $3\mathbf{i} + 4\mathbf{j}$ and passing through the point $(1, -4)$ at the moment that $t = 1$ second.

3 Suppose $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents a 1 km displacement due East and

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents a 1 km displacement due North.

The point $(0, 0)$ is the position of the Port Del Ayvd.

The position vector $\mathbf{r}$ of a ship is given by $\mathbf{r} = \begin{bmatrix} -20 \\ 32 \end{bmatrix} + t\begin{bmatrix} 12 \\ -5 \end{bmatrix}$ where t is the time in hours since 6:00 am.

a Find the distance between the ship and Port Del Ayvd at 6:00 am.

b Find the speed of the ship.

c Find the time when the ship will be due North of Port Del Ayvd.

4 Yacht A moves according to $x(t) = 4 + t$, $y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t$, $y(t) = -8 + t$, $t \geqslant 0$.

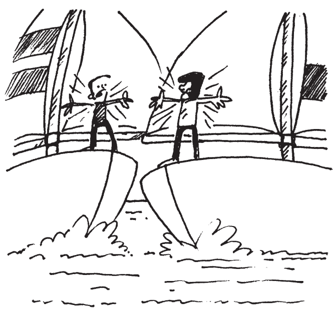
a Find the initial position of each yacht.

b Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant and state the speeds.

d If they start at 6:00 am, find the time when the yachts are closest to each other.

e Prove that the paths of the yachts are at right angles to each other.



5 Submarine P is at $(-5,\ 4)$ and fires a torpedo with velocity vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ at exactly 1.34 pm.

Submarine Q is at $(15,\ 7)$ and a minutes later can fire a torpedo only in the direction $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Distance units are kilometres and time units are minutes.

- **a** Show that the position of P's torpedo can be written as $P(x_1(t),\ y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.
- **b** What is the speed of P's torpedo?
- **c** Show that the position of Q's torpedo can be written in the form $x_2(t) = 15 - 4(t - a)$, $y_2(t) = 7 - 3(t - a)$.
- **d** Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo and at what time did the explosion occur?

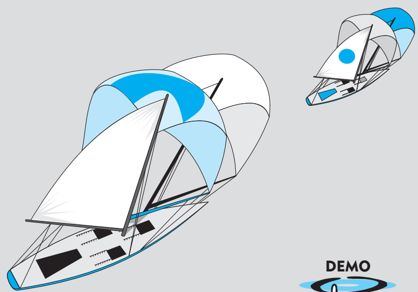
INVESTIGATION THE TWO YACHTS PROBLEM

Yacht A has initial position $(-10,\ 4)$

and has velocity vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Yacht B has initial position $(3,\ -13)$

and has velocity vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.



In this investigation you will plot the path of each yacht and determine the time when they are nearest, and the shortest distance they are apart.

DEMO

What to do:

1. Explain why the position of each yacht at time t is given by
$$\mathbf{r}_A = \begin{bmatrix} -10 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{bmatrix} 3 \\ -13 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$
2. On squared paper plot the path of the yachts when $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ ...$
3. Find the position vector of B relative to A, i.e., $\overrightarrow{AB}$.
4. Use **3** to show that if d is the distance between the yachts at any time t then $d^2 = 25t^2 - 214t + 458$.
5. Show that d^2 is a minimum when $t = 4.28$.
6. Hence, find the time when d is a minimum and then find the shortest distance.
7. Investigate other situations with different initial positions and velocity vectors. You should be able to create a situation where the yachts will collide.

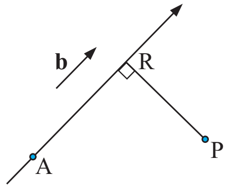
D THE CLOSEST DISTANCE

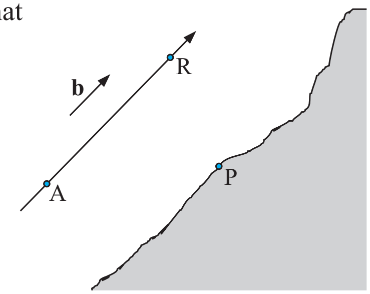
A ship sails through point A in the direction **b** past a port P. At that time will the ship R be closest to the port?

This occurs when PR is perpendicular to AR,

i.e., $\overrightarrow{PR} \bullet \mathbf{b} = 0$

{the scalar product is zero}.



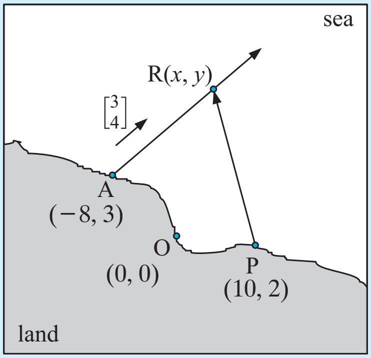


Example 6

If distances are measured in kilometres and a ship R is initially moving in the direction $\begin{bmatrix}3\\4\end{bmatrix}$ at a speed of 10 kmh^{-1}, find:

a an expression for the position of the ship in terms of t where t is the number of hours after leaving port A

b the time when the ship is closest to port P (10, 2).



a $\left|\begin{bmatrix}3\\4\end{bmatrix}\right| = \sqrt{3^2+4^2} = 5$ and since the speed is 10 kmh^{-1}, the ship's velocity vector is $2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$.

Relative to O(0, 0), $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-8\\3\end{bmatrix} + t\begin{bmatrix}6\\8\end{bmatrix}$$

$\therefore$ R is at $(-8+6t,\ 3+8t)$

b The ship is closest to P when $\overrightarrow{PR} \perp \begin{bmatrix}3\\4\end{bmatrix}$ $\quad\therefore\quad \overrightarrow{PR} \bullet \begin{bmatrix}3\\4\end{bmatrix} = 0$

$$\therefore \quad \begin{bmatrix}-8+6t-10\\3+8t-2\end{bmatrix} \bullet \begin{bmatrix}3\\4\end{bmatrix} = 0$$

i.e., $3(6t-18)+4(1+8t) = 0$

i.e., $18t-54+4+32t = 0$

i.e., $50t-50 = 0$

i.e., $t = 1$

i.e., 1 hour after leaving A.

EXERCISE 17D

1 Suppose $\mathbf{i}$ represents a displacement of 1 metre East and $\mathbf{j}$ represents a displacement of 1 m North. Time t, is measured in seconds.

A body has initial position $(-3, -2)$ and moves in a straight line with constant velocity $2\mathbf{i} + 4\mathbf{j}$. Find:

- **a** the body's position at any time t, $t \geqslant 0$
- **b** the times when it is **i** due East and **ii** due North of the origin
- **c** the coordinates of the axes intercepts.

2 An object moves in a straight line with constant velocity vector $-\mathbf{i} - 3\mathbf{j}$. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ represent a displacement of 1 metre. Time t is measured in seconds. Initially the object is at $(-2, 1)$. Find:

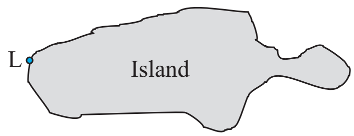
- **a** the object's position at time t, $t \geqslant 0$
- **b** the time when the object crosses the x-axis
- **c** the coordinate where the object crosses the x-axis.

In questions **3**, **4** and **5** a unit vector represents a displacement of 1 km. Time t is in hours.

3 An ocean liner is at $(6, -6)$, cruising at 10 km h^{-1} in the direction $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
A fishing boat is anchored at $(0, 0)$.

- **a** Find in terms of $\mathbf{i}$ and $\mathbf{j}$ the velocity vector of the liner.
- **b** Find the position vector of the liner at any time t hours after it has sailed from $(6, -6)$.
- **c** Find when the liner is due East of the fishing boat.
- **d** Find the time and position of the liner when it is nearest to the fishing boat.

4 A fishing trawler is moving with constant speed of $3\sqrt{2}$ km h^{-1} in the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Initially it is at the point $(-8, -5)$.
A lighthouse L on an island is at point $(0, 0)$.



- **a** Find in terms of $\mathbf{i}$ and $\mathbf{j}$:
  - **i** the initial position vector of the trawler
  - **ii** the direction vector of the trawler
  - **iii** the position vector of the trawler at any time t hours ($t \geqslant 0$).
- **b** Find the time at which the trawler is closest to the island.
- **c** Find whether the trawler will be breaking the law, given that it is not allowed within 4 km of the island.

5 Suppose unit vectors represent a displacement of 1 m. A remote controlled toy car moves in a straight line, starting at $(-3, -2)$.

After t seconds, its position (x, y) is given by the vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$.

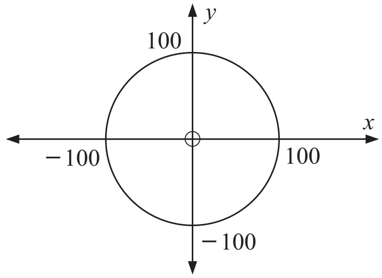
- **a** Find the position of the car after 4 seconds.
- **b** Find the speed of the car.

c At what time is the car closest to the controller who is standing at $(4, -1)$?

d Express the equation of the car's path in the form $ax + by = c$.

6 Let $\begin{bmatrix}1\\0\end{bmatrix}$ represent a 1 km due East displacement

and $\begin{bmatrix}0\\1\end{bmatrix}$ represent a 1 km due North displacement.



The control tower of an airport is at (0, 0). Aircraft within 100 km of (0, 0) will become visible on the radar screen at the control tower.

At 12:00 noon an aircraft is at A which is 200 km East and 100 km North of the control tower.

It is flying parallel to the vector $\mathbf{b} = \begin{bmatrix}-3\\-1\end{bmatrix}$ with a speed of $40\sqrt{10}$ km h^{-1}.

a Write down the velocity vector of the aircraft.

b Write a vector equation for the path of the aircraft (using t to represent the time in hours that have elapsed since 12:00 noon).

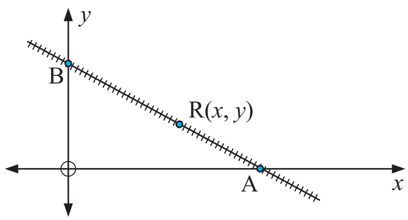
c Find the position of the aircraft at 1:00 pm.

d Show that at 1:00 pm the aircraft first becomes visible on the radar screen.

e Find the time when the aircraft will be closest to the control tower and find the distance between the aircraft and the control tower at this time.

f At what time will the aircraft disappear from the radar screen?

7 The diagram shows a railway track that has equation $2x + 3y = 36$.



The axes represent two long country roads. All distances are in kilometres.

a Find the coordinates of A and B.

b R(x, y) is any point on the railway track. Express the coordinates of point R in terms of x only.

c Some railway workers have set up a base camp at P(4, 0). Find $\overrightarrow{PR}$ and $\overrightarrow{AB}$.

d Hence, find the coordinates of the point on the railway track that would be closest to P. Find this distance.

8 A particle at P$(x(t), y(t))$ moves such that $x(t) = 2 - t$ and $y(t) = 1 + 3t$, $t \geqslant 0$, t in seconds. Distance units are centimetres.

a Find the initial position of P.

b Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.

c How close to $(0, 10)$ did the particle pass and at what time did this occur?

9 Point P$(x(t), y(t))$ moves such that $x(t) = 10 + at$ and $y(t) = 12 - 3t$, $t \geqslant 0$ where t is the time in seconds and distance units are centimetres.

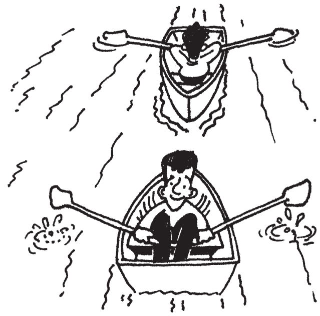
a Find the initial position of P.

b The speed of P is constant at 13 cm/s. Find a.

c In the case where $a < 0$, plot the motion of P over the first 4 seconds.

10 Boat A's position is given by $x(t) = 3 - t$, $y(t) = 2t - 4$ where the distance units are kilometres and the time units are hours.
Boat B's position is given by $x(t) = 4 - 3t$, $y(t) = 3 - 2t$.

a Find the initial position of each boat.
b Find the velocity vector of each boat.
c What is the angle between the paths of the boats?
d At what time are the boats closest to each other?



E GEOMETRIC APPLICATIONS OF $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

Vector equations of two intersecting lines can be **solved simultaneously** to find the point where the lines meet.

Example 7

Line 1 has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and

line 2 has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} + t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, s and t are scalars.

The two lines meet at E. Use vector methods to find the coordinates of E.

The lines meet where $\begin{bmatrix} -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} + t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\therefore \quad -2 + 3s = 15 - 4t$ and $1 + 2s = 5 + t$

$\therefore \quad 3s + 4t = 17 \quad \ldots\ldots (1)$ and $2s - t = 4 \quad \ldots\ldots (2)$

$$\begin{aligned} 3s + 4t &= 17 \\ 8s - 4t &= 16 \quad \{\text{when (2) is multiplied by 4}\} \\ \hline \therefore \quad 11s \quad &= 33 \end{aligned}$$

So, $s = 3$ and in (2) $2(3) - t = 4$, i.e., $t = 2$

Thus, using line 1, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

Checking in line 2, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

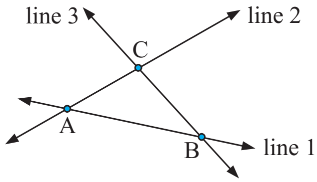
$\therefore$ the lines meet at (7, 7)

EXERCISE 17E

1 The triangle formed by the three lines is ABC.

Line 1 (AB) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + r\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, line 2 (AC) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

line 3 (BC) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ where r, s and t are scalars.



a Accurately, on a grid, draw the three lines.

b Hence, find the coordinates of A, B and C.

c Prove that ΔABC is isosceles.

d Use vector methods to *check* your answers to **b**.

2 A parallelogram is defined by four lines with equations:

Line 1 (AB) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + r\begin{bmatrix} 7 \\ 3 \end{bmatrix}$, line 2 (AD) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

line 3 (CD) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + t\begin{bmatrix} -7 \\ -3 \end{bmatrix}$, line 4 (CB) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + u\begin{bmatrix} -1 \\ -2 \end{bmatrix}$,

where r, s, t and u are scalars.

Lines 1 and 2 intersect at point A(-4, 6), lines 1 and 4 meet at B, lines 3 and 4 meet at C and lines 2 and 3 meet at D.

a Draw an accurate sketch of the four lines and the parallelogram formed by them. Label the vertices.

b From your diagram find the coordinates of B, C and D.

c Use vector methods to confirm your answers to **b**.

3 An isosceles triangle ABC is formed by these lines:

Line 1 (AB) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, line 2 (BC) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} + s\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and

line 3 (AC) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where r, s and t are scalars.

a Use vector methods to find the coordinates of A, B and C.

b Which two sides of the triangle are equal in length? Find their lengths.

4 Line QP has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + r\begin{bmatrix} 14 \\ 10 \end{bmatrix}$.

Line QR has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} 17 \\ -9 \end{bmatrix}$.

Line PR has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix} + t\begin{bmatrix} 5 \\ -7 \end{bmatrix}$.

Triangle PQR is formed by these lines. r, s and t are scalars.

a Use vector methods to find the coordinates of P, Q and R.

b Find vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and evaluate $\overrightarrow{PQ} \bullet \overrightarrow{PR}$.

c Hence, find the size of $\measuredangle$QPR.

d Find the area of ΔPQR.

5 Quadrilateral ABCD is formed by these lines:

Line 1 (AB) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + r\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, line 2 (BC) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} + s\begin{bmatrix} -8 \\ 32 \end{bmatrix}$,

line 3 (CD) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \end{bmatrix} + t\begin{bmatrix} -8 \\ -2 \end{bmatrix}$ and line 4 (AD) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + u\begin{bmatrix} -3 \\ 12 \end{bmatrix}$

where r, s, t and u are scalars.

Lines 1 and 2 meet at B, lines 2 and 3 at C, lines 3 and 4 at D, lines 1 and 4 at A(2, 5).

a Use vector methods to find the coordinates of B, C and D.

b Write down vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ and hence find:

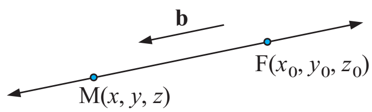
i $|\overrightarrow{AC}|$ **ii** $|\overrightarrow{DB}|$ **iii** $\overrightarrow{AC} \bullet \overrightarrow{DB}$

c What do the answers to **b** tell you about quadrilateral ABCD?

F LINES IN SPACE

A **line** in space is *a straight line* which continues indefinitely in both directions and contains a continuous infinite set of points.

Suppose $M(x, y, z)$ is a point which is free to move on a line containing a fixed point $F(x_0, y_0, z_0)$.



If $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a direction vector of the line, then

Now $\overrightarrow{FM} = t\mathbf{b}$ for some scalar t

$$\therefore \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the **vector equation** of the line,

and $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ are its **parametric equations**.

Note: $t = \dfrac{x - x_0}{a}$, $t = \dfrac{y - y_0}{b}$, $t = \dfrac{z - z_0}{c}$

$\therefore \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ {called the **cartesian equations** of the line}.

Example 8

Find the vector equation and the parametric equations of the line through $(1, -2, 3)$ in the direction $4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

The vector equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, t in $\mathcal{R}$.

The parametric equations are: $x = 1 + 4t$, $y = -2 + 5t$, $z = 3 - 6t$, t in $\mathcal{R}$.

EXERCISE 17F

1 Find the vector equation of the line:

- **a** parallel to $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and through the point $(1, 3, -7)$
- **b** through $(0, 1, 2)$ and with direction vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- **c** parallel to the X-axis and through the point $(-2, 2, 1)$.

2 Find the parametric equations of the line:

- **a** parallel to $\begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$ and through the point $(5, 2, -1)$
- **b** parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and through the point $(0, 2, -1)$
- **c** perpendicular to the XOY plane and through $(3, 2, -1)$.

Example 9

Find the parametric equations of the line through $A(2, -1, 4)$ and $B(-1, 0, 2)$.

We require a direction vector for the line ($\overrightarrow{AB}$ or $\overrightarrow{BA}$)

$$\overrightarrow{AB} = \begin{bmatrix} -1-2 \\ 0 - -1 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

Using the point A, the equations are: $x = 2 - 3t$, $y = -1 + t$, $z = 4 - 2t$ (t in $\mathcal{R}$)

[Using the point B, the equations are: $x = -1 - 3s$, $y = s$, $z = 2 - 2s$, (s in $\mathcal{R}$)]

Note: Both sets of equations generate the same set of points and $s = t - 1$.

3 Find the parametric equations of the line through:

- **a** $A(1, 2, 1)$ and $B(-1, 3, 2)$
- **b** $C(0, 1, 3)$ and $D(3, 1, -1)$
- **c** $E(1, 2, 5)$ and $F(1, -1, 5)$
- **d** $G(0, 1, -1)$ and $H(5, -1, 3)$

4 Find the coordinates of the point where the line with parametric equations $x = 1 - t$, $y = 3 + t$ and $z = 3 - 2t$ meets:

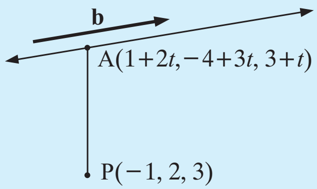
a the XOY plane **b** the YOZ plane **c** the XOZ plane.

5 Find points on the line with parametric equations $x = 2 - t$, $y = 3 + 2t$ and $z = 1 + t$ which are $5\sqrt{3}$ units from the point $(1, 0, -2)$.

Example 10

Find the coordinates of the foot of the perpendicular from P$(-1, 2, 3)$ to the line with parametric equations $x = 1 + 2t$, $y = -4 + 3t$, $z = 3 + t$.

A$(1 + 2t, \ -4 + 3t, \ 3 + t)$ is any point on the given line.



$$\overrightarrow{PA} = \begin{bmatrix} 1 + 2t - -1 \\ -4 + 3t - 2 \\ 3 + t - 3 \end{bmatrix} = \begin{bmatrix} 2 + 2t \\ -6 + 3t \\ t \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is the direction vector of the line.

Now as $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, $\overrightarrow{PA} \bullet \mathbf{b} = 0$

$$\therefore \begin{bmatrix} 2 + 2t \\ -6 + 3t \\ t \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0$$

$$\therefore 2(2 + 2t) + 3(-6 + 3t) + 1(t) = 0$$

$$\therefore 4 + 4t - 18 + 9t + t = 0$$

$$\therefore 14t = 14$$

$$\therefore t = 1$$

and substituting $t = 1$ into the parametric equations we obtain the foot of the perpendicular $(3, -1, 4)$.

Note: As $t = 1$, $\overrightarrow{PA} = [4, -3, 1]$ and $|\overrightarrow{PA}|$
$= \sqrt{16 + 9 + 1}$
$= \sqrt{26}$ units

$\therefore$ the shortest distance from P to the line is $\sqrt{26}$ units.

6 Find the coordinates of the foot of the perpendicular:

a from $(1, 1, 2)$ to the line with equations $x = 1 + t$, $y = 2 - t$, $z = 3 + t$

b from $(2, 1, 3)$ to the line with vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

7 Find the distance from:

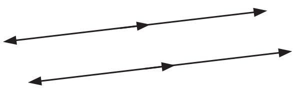
a $(3,\ 0,\ -1)$ to the line with equations $x = 2 + 3t, \quad y = -1 + 2t, \quad z = 4 + t$

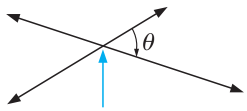
b $(1,\ 1,\ 3)$ to the line with vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

G LINE CLASSIFICATION

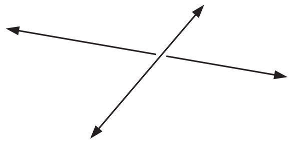
Two lines in space are either **parallel**, **intersecting** or **skew**,

i.e.,





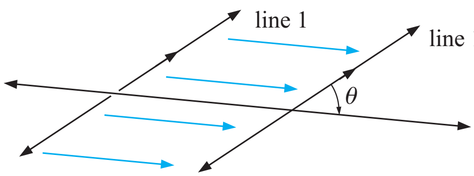
or



Skew lines are any lines which are neither parallel nor intersecting.

- If the lines are **parallel**, the angle between them is 0^o.
- If the lines are **intersecting**, the angle between them is θ^o, as shown.
- If the lines are **skew**, there is still an angle that one line makes with the other and if we translate one line to intersect the other, the angle between the original lines is defined as the angle between the intersecting lines,

  i.e.,



Example 11

Line 1 has equations $x = -1 + 2s, \quad y = 1 - 2s \quad$ and $\quad z = 1 + 4s$.
Line 2 has equations $x = 1 - t, \quad y = t \quad$ and $\quad z = 3 - 2t$.
Show that line 1 and line 2 are parallel.

Line 1 is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ with direction vector $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Likewise, line 2 has direction vector $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$,

and as $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, then lines 1 and 2 are parallel.

{If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then $\mathbf{a} \parallel \mathbf{b}$.}

Example 12

Line 1 has equations $x = -1 + 2s$, $y = 1 - 2s$ and $z = 1 + 4s$.
Line 2 has equations $x = 1 - t$, $y = t$ and $z = 3 - 2t$.
Line 3 has equations $x = 1 + 2u$, $y = -1 - u$, $z = 4 + 3u$.

a Show that line 2 and line 3 intersect and find the angle between them.
b Show that line 1 and line 3 are skew.

a Equating x, y, and z values in line 2 and 3 gives

$$\underbrace{\begin{aligned} 1 - t &= 1 + 2u \qquad & t &= -1 - u \\ \therefore\ t &= -2u, \qquad & \therefore\ t &= -1 - u, \end{aligned}}\qquad \begin{aligned} &\text{and} \quad 3 - 2t = 4 + 3u \\ &\text{and} \quad 3u + 2t = -1 \quad \text{..... (1)} \end{aligned}$$

Solving these we get
$$\begin{aligned} -2u &= -1 - u \\ \therefore\ -u &= -1 \\ \therefore\ u &= 1 \quad \text{and so} \quad t = -2 \end{aligned}$$

Checking in (1) $3u + 2t = 3(1) + 2(-2) = 3 - 4 = -1$ ✓

$\therefore$ $u = 1$, $t = -2$ satisfies all three equations, a *common solution.*

Using $u = 1$, they meet at $(1 + 2(1), -1 - (1), 4 + 3(1))$ i.e., $(3, -2, 7)$

The direction of line 2 could be defined by $\mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

The direction of line 3 could be defined by $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Now $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned} \therefore\ -2 - 1 - 6 &= \sqrt{1 + 1 + 4}\ \sqrt{4 + 1 + 9}\cos\theta \\ \therefore\ -9 &= \sqrt{6}\sqrt{14}\cos\theta \\ \therefore\ \cos\theta &= \frac{-9}{\sqrt{84}} \end{aligned}$$

So, $\theta \doteqdot 169.11^o$ i.e., 169^o (to 3 s.f.)

b Equating x, y, and z values in line 1 and 3 gives

$$\underbrace{\begin{aligned} -1 + 2s &= 1 + 2u \qquad & 1 - 2s &= -1 - u \\ \therefore\ 2s - 2u &= 2, \qquad & \therefore\ -2s + u &= -2, \end{aligned}}\qquad \begin{aligned} &\text{and} \quad 1 + 4s = 4 + 3u \\ &\text{and} \quad 4s - 3u = 3 \quad \text{..... (1)} \end{aligned}$$

Solving these we get
$$\begin{aligned} 2s - 2u &= 2 \\ \therefore\ -2s + u &= -2 \\ \hline \therefore\ -u &= 0 \quad \{\text{adding them}\} \end{aligned}$$

$\therefore$ $u = 0$ and so $2s = 2$ i.e., $s = 1$

Checking in (1), $4s - 3u = 4(1) - 3(0) = 4 \neq 3$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines cannot meet, and as they are not parallel they must be skew.

Note: Even though lines 1 and 3 in **Example 12** were skew we can still find the angle one makes with the other using the same method as in **b**.

PERPENDICULAR AND PARALLEL TESTS

Non–zero vectors $\mathbf{v}$ and $\mathbf{w}$ are
- perpendicular if $\mathbf{v} \bullet \mathbf{w} = 0$
- parallel if $\mathbf{v} \bullet \mathbf{w} = \pm |\mathbf{v}|\,|\mathbf{w}|$

Proof: If perpendicular, $\theta = 90^o$

and as $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$

then $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos 90^o$
$= |\mathbf{v}|\,|\mathbf{w}|\, 0$
$= 0$

If parallel, $\theta = 0^o$ or 180^o

where $\cos\theta = 1$ or -1

$\therefore \quad \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \times \pm 1$

i.e., $\mathbf{v} \bullet \mathbf{w} = \pm |\mathbf{v}|\,|\mathbf{w}|$

EXERCISE 17G

1 Classify the following line pairs as either parallel, intersecting or skew and in each case find the measure of the angle acute between them:

a $x = 1 + 2t, \quad y = 2 - t, \quad z = 3 + t$ and $x = -2 + 3s, \quad y = 3 - s, \quad z = 1 + 2s$

b $x = -1 + 2t, \quad y = 2 - 12t, \quad z = 4 + 12t$
and $x = 4s - 3, \quad y = 3s + 2, \quad z = -s - 1$

c $x = 6t, \quad y = 3 + 8t, \quad z = -1 + 2t$ and $x = 2 + 3s, \quad y = 4s, \quad z = 1 + s$

d $x = 2 - y = z + 2$ and $x = 1 + 3s, \quad y = -2 - 2s, \quad z = 2s + \frac{1}{2}$

e $x = 1 + t, \quad y = 2 - t, \quad z = 3 + 2t$ and $x = 2 + 3s, \quad y = 3 - 2s, \quad z = s - 5$

f $x = 1 - 2t, \quad y = 8 + t, \quad z = 5$ and $x = 2 + 4s, \quad y = -1 - 2s, \quad z = 3$

REVIEW SET 17A

1 Find **a** the vector equation **b** the parametric equations

of the line that passes through $(-6, 3)$, with direction $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

2 Find the vector equation of the line which cuts the y-axis at $(0, 8)$ and has direction $5\mathbf{i} + 4\mathbf{j}$.

3 $(-3, m)$ lies on the line with vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \end{bmatrix} + t\begin{bmatrix} -7 \\ 4 \end{bmatrix}$. Find m.

4 A particle at $P(x(t), y(t))$ moves such that $x(t) = -4 + 8t$ and $y(t) = 3 + 6t$, $t \geqslant 0$, t in seconds. Distance units are metres. Find the:

a initial position of P
b position of P after 4 seconds
c speed of P
d velocity vector of P.

5 Find the velocity vector of an object that is moving in the direction $3\mathbf{i} - \mathbf{j}$ with a speed of 20 kmh^{-1}.

6 A yacht is sailing at a constant speed of $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll.

- **a** Find in terms of $\mathbf{i}$ and $\mathbf{j}$:
  - **i** the initial position vector of the yacht
  - **ii** the direction vector of the yacht
  - **iii** the position vector of the yacht at any time t hours ($t \geqslant 0$).
- **b** Find the time when the yacht is closest to the beacon
- **c** Find whether there is a possibility that the yacht could hit the reef around the atoll given that the atoll has a radius of 8 km.

7 Submarine X23 is at (2, 4) and fires a torpedo with velocity vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ at exactly 2.17 pm.

Submarine Y18 is at (11, 3) and has velocity vector $\begin{bmatrix} -1 \\ a \end{bmatrix}$.

It fires a torpedo 2 minutes later to intercept the torpedo from X23. Given that the interception occurs:

- **a** find $x_1(t)$ and $y_1(t)$ for submarine X23
- **b** find $x_2(t)$ and $y_2(t)$ for submarine Y18.
- **c** At what time does the interception occur?
- **d** What was the direction and speed of the interception torpedo?

8 Trapezium (trapezoid) KLMN is formed by these lines:

Line 1 (KL) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 19 \end{bmatrix} + p\begin{bmatrix} 5 \\ -2 \end{bmatrix}$, line 2 (ML) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -5 \end{bmatrix} + q\begin{bmatrix} -11 \\ 16 \end{bmatrix}$

line 3 (NK) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ and line 4 (MN) is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

where p, q, r and s are scalars. Lines 1 and 2 meet at L(22, 11).

Lines 1 and 3 meet at K, lines 2 and 4 at M, lines 3 and 4 at N.

- **a** Which two lines are parallel? Why?
- **b** Which lines are perpendicular? Why?
- **c** Use vector methods to find the coordinates of K, M and N.
- **d** Calculate the area of trapezium KLMN.

REVIEW SET 17B

1 A is $(3, 2, -1)$ and $B(-1, 2, 4)$.

- **a** Write down the vector equation of the line through A and B.
- **b** Find *two* points on the line AB which are $2\sqrt{41}$ units from A.

2 For $A(3, -1, 1)$ and $B(0, 2, -1)$, find the:

- **a** vector equation of the line passing through A and B
- **b** the coordinates of P which divides BA in the ratio $2 : 5$.

3 P$(-1, 2, 3)$ and Q$(4, 0, -1)$ are two points in space. Find:

a $\overrightarrow{\mathrm{PQ}}$

b the angle that $\overrightarrow{\mathrm{PQ}}$ makes with the X-axis.

4 Given the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 16 \\ 7 \end{bmatrix}$ and $x = 15+3t$, $y = 29+8t$, $z = 5-5t$:

a show that they are skew

b find the acute angle between them.

5 Find the coordinates of the foot of the perpendicular from Q$(-1, 2, 3)$ to the line $x = 2 - t$, $y = 3 + t$, $z = -2t$.

6 P$(2, 0, 1)$, Q$(3, 4, -2)$ and R$(-1, 3, 2)$ are three points in space. Find:

a $\overrightarrow{\mathrm{PQ}}$ and $|\,\overrightarrow{\mathrm{PQ}}\,|$

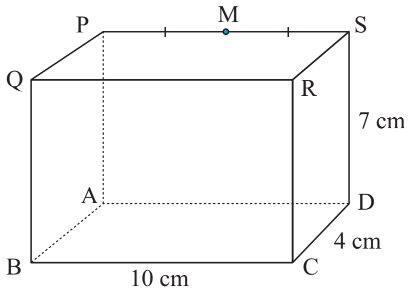
b the parametric equations of the line through P and Q.

7 Consider the lines with equations $x = 3 + 2\lambda$, $y = 4 + \lambda$, $z = -1 - 2\lambda$ and $x = -1 + 3t$, $y = 2 + 2t$, $z = 3 - t$.

a Are the lines parallel, intersecting or skew?

b Determine the cosine of the acute angle between the lines.

8



Use vector methods to determine the measure of angle QDM given that M is the midpoint of PS of the rectangular prism.

Chapter 18

Descriptive statistics

Contents:

A Continuous numerical data and histograms

B Measuring the centre of data

Investigation: Merits of the mean and median

C Cumulative data

D Measuring the spread of data

E Statistics using technology

F Variance and standard deviation

G The significance of standard deviation

Review set 18A

Review set 18B

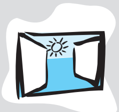
BACKGROUND KNOWLEDGE IN STATISTICS

Before starting this course you should make sure that you have a good understanding of the necessary background knowledge.

Click on the icon alongside to obtain a printable set of exercises and answers on this background knowledge.

BACKGROUND KNOWLEDGE

THE PEA PROBLEM



A farmer wishes to investigate the effect of a new organic fertiliser on his crops of peas. He is hoping to improve the crop yield by using the fertiliser. He set up a small garden which was subdivided into two equal plots and planted many peas. Both plots were treated the same except for the use of the fertiliser on one, but not the other. All other factors such as watering were as normal.

A random sample of 150 pods was harvested from each plot at the same time and the number of peas in each pod counted. The results were:

Without fertiliser

4 6 5 6 5 6 4 6 4 9 5 3 6 8 5 4 6 8 6 5 6 7 4 6 5 2 8 6 5 6 5 5 5 4 4 4 6 7 5 6
7 5 5 6 4 8 5 3 7 5 3 6 4 7 5 6 5 7 5 7 6 7 5 4 7 5 5 5 6 6 5 6 7 5 8 6 8 6 7 6
6 3 7 6 8 3 3 4 4 7 6 5 6 4 5 7 3 7 7 6 7 7 4 6 6 5 6 7 6 3 4 6 6 3 7 6 7 6 8 6
6 6 6 4 7 6 6 5 3 8 6 7 6 8 6 7 6 6 6 8 4 4 8 6 6 2 6 5 7 3

With fertiliser

6 7 7 4 9 5 5 5 8 9 8 9 7 7 5 8 7 6 6 7 9 7 7 7 8 9 3 7 4 8 5 10 8 6 7 6 7 5 6 8
7 9 4 4 9 6 8 5 8 7 7 4 7 8 10 6 10 7 7 7 9 7 7 8 6 8 6 8 7 4 8 6 8 7 3 8 7 6 9 7
6 9 7 6 8 3 9 5 7 6 8 7 9 7 8 4 8 7 7 7 6 6 8 6 3 8 5 8 7 6 7 4 9 6 6 6 8 4 7 8
9 7 7 4 7 5 7 4 7 6 4 6 7 7 6 7 8 7 6 6 7 8 6 7 10 5 13 4 7 7

For you to consider:

- Can you state clearly the problem that the farmer wants to solve?
- How has the farmer tried to make a fair comparison?
- How could the farmer make sure that his selection is at random?
- What is the best way of organising this data?
- What are suitable methods of display?
- Are there any abnormally high or low results and how should they be treated?
- How can we best indicate the most typical pod size?
- How can we best indicate the spread of possible pod sizes?
- What is the best way to show 'typical pod size' and the spread?
- Can a satisfactory conclusion be made?

A CONTINUOUS NUMERICAL DATA AND HISTOGRAMS

A **continuous numerical variable** can theoretically take any value on part of the number line. A continuous variable often has to be **measured** so that data can be recorded.

Examples of continuous numerical variables are:

The height of Year 10 students: the variable can take any value from about 140 cm to 200 cm.

The speed of cars on a stretch of highway: the variable can take any value from 0 km/h to the fastest speed that a car can travel, but is most likely to be in the range 30 km/h to 120 km/h.

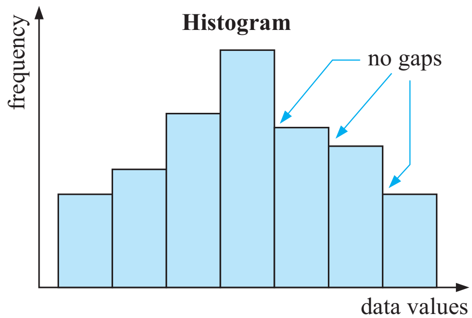
ORGANISATION AND DISPLAY OF CONTINUOUS DATA

When data is recorded for a continuous variable there are likely to be many different values so this data is organised by grouping into **class intervals**. A special type of graph, called a **histogram**, is used to display the data.

A histogram is similar to a column graph but, to account for the continuous nature of the variable, a number line is used for the horizontal axis and the 'columns' are joined together.

An example is given alongside:

Note: The **modal class** (the class of values that appears most often) is easy to identify from a histogram.



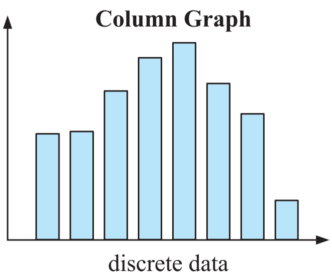
If the class intervals are the same size then the frequency is represented by the height of the 'columns'.

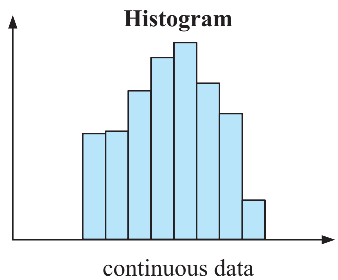
SUMMARY (COLUMN GRAPHS AND HISTOGRAMS)

Column graphs and histograms both have the following features:

- on the **vertical axis** we have the **frequency** of occurrence
- on the **horizontal axis** we have the range of scores
- **column widths are equal** and the height varies according to frequency.

Histograms are used whenever the data is **continuous** and have **no gaps between the columns**.





CASE STUDY — DRIVING A GOLF BALL

While Norm Gregory was here for the golf championship, I decided to ask for his cooperation in a data gathering exercise. He agreed and so I asked him to hit 30 balls in succession with his driver. I then measured how far each ball travelled in metres. The data was as follows:

244.6	245.1	248.0	248.8	250.0
251.1	251.2	253.9	254.5	254.6
255.9	257.0	260.6	262.8	262.9
263.1	263.2	264.3	264.4	265.0
265.5	265.6	266.5	267.4	269.7
270.5	270.7	272.9	275.6	277.5

This type of data must be **grouped** before a histogram can be drawn.

In forming groups, find the lowest and highest values, and then make the group width such that you achieve about 6 to 12 groups. In this case the lowest value is 244.6 m while the largest is 277.5 m. This gives a range of approximately 35 m, hence a group width of 5 will give eight groups.

We will use the following method of grouping. The group '240 - ' actually means that any data value 240 but $<$ 245 can fit in this group. Similarly the group '260 - ' will contain data 260 but $<$ 265. This technique creates a home for every number $\geqslant$ 240 but $<$ 280. Groups should all be of the same width.

A tally column is used to count the data that falls in a given group in an efficient way. Do not try to determine the number of data values in the 240- group first off. Simply place a vertical stroke in the tally column to register an entry as you work your way through the data from start to finish as it is presented to you. Every fifth entry in a group is marked with a diagonal line through the previous four so groups of five can be counted quickly.

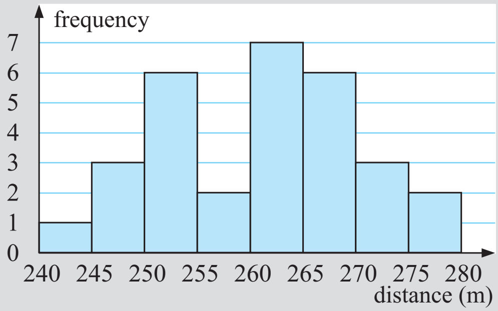
A frequency column summarises the number of data values in each group. The relative frequency column measures the percentage of the total number of data values in each group. Here, percentages offer an easier way to compare the number of balls Norm hit 'over 270 m but under 275 m' to the number he hit 'under 245 m'.

Norm Gregory's 30 drives

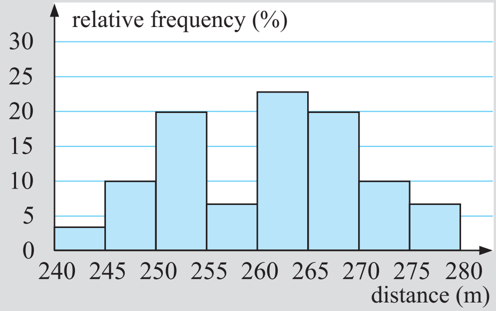
Distance (m)	*Tally*	*Frequency* (f)	*% Relative frequ.* (rf)
240 -	|	1	3.3
245 -	|||	3	10.0
250 -	~~||||~~ |	6	20.0
255 -	||	2	6.7
260 -	~~||||~~ ||	7	23.3
265 -	~~||||~~ |	6	20.0
270 -	|||	3	10.0
275 - (but $<$ 280)	||	2	6.7
	Totals	30	100.0

From this table two histograms can be drawn, firstly a **frequency histogram**, and secondly a **relative frequency histogram**. They look as follows. Note, all histograms require a title.

A frequency histogram displaying the distribution of 30 of Norm Gregory's drives.



A relative frequency histogram displaying the distribution of 30 of Norm Gregory's drives.



The advantage of the relative frequency histogram is best seen when you wish to compare distributions with different numbers of data values. Using percentages allows for a fair comparison.

Notice how the horizontal axis is labelled. The left edge of each bar is the first possible entry for that group.

Example 1

The weight of parcels sent on a particular day from a post office is recorded, in kilograms:

2.1, 3.0, 0.6, 1.5, 1.9, 2.4, 3.2, 4.2, 2.6, 3.1, 1.8, 1.7, 3.9, 2.4, 0.3, 1.5, 1.2

Organise the data using a frequency table and graph the data.

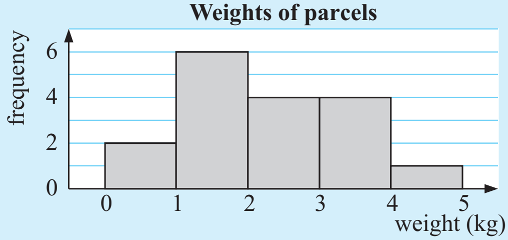
The data is *continuous* because the weight could be any value from 0.1 kg up to 5 kg.

The lowest weight recorded is 0.3 kg and the heaviest is 4.2 kg so we will use class intervals of 1 kg. The class interval 1- would include all weights from 1 kg up to, but not including 2 kg.

Weight (kg)	*Frequency*
0 -	2
1 -	6
2 -	4
3 -	4
4 - < 5	1

A histogram is used to graph this continuous data.

Weights of parcels



A stemplot could also be used to organise the data:

Stem	*Leaf*
0	3 6
1	2 5 5 7 8 9
2	1 4 4 6
3	0 1 2 9
4	2

Scale: 1 | 2 means 1.2 kg.

Note: The modal class is (1-< 2) kg as this occurred most frequently.

EXERCISE 18A

1 A frequency table for the heights of a basketball squad is given below.

Height (cm)	*Frequency*
170 -	1
175 -	8
180 -	9
185 -	11
190 -	9
195 -	3
200 - < 205	3

a Explain why 'height' is a continuous variable.

b Construct a histogram for the data. The axes should be carefully marked and labelled and include a heading for the graph.

c What is the modal class? Explain what this means.

d Describe the distribution of the data.

2 A school has conducted a survey of 60 students to investigate the time it takes for students to travel to school. The following data gives the travel time to the nearest minute:

12	15	16	8	10	17	25	34	42	18	24	18	45	33	38
45	40	3	20	12	10	10	27	16	37	45	15	16	26	32
35	8	14	18	15	27	19	32	6	12	14	20	10	16	14
28	31	21	25	8	32	46	14	15	20	18	8	10	25	22

a Is travel time a discrete or continuous variable?

b Construct an ordered stemplot for the data using stems 0, 1, 2, ...

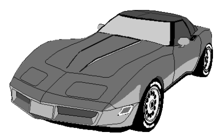
c Describe the distribution of the data.

d Copy and complete:
"The modal travelling time was between ..... and ..... minutes."

3 For the following data, state whether a histogram or a column graph should be used and draw the appropriate graph.

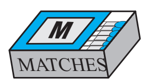
a Most appealing car colour data.

Colour	white	red	blue	green	other
Frequency	38	27	19	18	11



b The number of matches in 30 match boxes data.

Number of matches per box	47	49	50	51	52	53	55
Frequency	1	1	9	12	4	2	1



c The heights of 25 hockey players (to the nearest cm) data

Height (cm)	120 – 129	130 – 139	140 – 149	150 – 159	160 – 169
Frequency	1	2	7	14	1

d The time taken to make a pizza (to the nearest min.) data.

Time taken (min)	6	7	8	9	10	11	12	13
Frequency	1	0	1	3	11	22	7	1

e 50 marathon runners have 'best times' as recorded in the given table.

Time (min)	120 – 129	130 – 139	140 – 149	150 – 159	160 – 169
frequency	1	10	26	11	2

4

height (mm)	*frequency*
300 - 324	12
325 - 349	18
350 - 374	42
375 - 399	28
400 - 424	14
425 - 449	6

A plant inspector takes a random sample of two week old seedlings from a nursery and measures their height to the nearest mm.

The results are shown in the table alongside.

a Represent the data on a histogram.

b How many of the seedlings are 400 mm or more?

c What percentage of the seedlings are between 349 and 400 mm?

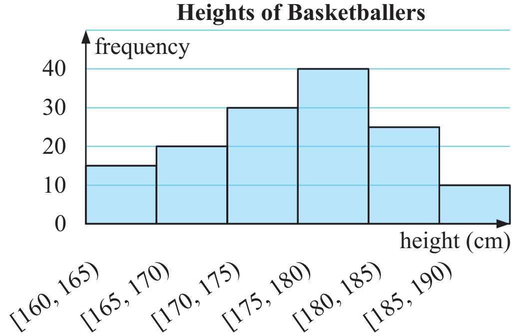
d The total number of seedlings in the nursery is 1462. Estimate the number of seedlings which measure **i** less than 400 mm **ii** between 374 and 425 mm.

5 The histogram shows the height (cm) of some of the basketballers in a local competition.

a How many basketballers were involved in the survey?

b How many of the basketballers were less than 180 cm tall?

c What percentage of the basketballers were 175 cm or taller?



B MEASURING THE CENTRE OF DATA

A better picture of the data in a data set can be seen if we can locate the **middle (centre)** of the data and have an indication of its **spread**. Knowing one of these without the other is often of little use.

There are *three statistics* that are used to measure the **centre** of a data set. These are: the **mean**, the **median** and the **mode**.

THE MEAN

The **mean** of a data set is the statistical name for the arithmetic average and can be found by dividing the sum of the data by the number of data,

$$\text{i.e.,}\quad \textbf{mean} = \frac{\textbf{sum of all data values}}{\textbf{the number of data values}}.$$

The mean gives us a single number which indicates a centre of the data set.

For example, a mean test mark of 73% tells us that there are several marks below 73% and several above it with 73% at the centre. 73% does not have to be one of the data set values.

If we let
x be a data value
n be the number of data values in the sample, or population
$\sum$ mean "the sum of"
$\overline{x}$ represent the mean of a **sample** and
μ represent the mean of a **population**

then we have: $\mu = \dfrac{\sum x}{n}$ or $\overline{x} = \dfrac{\sum x}{n}$.

THE MEDIAN

The **median** is the *middle value* of an ordered data set.

An ordered data set is obtained by listing the data, usually from smallest to largest. The median splits the data in two halves. Half the data are less than or equal to the median and half are greater than or equal to it.

For example, if the median mark for a test is 73% then you know that half the class scored less than or equal to 73% and half scored greater than or equal to 73%.

Note: For an **odd number** of data, the median is one of the data.

For an **even number** of data, the median is the average of the two middle values and may not be one of the original data.

Here is a **rule for finding the median** data values:

If there are n data values, find $\dfrac{n+1}{2}$. The median is the $\left(\dfrac{n+1}{2}\right)$th data value.

For example:

When $n = 13$, $\dfrac{13+1}{2} = 7$, $\therefore$ median = 7th ordered data value.

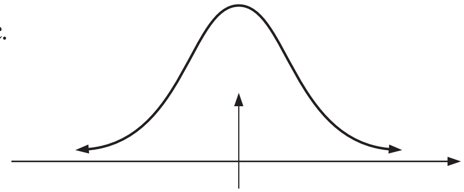
When $n = 14$, $\dfrac{14+1}{2} = 7.5$, $\therefore$ median = average of 7th and 8th ordered data values.

THE MERITS OF THE MEAN AND MEDIAN USED AS A MEASURE OF CENTRE

- Because of the way that the mean is calculated, for some data sets it is a very **poor** measure of a distribution's centre.
- All measures of centre are useless to support an argument if only they are quoted.
- The median is the only measure of centre that will locate the true centre regardless of the data set's features. It is unaffected by the presence of extreme values. It is called a resistant measure of centre.
- The mean is an accurate measure of centre if the distribution is symmetrical or approximately symmetrical. If it is not, then unbalanced high or low values will *drag* the mean toward them and hence cause the mean to be an inaccurate measure of centre. It is called a non-resistant measure of centre. *If it is considered inaccurate, it should not be used in discussion.*
- The following diagrams show the approximate relative postions of the mean and median for the more common shaped distributions.

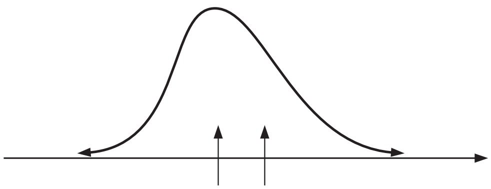
THE RELATIONSHIP BETWEEN THE MEAN AND THE MEDIAN FOR DIFFERENT DISTRIBUTIONS

First of all consider a distribution that is **symmetric**.

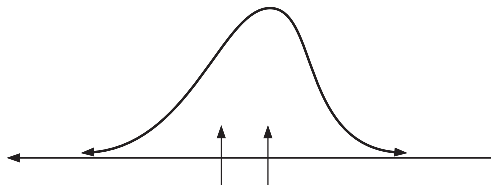


Note: Other symmetrical shapes exist other than this '**bell**' type shape. Any symmetrical distribution will have a mean and median that are approximately equal.

positively skewed distribution



negatively skewed distribution



Hence if the data set has symmetry, both the mean and the median should accurately measure the centre of the distribution.

Note: The mean is influenced by all data values in the data set whereas the median is not.

INVESTIGATION — MERITS OF THE MEAN AND MEDIAN

Recall the data gained from Norm Gregory while he was here for the golf championship. The data was as follows:

244.6	245.1	248.0	248.8	250.0
251.1	251.2	253.9	254.5	254.6
255.9	257.0	260.6	262.8	262.9
263.1	263.2	264.3	264.4	265.0
265.5	265.6	266.5	267.4	269.7
270.5	270.7	272.9	275.6	277.5

TI

C

STATISTICS PACKAGE

What to do:

1 Enter the data as a List into a **graphics calculator** or use the **Statistics package** supplied.

- **a** Produce a histogram of the data. Set the X values from 240 to 280 with an increment of 5. Set the Y values from 0 to 30.
- **b** Comment on the shape of the distribution.
- **c** Find **i** the median **ii** the mean
- **d** Compare the mean and the median. Is the mean an accurate measure of the centre?

2 The mode is meaningless because no value occurs more than once.
  a Verify from the histogram above that the mode appears to be the category [260-265).
  b What would the mode be if our intervals were 2 m, starting at 240 m?

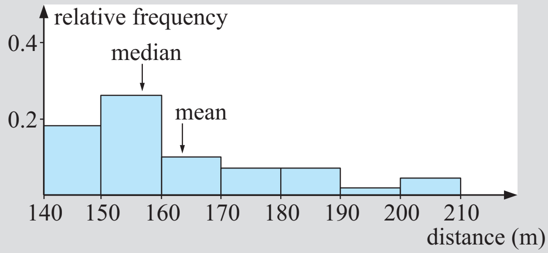
3 Now what would have happened if Norm had hit a few very bad drives? Let us say that his three shortest drives were very short!
  a Change the three shortest drives to 82.1 m, 103.2 m and 111.1 m.
  b Repeat **1 a**, **b**, **c** and **d** but set the X values from 75 to 300 with an increment of 25 for the histogram.

4 What would have happened if Norm had hit a few super long balls in addition to the very bad ones? Let us imagine that the longest balls he hit were very long.
  a Change the three longest drives to 403.9 m, 415.5 m and 420.0 m.
  b Repeat **1 a**, **b**, **c** and **d** but set the X values from 50 to 450 with an increment of 50 for the histogram.

While collecting the data from Norm, I decided to have a hit as well. I hit 30 golf balls with my driver. The relative frequency histogram reveals the results below.

This distribution is clearly positively skewed. This suggests that most of my hits were around the 140 to 170 metres but a few were around the 170 to 210 metres. The mean would not be an accurate measure of the centre of this distribution due to the few higher scores. Indeed the mean is 163.66 m compared to the median of 157.50 m.

Note that you have not been supplied with the data in this case. The idea is for you to get a feel for the data set from the histogram.



THE MODE

The **mode** is the most frequently occurring value in the data set.

UNGROUPED DATA

Example 2

The number of trucks using a road over a 13-day period is: 4 6 3 2 7 8 3 5 5 7 6 6 4.
For this data set, find: **a** the mean **b** the median **c** the mode.

a $\text{mean} = \dfrac{4+6+3+2+7+8+3+5+5+7+6+6+4}{13}$ ← sum of the data / 13 data values

$\doteqdot 5.08$ trucks

b The ordered data set is: ~~2 3 3 4 4 5~~ $\boxed{5}$ ~~6 6 6 7 7 8~~ $\{\text{as } n = 13, \frac{n+1}{2} = 7\}$

$\therefore$ median = 5 trucks

c 6 is the score which occurs the most often $\therefore$ mode = 6 trucks

For the truck data of **Example 2**, how are the measures of the middle affected if on the 14th day the number of trucks was 7?

We expect the mean to rise as the new data value is greater than the old mean.

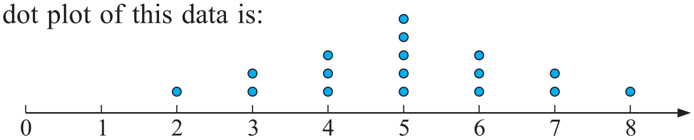
In fact, new mean $= \dfrac{66+7}{14} = \dfrac{73}{14} \doteqdot 5.21$ trucks

The new ordered data set would be: $2\ 3\ 3\ 4\ 4\ 5\ \underbrace{5\ 6}_{\text{two middle scores}}\ 6\ 6\ 7\ 7\ 7\ 8$

$\therefore$ median $= \dfrac{5+6}{2} = 5.5$ trucks

This new data set has two modes. The modes are 6 and 7 trucks and we say that the data set is **bimodal**.

Note:
- If a data set has three or more modes, we do not use the mode as a measure of the middle.
- Consider the data: 4 2 5 6 7 4 5 3 5 4 7 6 3 5 8 6 5.
  The dot plot of this data is:



  For this data the mean, median and mode are all 5.

  Equal values (or approximately equal values) of the mean, mode and median can indicate a *symmetrical distribution* of data.

EXERCISE 18B.1

1 Find the **i** mean **ii** median **iii** mode for each of the following data sets:

a 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 9

b 10, 12, 12, 15, 15, 16, 16, 17, 18, 18, 18, 18, 19, 20, 21

c 22.4, 24.6, 21.8, 26.4, 24.9, 25.0, 23.5, 26.1, 25.3, 29.5, 23.5

d 127, 123, 115, 105, 145, 133, 142, 115, 135, 148, 129, 127, 103, 130, 146, 140, 125, 124, 119, 128, 141

2 Consider the following two data sets:

Data set A: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 10
Data set B: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 15

a Find the mean for both Data set A and Data set B.

b Find the median of both Data set A and Data set B.

c Explain why the mean of Data set A is less than the mean of Data set B.

d Explain why the median of Data set A is the same as the median of Data set B.

3 The annual salaries of ten office workers are:

\$23 000, \$46 000, \$23 000, \$38 000, \$24 000,
\$23 000, \$23 000, \$38 000, \$23 000, \$32 000

a Find the mean, median and modal salaries of this group.

b Explain why the mode is an unsatisfactory measure of the middle in this case.

c Is the median a satisfactory measure of the middle of this data set?

4 The following raw data is the daily rainfall (to the nearest millimetre) for the month of July 2001 in the desert:

3, 1, 0, 0, 0, 0, 0, 2, 0, 0, 3, 0, 0, 0, 7, 1, 1, 0, 3, 8, 0, 0, 0, 42, 21, 3, 0, 3, 1, 0, 0

a Find the mean, median and mode for the data.

b Give a reason why the median is not the most suitable measure of centre for this set of data.

c Give a reason why the mode is not the most suitable measure of centre for this set of data.

d Are there any outliers in this data set?

e On some occasions outliers are removed because they are not typical of the rest of the data and are often due to errors in observation and/or calculation. If the outliers in the data set were accurately found, should they be removed before finding the measures of the middle?

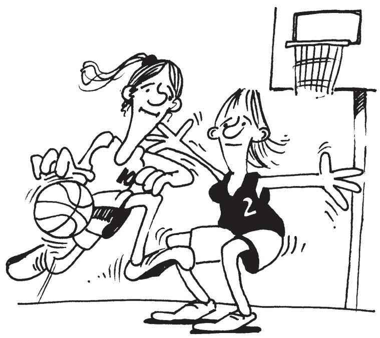
5 A basketball team scored 43, 55, 41 and 37 goals in their first four matches.

a What is the mean number of goals scored for the first four matches?

b What score will the team need to shoot in the next match so that they maintain the same mean score?

c The team shoots only 25 goals in the fifth match. What is the mean number of goals scored for the five matches?

d The team shoots 41 goals in their sixth and final match. Will this increase or decrease their previous mean score? What is the mean score for all six matches?



Example 3

The mean of five scores is 12.2. What is the sum of the scores?

Let S = sum of scores $\quad \therefore \quad \dfrac{S}{5} = 12.2$

$\therefore \quad S = 12.2 \times 5 = 61$

i.e., the sum of scores is 61.

6 The mean of 10 scores is 11.6. What is the sum of the scores?

7 While on an outback safari, Bill drove, on average, 262 km per day for a period of 12 days. How far did Bill drive in total while on safari?

8 The mean monthly sales for a clothing store are \$15 467. Calculate the total sales for the store for the year.

Example 4

Find x if 10, 7, 3, 6 and x have a mean of 8.

There are 5 scores.

$$\therefore \quad \frac{10+7+3+6+x}{5} = 8$$

$$\therefore \quad \frac{26+x}{5} = 8$$

$$\therefore \quad 26+x = 40$$

$$\therefore \quad x = 14$$

9 Find x if 5, 9, 11, 12, 13, 14, 17 and x have a mean of 12.

10 Find a, given that 3, 0, a, a, 4, a, 6, a and 3 have a mean of 4.

11 Over the complete assessment period, Gitta averaged 35 out of a possible 40 marks for her maths tests. However, when checking her files, she could only find 7 of the 8 tests. For these she scored 29, 36, 32, 38, 35, 34 and 39. Determine how many marks out of 40 she scored for the eighth test.

12 On the first five days of his holiday Oskar drove an average of 256 kilometres per day and on the next three days he drove an average of 172 kilometres per day.

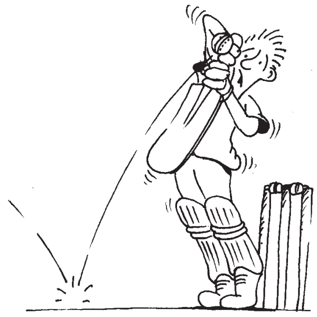
a What is the total distance that Oskar drove in the first five days?

b What is the total distance that Oskar drove in the next three days?

c What is the mean distance travelled per day, by Oskar, over the eight days?

13 A sample of 10 measurements has a mean of 15.7 and a sample of 20 measurements has a mean of 14.3. Find the mean of all 30 measurements.

14 A cricketer has scored an average of 25.4 runs in his last 10 innings. He scores 58 and 16 runs in his next two innings. What is his new batting average?



15 Jane had seven spelling tests, each with twelve words, but could only find the results of five of them. These were: 9, 5, 7, 9 and 10. She asked her teacher for the other two results and the teacher said that the mode of her scores was 9 and the mean was 8. What are the two missing results, given that Jane knows that her worst result was a 5?

DISCUSSION

Which of the measures of the middle is more affected by the presence of an outlier? Develop at least two examples to show how the measures of the middle can be altered by outliers.

MEASURES OF THE CENTRE FROM OTHER SOURCES

When the same data appear several times we often summarise the data in table form. Consider the data of the **given table**:

Data value	*Frequency*	*Product*
3	1	$3 \times 1 = 3$
4	1	$4 \times 1 = 4$
5	3	$5 \times 3 = 15$
6	7	$6 \times 7 = 42$
7	15	$7 \times 15 = 105$
8	8	$8 \times 8 = 64$
9	5	$9 \times 5 = 45$
Total	40	278

We can find the measures of the centre directly from the table.

The mode

The mode is 7. There are 15 of data value 7 which is more than any other data value.

The mean

Adding a **'Product' column** to the table helps to add all scores. For example, there are 15 data of value 7 and these add to $15 \times 7 = 105$.

So, $\text{mean} = \dfrac{278}{40} = 6.95$

The median

There are 40 data values, an even number, so there are *two middle* data values. What are they? How do we find them from the table?

As the sample size $n = 40$, $\dfrac{n+1}{2} = \dfrac{41}{2} = 20.5$

$\therefore$ the median is the average of the 20th and 21st data values.

In the table, the blue numbers show us accumulated values.

Data value	*Frequency*		
3	1	1	← one number is 3
4	1	2	← two numbers are 4 or less
5	3	5	← five numbers are 5 or less
6	7	12	← 12 numbers are 6 or less
7	15	27	← 27 numbers are 7 or less
8	8		
9	5		
Total	40		

We can see that the 20th and 21st data values (in order) are both 7's,

$\therefore$ $\text{median} = \dfrac{7+7}{2} = 7$

Notice that in this example the distribution is clearly skewed even though the mean, median and mode are nearly equal. So, we must be careful in saying that equal values of these measures of the middle enable us to say with certainty that the distribution is symmetric.

Example 5

Julie and Annika both play goalshooter for their respective netball teams. Their performances for the season were as follows:

Julie played 11 games and scored 14, 22, 17, 31, 15, 19, 24, 28, 26, 35, 29 goals.
Annika played 8 games and scored 17, 21, 36, 19, 16, 28, 26, 32 goals.

Who had the higher mean, Julie or Annika?

$$\text{Julie's mean} = \frac{14+22+17+31+\ ....\ +35+29}{11} = \frac{260}{11} \doteqdot 23.64 \text{ goals}$$

$$\text{Annika's mean} = \frac{17+21+36+19+16+28+26+32}{8} = \frac{195}{8} \doteqdot 24.38 \text{ goals}$$

Annika had the higher mean number of goals.

Example 6

The table alongside shows the number of aces served by tennis players in their first set of a tournament.

Number of aces	1	2	3	4	5	6
Frequency	4	11	18	13	7	2

Determine the mean number of aces in the first set.

No. of aces (x)	*Frequency* (f)	*Product* ($f\,x$)
1	4	4
2	11	22
3	18	54
4	13	52
5	7	35
6	2	12
Total	55	179

2 aces occurred 11 times.
Instead of adding
$2+2+\ ....\ +2$, 11 times
we simply calculate
11×2.

$$\overline{x} = \frac{\sum fx}{\sum f} = \frac{\text{sum of all data values}}{\text{the number of data values}}$$

$$= \frac{179}{55}$$

$$\doteqdot 3.25 \text{ aces}$$

Note: $\overline{x} = \dfrac{\sum_{i=1}^{6} f_i x_i}{\sum_{i=1}^{6} f_i}$ has been abbreviated to $\overline{x} = \dfrac{\sum fx}{\sum f}$.

Example 7

Find the median number of peas in a pod:

a 3, 6, 5, 7, 7, 4, 6, 5, 6, 7, 6, 8, 10, 7, 8

b 3, 6, 5, 7, 7, 4, 6, 5, 6, 7, 6, 8, 10, 7, 8, 9

a The ordered data set is:

$\cancel{3\ 4\ 5\ 5\ 6\ 6\ 6}\ \boxed{6}\ \cancel{7\ 7\ 7\ 7\ 8\ 8\ 10}$ (15 of them)

{as $n = 15$, $\dfrac{n+1}{2} = 8$ $\therefore$ the median is the 8th data value}

$\therefore$ the median $= 6$ peas

b The ordered data set is:

$\cancel{3\ 4\ 5\ 5\ 6\ 6\ 6}\ \boxed{6\ \ 7}\ \cancel{7\ 7\ 7\ 8\ 8\ 9\ 10}$ (16 of them)

{as $n = 16$, $\dfrac{n+1}{2} = 8.5$ $\therefore$ the median is the average of the 8th and 9th data values}

$\therefore$ the median $= \dfrac{6+7}{2} = 6.5$ peas

Example 8

Find the median of the data given in the table that shows the number of people on each table at a restaurant:

Number of people	5	6	7	8	9	10	11	12
Frequency	1	0	3	9	12	7	4	2

The total number in the data set is the frequency sum and this is $n = 38$.
In this table the data is already ordered.

As $\dfrac{n+1}{2} = \dfrac{39}{2} = 19.5$, the median is the average of the 19th and 20th data values.

Number of people	5	6	7	8	9	10	11	12
Frequency	1	0	3	9	12	7	4	2

13 data values of 8 or less

the 14th to the 25th are all 9's

$\therefore$ median $= \dfrac{9+9}{2} = 9$ people

Example 9

In a class of 20 students the results of a spelling test out of 10 are shown in the table.

Calculate the:

a mean
b median
c mode

Score	*Number of students*
5	1
6	2
7	4
8	7
9	4
10	2
Total	20

a There are 20 scores.

	sum of scores
1 is a 5	5
2 are 6's	12
4 are 7's	28
7 are 8's	56
4 are 9's	36
2 are 10's	20
Total	157

The mean score

$$= \frac{\text{total of scores}}{20}$$

$$= \frac{157}{20}$$

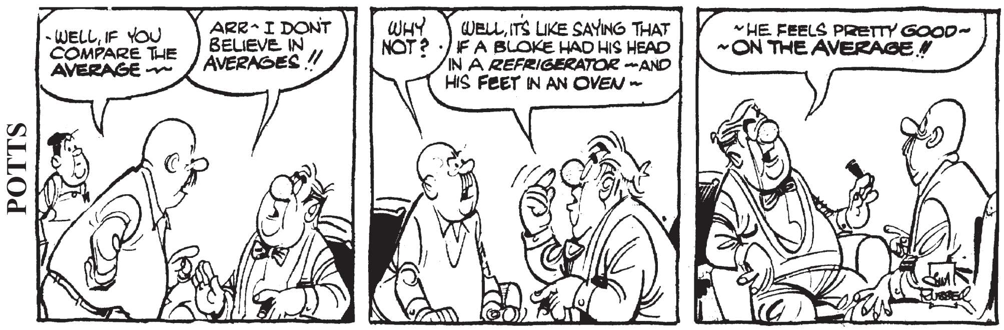
$$= 7.85$$

b There are 20 scores, and so the median is the average of the 10th and 11th scores.

Score	*Number of students*	
5	1	← 1st student
6	2	← 2nd and 3rd student
7	4	← 4th, 5th, 6th and 7th student
8	7	← 8th, 9th, **10th**, **11th**, 12th, 13th, 14th student
9	4	
10	2	

The 10th and 11th students both scored 8 $\therefore$ median $= 8$.

c Looking down the 'number of students' column, the highest frequency is 7. This corresponds to a score of 8, $\therefore$ mode $= 8$.



The publishers acknowledge the late Mr Jim Russell, General Features for the reproduction of this cartoon

EXERCISE 18B.2

1 The table alongside shows the results when 3 coins were tossed simultaneously 30 times. The number of heads appearing was recorded. Calculate the:

a mode **b** median **c** mean.

Number of heads	*Number of times occurred*
0	4
1	12
2	11
3	3
Total	30

2 The following frequency table records the number of phone calls made in a day by 50 fifteen-year-olds.

No. of phone calls	*Frequency*
0	5
1	8
2	13
3	8
4	6
5	3
6	3
7	2
8	1
9	0
10	0
11	1

a For this data, find the:
i mean **ii** median **iii** mode.

b Construct a column graph for the data and show the position of the measures of centre (mean, median and mode) on the horizontal axis.

c Describe the distribution of the data.

d Why is the mean larger than the median for this data?

e Which measure of centre would be the most suitable for this data set?

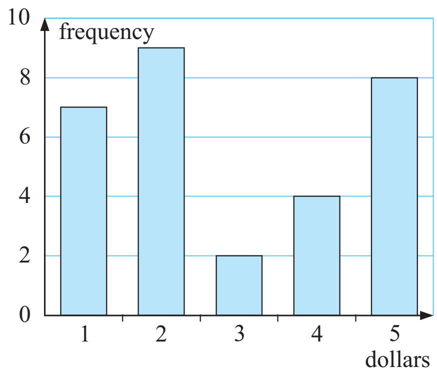
3 The frequency column graph alongside gives the value of donations for the Heart Foundation collected in a particular street.

a Construct a frequency table from the graph.

b Determine the total number of donations.

c For the donations find the:
i mean **ii** median
iii mode.

d Which of the measures of central tendency can be found easily using the graph only?



4 A company claims that their match boxes contain, on average, 50 matches per box. On doing a survey, the Consumer Protection Society recorded the following results:

Number in a box	*Frequency*
47	5
48	4
49	11
50	6
51	3
52	1
Total	30

a For the data calculate the:
i mode **ii** median **iii** mean.

b Do the results of this survey support the company's claim?

c In court for 'false advertising', the company won their case against the Consumer Protection Society. Suggest why and how they did this.

5 Families at a school were surveyed. The number of children in each family was recorded. The results of the survey are shown alongside.

Number of Children	*Frequencies*
1	5
2	28
3	15
4	8
5	2
6	1
Total	59

a Calculate the:
  i mean **ii** mode **iii** median.

b The average Australian family has 2.2 children. How does this school compare to the national average?

c The data set is skewed. Is the skewness positive or negative?

d How has the skewness of the data affected the measures of this middle?

6 For the data displayed in the following stem-and-leaf plots find the:
  i mean **ii** median **iii** mode.

a

Stem	*Leaf*
5	3 5 6
6	0 1 2 4 6 7 9
7	3 3 6 8
8	4 7
9	1

where 5 | 3 means 53

b

Stem	*Leaf*
3	7
4	0 4 8 8
5	0 0 1 3 6 7 8 9
6	0 3 6 7 7 7
7	0 6 9
8	1

where 3 | 7 means 3.7

7 Revisit **The Pea Problem** on page **420**.

a Use the frequency table for the *Without fertiliser* data to find the:
  i mean **ii** mode **iii** median number of peas per pod.

b Use a frequency table for the *With fertiliser* data to find the:
  i mean **ii** mode **iii** median number of peas per pod.

c Which of the measures of 'the centre' is appropriate to use in a report on this data?

d Has the application of fertiliser significantly improved the number of peas per pod?

8 Below are the points scored by two basketball teams over a 12 match series:

Team A: 91, 76, 104, 88, 73, 55, 121, 98, 102, 91, 114, 82
Team B: 87, 104, 112, 82, 64, 48, 99, 119, 112, 77, 89, 108

Which team had the higher mean score?

9 Select the mode(s) for the following sets of numbers:

a 44, 42, 42, 49, 47, 44, 48, 47, 49, 41, 45, 40, 49

b 148, 144, 147, 147, 149, 148, 146, 144, 145, 143, 142, 144, 147

c 25, 21, 20, 24, 28, 27, 25, 29, 26, 28, 22, 25

10 Calculate the median value for the following data:

a 21, 23, 24, 25, 29, 31, 34, 37, 41

b 105, 106, 107, 107, 107, 107, 109, 120, 124, 132

c 173, 146, 128, 132, 116, 129, 141, 163, 187, 153, 162, 184

11 A survey of 50 students revealed the following number of siblings per student:

1, 1, 3, 2, 2, 2, 0, 0, 3, 2, 0, 0, 1, 3, 3, 4, 0, 0, 5, 3, 3, 0, 1, 4, 5,
1, 3, 2, 2, 0, 0, 1, 1, 5, 1, 0, 0, 1, 2, 2, 1, 3, 2, 1, 4, 2, 0, 0, 1, 2

a What is the modal number of siblings per student?
b What is the mean number of siblings per student?
c What is the median number of siblings per student?

12 This table shows the average monthly rainfall for a city in the Southern Hemisphere.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Av. rainfall (mm)	16	34	38	41	98	172	166	159	106	71	52	21

Calculate the mean average monthly rainfall for this city.

13 The selling prices of the last 10 houses sold in a certain district were as follows:

\$146 400, \$127 600, \$211 000, \$192 500,
\$256 400, \$132 400, \$148 000, \$129 500,
\$131 400, \$162 500

a Calculate the mean and median selling prices of these houses and comment on the results.
b Which measure would you use if you were:
  i a vendor wanting to sell your house
  ii looking to buy a house in the district?

14 51 packets of chocolate almonds were opened and their contents counted. The following table gives the distribution of the number of chocolates per packet sampled.

Find the mean, mode and median of the distribution.

Number in packet	*Frequency*
32	6
33	8
34	9
35	13
36	10
37	3
38	2

15 A sample of 12 measurements has a mean of 16.5 and a sample of 15 measurements has a mean of 18.6. Find the mean of all 27 measurements.

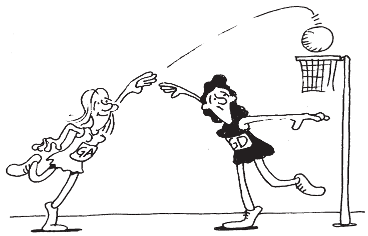
16 The table alongside compares the mass at birth of some guinea pigs with their mass when they were two weeks old.

a What was the mean birth mass?
b What was the mean mass after two weeks?
c What was the mean increase over the two weeks?

Guinea Pig	*Mass (g) at birth*	*Mass (g) at 2 weeks*
A	75	210
B	70	200
C	80	200
D	70	220
E	74	215
F	60	200
G	55	206
H	83	230

17 15 of 31 measurements are below 10 cm and 12 measurements are above 11 cm. Find the median if the other 4 measurements are 10.1 cm, 10.4 cm, 10.7 cm and 10.9 cm.

18 Towards the end of season, a netballer had played 14 matches and had an average of 16.5 goals per game. In the final two matches of the season the netballer threw 21 goals and 24 goals. Find the netballer's new average.



19 The mean and median of a set of 9 measurements are both 12. If 7 of the measurements are 7, 9, 11, 13, 14, 17 and 19, find the other two measurements.

20 Two brands of matches claim that their boxes contain, on average, 50 matches per box. On doing a survey the Consumer Protection Society (C.P.S.) recorded the following results:

Brand A

number in a box	46	47	48	49	50	51	52	53	55
frequency	1	1	2	7	10	20	15	3	1

Brand B

number in a box	48	49	50	51	52	53	54
frequency	3	17	30	7	2	1	1

a Find the average contents of Brands A and B.

b Would it be 'fair' of the C.P.S. to prosecute the manufacturers of either brand, based on these statistics?

21 A magazine is designed to offer advice to consumers. Its editor decides to test the weights of loaves of bread of two different brands, each of which guarantees that the minimum weight is 400 g. 100 loaves of each brand were weighed and the distributions of weights were:

Brand X

weight (to nearest 10 g)	370	380	390	400	410	420	430
number of loaves	2	5	32	35	20	5	1

Brand Y

weight (to nearest 10 g)	380	390	400	410	420	430
number of loaves	3	21	43	24	7	2

Which brand will the magazine recommend to its readers if its decision is based on average weights?

22 In an office of 20 people there are only 4 salary levels paid:
\$50 000 (1 person), \$42 000 (3 people), \$35 000 (6 people), \$28 000 (10 people).

a Calculate:

i the median salary **ii** the modal salary **iii** the mean salary.

b Which measure of central tendency would be used by a top salary earner if she is the boss and is against a pay rise for the other employees?

GROUPED DATA

When information has been gathered in classes we use the **midpoint** of the class to represent all scores within that interval.

We are assuming that the scores within each class are evenly distributed throughout that interval. The mean calculated will therefore be an **approximation** to the true value.

Example 10

Find the approximate mean of the *ages of bus drivers* data, to the nearest year:

age (yrs)	21-25	26-30	31-35	36-40	41-45	46-50	51-55
frequency	11	14	32	27	29	17	7

age (yrs)	*frequency* (f)	*midpoint* (x)	$f\,x$
21-25	11	23	253
26-30	14	28	392
31-35	32	33	1056
36-40	27	38	1026
41-45	29	43	1247
46-50	17	48	816
51-55	7	53	371
Total	137		5161

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{5161}{137} \doteqdot 37.7$$

EXERCISE 18B.3

1 Find the approximate mean for each of the following distributions:

a

Score (x)	*Frequency* (f)
1-5	7
6-10	12
11-15	15
16-20	10
21-25	11

b

Score (x)	*Frequency* (f)
40-42	2
43-45	1
46-48	5
49-51	6
52-54	12
55-57	3

2 50 students sit a mathematics test and the results are as follows:

Score	0-9	10-19	20-29	30-39	40-49
Frequency	2	5	7	27	9

Find an estimate of the mean score.

3 Following is a record of the number of goals Chloë has scored in her basketball matches.

15	8	6	10	0	9	2	16	11	23	14	13	17	16	20	12	13
12	10	3	13	5	18	14	19	4	15	15	19	19	14	6	11	29
8	9	3	20	9	25	7	15	19	21	23	12	17	22	14	26	

a Find the mean number of goals per match.

b Estimate the mean by grouping the data into:
  i intervals 0-4, 5-9, 10-14, etc. **ii** intervals 0-8, 9-16, 17-24, 25-30.

c Comment on your answers from **a** and **b**.

4 The table shows the length of newborn babies at a hospital over a one week period.
Find the approximate mean length of the newborn babies.

Length (mm)	*frequency*
400 to 424	2
425 to 449	7
450 to 474	15
475 to 499	31
500 to 524	27
525 to 549	12
550 to 574	4
575 to 599	1

5 The table shows the petrol sales in one day by a number of city service stations.

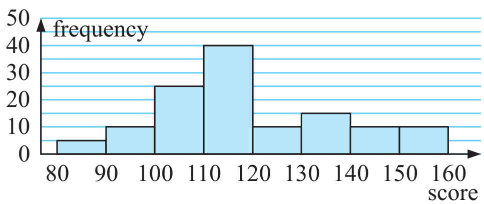
a How many service stations were involved in the survey?

b Estimate the total amount of petrol sold for the day by the service stations.

c Find the approximate mean sales of petrol for the day.

Thousands of litres (l)	*frequency*
2000 to 2999	4
3000 to 3999	4
4000 to 4999	9
5000 to 5999	14
6000 to 6999	23
7000 to 7999	16

6 This histogram illustrates the results of an aptitude test given to a group of people seeking positions in a company.



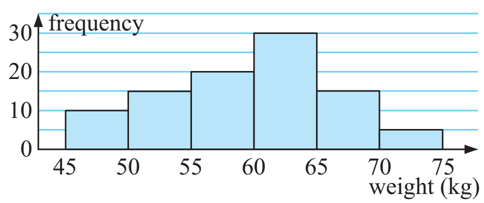
a How many people sat for the test?

b Find an estimate of the mean score for the test.

c What fraction of the people scored less than 100 for the test?

d If the top 20% of the people are offered positions in the company, estimate the minimum mark required.

7



The histogram shows the weights (in kg) of a group of year 10 students at a country high school.

a How many students were involved in the survey?

b Calculate the mean weight of the students.

c How many students weigh less than 56 kg?

d What percentage of students weigh between 50 and 60 kg?

e If a student was selected at random, what would be the chance that the student weighed less than 60 kg?

C CUMULATIVE DATA

Sometimes it is useful to know the number of scores that lie above or below a particular value. In such situations it is convenient to construct a **cumulative frequency distribution table** and use a graph called a **cumulative frequency polygon** to represent the data.

Example 11

The data shown gives the weights of 120 male footballers.

a Construct a cumulative frequency distribution table.

b Represent the data on a cumulative frequency polygon.

c Use your graph to estimate the:

- **i** median weight
- **ii** number of men weighing less than 73 kg
- **iii** number of men weighing more than 92 kg.

Weight (w kg)	*frequency*
$55 \leqslant w < 60$	2
$60 \leqslant w < 65$	3
$65 \leqslant w < 70$	12
$70 \leqslant w < 75$	14
$75 \leqslant w < 80$	19
$80 \leqslant w < 85$	37
$85 \leqslant w < 90$	22
$90 \leqslant w < 95$	8
$95 \leqslant w < 100$	2
$100 \leqslant w < 105$	1

a

Weight (w kg)	*frequency*	*cumulative frequency*
$55 \leqslant w < 60$	2	2
$60 \leqslant w < 65$	3	5
$65 \leqslant w < 70$	12	17
$70 \leqslant w < 75$	14	31
$75 \leqslant w < 80$	19	50
$80 \leqslant w < 85$	37	87
$85 \leqslant w < 90$	22	109
$90 \leqslant w < 95$	8	117
$95 \leqslant w < 100$	2	119
$100 \leqslant w < 105$	1	120

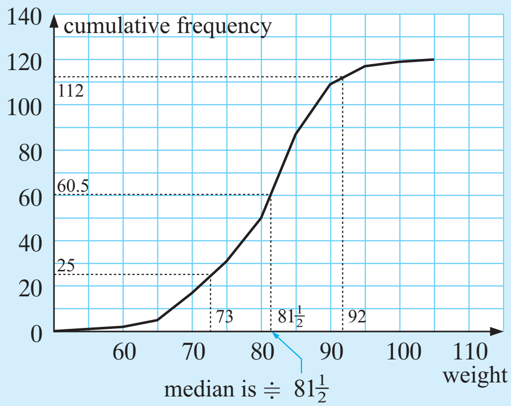
this is $2 + 3$

this is $2 + 3 + 12$, etc.

this 50 means that there are 50 players who weigh less than 80 kg

Note: The cumulative frequency gives a *running total* of the number of players up to certain weights.

b **Cumulative frequency polygon of footballers' weights**



c **i** The median is the average of the 60th and 61st weights. Call it 60.5. Reading from the graph, the median $\doteqdot 81.5$.

ii There are 25 men who weigh less than 73 kg.

iii There are $120 - 112 = 8$ men who weigh more than 92 kg.

EXERCISE 18C

1 Calculate the median of the following distributions:

a

score	1	2	3	4	5	6
frequency	25	11	8	5	4	1

b

score	5	6	7	8	9	10
frequency	1	3	11	12	8	2

2 This table indicates the number of errors in randomly chosen pages of a telephone directory:

number of errors	0	1	2	3	4	5	6
frequency	67	35	17	8	11	2	1

Find the median number of errors.

3 The following data shows the lengths of 30 trout caught in a lake during a fishing competition. Measurements are to the nearest centimetre.

31 38 34 40 24 33 30 36 38 32 35 32 36 27 35
40 34 37 44 38 36 34 33 31 38 35 36 33 33 28

a Construct a cumulative frequency table for trout lengths, x cm, using the following intervals $24 \leqslant x < 27$, $27 \leqslant x < 30$, ... etc.

b Draw a cumulative frequency graph.

c Use **b** to find the median length.

d Use the original data to find its median and compare your answer with **c**. Comment!

4 In an examination the following scores were achieved by a group of students:

Draw a cumulative frequency graph of the data and use it to find:

a the median examination mark

b how many students scored less than 65 marks

c how many students scored between 50 and 70 marks

d how many students failed, given that the pass mark was 45

e the credit mark, given that the top 16% of students were awarded credits.

score	*frequency*
$10 \leqslant x < 20$	2
$20 \leqslant x < 30$	5
$30 \leqslant x < 40$	7
$40 \leqslant x < 50$	21
$50 \leqslant x < 60$	36
$60 \leqslant x < 70$	40
$70 \leqslant x < 80$	27
$80 \leqslant x < 90$	9
$90 \leqslant x < 100$	3

5 The following frequency distribution was obtained by asking 50 randomly selected people the size of their shoes.

shoe size	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$	9	$9\frac{1}{2}$	10
frequency	1	1	0	3	5	13	17	7	2	0	1

Draw a cumulative frequency graph of the data and use it to find:

a the median shoe size

b how many people had a shoe size of: **i** $7\frac{1}{2}$ or more **ii** 8 or less.

6 In a cross-country race, the times (in minutes) of 80 competitors were recorded as follows:

Draw a cumulative frequency graph of the data and use it to find:

a the median time

b the approximate number of runners whose time was not more than 28 minutes

c the approximate time in which the fastest 25 runners competed.

Score	*frequency*
$20 \leqslant t < 25$	15
$25 \leqslant t < 30$	33
$30 \leqslant t < 35$	21
$35 \leqslant t < 40$	10
$40 \leqslant t < 45$	1

7 The following table gives the age groups of car drivers involved in an accident in a city for a given year.

Draw a cumulative frequency graph of the data and use it to find:

a the median age of the drivers involved in the accidents

b the percentage of drivers, with ages of 23 or less, involved in accidents.

c Estimate the probability that a driver involved in an accident is:

  i aged less than or equal to 27 years

  ii aged 27 years.

Age (in years)	*No. of accidents*
$16 \leqslant x < 20$	59
$20 \leqslant x < 25$	82
$25 \leqslant x < 30$	43
$30 \leqslant x < 35$	21
$35 \leqslant x < 40$	19
$40 \leqslant x < 50$	11
$50 \leqslant x < 60$	24
$60 \leqslant x < 80$	41

8 The table below gives the distribution of the life of electric light globes.

Draw a cumulative frequency graph of the data and use it to estimate:

a the median life of a globe

b the percentage of globes which have a life of 2700 hours or less

c the number of globes which have a life between 1500 and 2500 hours

d the probability that a globe fails before 800 hours of use.

Life (hours)	*Number of globes*
$0 \leqslant l < 500$	5
$500 \leqslant l < 1000$	17
$1000 \leqslant l < 2000$	46
$2000 \leqslant l < 3000$	79
$3000 \leqslant l < 4000$	27
$4000 \leqslant l < 5000$	4

9 The following table is a summary of the distance (to the nearest metre) a cricket ball was thrown by a number of different students.

Distance (m)	$20 \leqslant d < 30$	$30 \leqslant d < 40$	$40 \leqslant d < 50$	$50 \leqslant d < 60$	$60 \leqslant d < 70$	$70 \leqslant d < 80$
frequency	4	17	38	23	17	2

Draw a cumulative frequency graph of the data and use it to find:

a the median distance thrown by the students

b the number of students who threw the ball less than 35 m

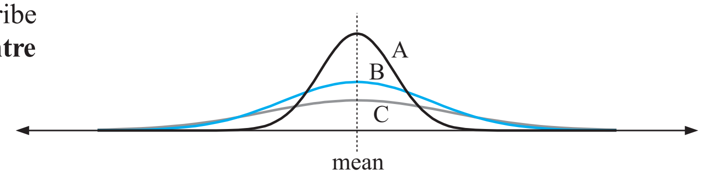
c the number of students who threw the ball between 45 and 65 m.

d If only students who threw the ball further than 53 m were considered for further coaching, how many students were considered?

e If a student is selected at random from the group, determine the likelihood that he (or she) threw the ball less than 32 m.

D MEASURING THE SPREAD OF DATA

We use two measures to describe a distribution. These are its **centre** and its **variability** (or **spread**).



The above distributions have the same mean, but clearly they have a different spread. For example, the A distribution has most scores close to the mean whereas the C distribution has greater spread.

Consequently, we need to consider measures of variability (spread), and we will examine three of these measures: the **range**, the **interquartile range** (**IQR**) and the **standard deviation**.

THE RANGE

For a given set of data the **range** is the difference between the maximum (largest) and the minimum (smallest) data values.

Example 12

A greengrocer chain is to purchase apples from two different wholesalers. They take six random samples of 50 apples to examine them for skin blemishes. The counts for the number of blemished apples are:

Wholesaler Redapp	5	17	15	3	9	11
Wholesaler Pureapp	10	13	12	11	12	11

What is the range from each wholesaler?

Wholesaler Redapp Range $= 17 - 3 = 14$

Wholesaler Pureapp Range $= 13 - 10 = 3$

Note: The **range** is not considered to be a particularly reliable measure of spread as it uses only two data values.

THE UPPER AND LOWER QUARTILES AND THE INTERQUARTILE RANGE

The median divides the ordered data set into two halves and these halves are divided in half again by the **quartiles**.

The middle value of the lower half is called the **lower quartile**. One-quarter, or 25%, of the data have a value less than or equal to the lower quartile. 75% of the data have values greater than or equal to the lower quartile.

The middle value of the upper half is called the **upper quartile**. One-quarter, or 25%, of the data have a value greater than or equal to the upper quartile. 75% of the data have values less than or equal to the upper quartile.

interquartile range = upper quartile − lower quartile

The interquartile range is the range of the middle half (50%) of the data.

The data set has been divided into quarters by the lower quartile (Q_1), the median (Q_2) and the upper quartile (Q_3).

So, the **interquartile range**, $\mathbf{IQR = Q_3 - Q_1}$.

Example 13

For the data set: 6, 4, 7, 5, 3, 4, 2, 6, 5, 7, 5, 3, 8, 9, 3, 6, 5 find the

a median **b** lower quartile **c** upper quartile **d** interquartile range

The ordered data set is:

$\cancel{2\ 3\ 3\ 3\ 4\ 4\ 5\ 5}\ \boxed{5}\ \cancel{5\ 6\ 6\ 6\ 7\ 7\ 8\ 9}$ (17 of them)

a The median $= \left(\frac{17+1}{2}\right)$th score $=$ 9th score $= 5$

b/c As the median is a data value we now ignore it and split the remaining data into two

$$\underbrace{2\ 3\ 3\ \boxed{3\ 4}\ 4\ 5\ 5}_{\text{lower}} \qquad \underbrace{5\ 6\ 6\ \boxed{6\ 7}\ 7\ 8\ 9}_{\text{upper}}$$

$Q_1 =$ median of lower half $= \dfrac{3+4}{2} = 3.5$

$Q_3 =$ median of upper half $= \dfrac{6+7}{2} = 6.5$

d $\text{IQR} = Q_3 - Q_1 = 6.5 - 3.5 = 3$

Example 14

For the data set: 11, 6, 7, 8, 13, 10, 8, 7, 5, 2, 9, 4, 4, 5, 8, 2, 3, 6 find

a the median **b** the lower quartile

c the upper quartile **d** the interquartile range.

The ordered data set is:

$\cancel{2\ 2\ 3\ 4\ 4\ 5\ 5\ 6}\ \boxed{6\ \ 7}\ \cancel{7\ 8\ 8\ 8\ 9\ 10\ 11\ 13}$ (18 of them)

a As $n = 18$, $\dfrac{n+1}{2} = 9.5$

$\therefore$ median $= \dfrac{\text{9th value} + \text{10th value}}{2} = \dfrac{6+7}{2} = 6.5$

b/c As the median is not a data value we split the data into two

$$\underbrace{2\ 2\ 3\ 4\ \boxed{4}\ 5\ 5\ 6\ 6}_{\text{lower}} \qquad \underbrace{7\ \ 7\ \ 8\ \ 8\ \ \boxed{8}\ \ 9\ 10\ 11\ 13}_{\text{upper}}$$

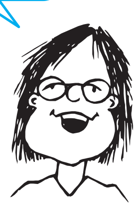
$\therefore$ $Q_1 = 4$, $Q_3 = 8$

d $\text{IQR} = Q_3 - Q_1$
$= 8 - 4$
$= 4$

Note: Some computer packages (e.g. MS Excel) calculate quartiles in a different way to this example.

EXERCISE 18D.1

Small sample, rounded continuous data, can often be treated in the same way as discrete data for the purpose of analysis.



1 For each of the following data sets, find:

- **i** the median (make sure the data is ordered)
- **ii** the upper and lower quartiles
- **iii** the range
- **iv** the interquartile range.

a 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 9

b 10, 12, 15, 12, 24, 18, 19, 18, 18, 15, 16, 20, 21, 17, 18, 16, 22, 14

c 21.8, 22.4, 23.5, 23.5, 24.6, 24.9, 25, 25.3, 26.1, 26.4, 29.5

d 127, 123, 115, 105, 145, 133, 142, 115, 135, 148, 129, 127, 103, 130, 146, 140, 125, 124, 119, 128, 141

2 The time spent (in minutes) by 20 people in a queue at a bank waiting to be attended by a teller, has been recorded as follows:

3.4	2.1	3.8	2.2	4.5	1.4	0
0	1.6	4.8	1.5	1.9	0	3.6
5.2	2.7	3.0	0.8	3.8	5.2	

a Find the median waiting time and the upper and lower quartiles.

b Find the range and interquartile range of the waiting time.

c Copy and complete the following statements:

- **i** "50% of the waiting times were greater than .... minutes."
- **ii** "75% of the waiting times were less than ...... minutes."
- **iii** "The minimum waiting time was .... minutes and the maximum waiting time was ..... minutes. The waiting times were spread over ...... minutes."

3

Stem	*Leaf*
0	3 4 7 9
1	0 3 4 6 7 8
2	0 0 3 5 6 9 9 9
3	1 3 7 8
4	2

3 | 7 means 37

For the data set given, find:

- **a** the minimum value
- **b** the maximum value
- **c** the median
- **d** the lower quartile
- **e** the upper quartile
- **f** the range
- **g** the interquartile range.

4 The heights of 20 six-year-olds are recorded in the following stem-and-leaf plot:

Stem	*Leaf*
10	9
11	1 3 4 4 8 9
12	2 2 4 4 6 8 9 9
13	1 2 5 8 8

10 | 9 reads 109 cm

a Find:

- **i** the median height
- **ii** the upper and lower quartiles of the data.

b Copy and complete the following statements:

- **i** "Half of the children are no more than ..... cm tall."
- **ii** "75% of the children are no more than ......cm tall."

c Find the: **i** range **ii** interquartile range for the height of six year olds.

d Copy and complete:
"The middle 50% of the children have heights spread over ..... cm."

5 Revisit **The Pea Problem** on page **420**.

a For the *Without fertiliser* data, find:
- **i** the range
- **ii** the median
- **iii** the lower quartile
- **iv** the upper quartile
- **v** the interquartile range

b Repeat **a** for the *With fertiliser* data.

c Reconsider the questions posed. Amend your solutions where appropriate.

BOX-AND-WHISKER PLOTS

A **box-and-whisker plot** (or simply a **boxplot**) is a visual display of some of the descriptive statistics of a data set. It shows:

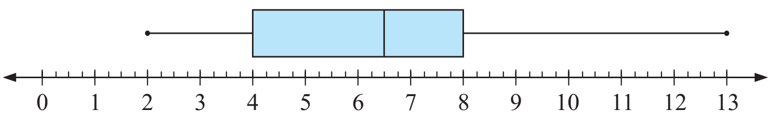
- the minimum value (Min_x)
- the lower quartile (Q_1)
- the median (Q_2)
- the upper quartile (Q_3)
- the maximum value (Max_x)

These five numbers form what is known as the **five-number summary** of a data set.

In **Example 14** on page **446** the **five-number summary** is

minimum $= 2$
$Q_1 = 4$
median $= 6.5$
$Q_3 = 8$
maximum $= 13$

and the corresponding boxplot is:



Note:
- The rectangular box represents the 'middle' half of the data set.
- The lower whisker represents the 25% of the data with smallest values.
- The upper whisker represents the 25% of the data with greatest values.

Example 15

For the data set: 4 5 9 5 1 7 8 7 3 5 6 3 4 3 2 5

a construct the five-number summary

b draw a boxplot

c find the **i** range **ii** interquartile range

d the percentage of data values above 3.

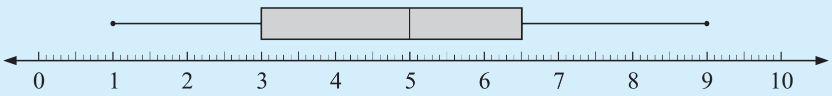
a The ordered data set is

1 2 3 3 3 4 4 5 5 5 5 6 7 7 8 9 (16 of them)

$Q_1 = 3$ median $= 5$ $Q_3 = 6.5$

So the **5-number summary** is: $\begin{cases} \text{min. value} = 1 & Q_1 = 3 \\ \text{median} = 5 & Q_3 = 6.5 \\ \text{max. value} = 9 & \end{cases}$

b



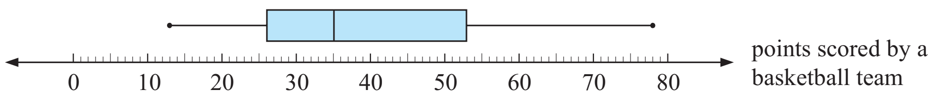
c **i** range $=$ max. value $-$ min. value
$= 9 - 1$
$= 8$

ii $\text{IQR} = Q_3 - Q_1$
$= 6.5 - 3$
$= 3.5$

d 75% of the data values are above 3.

EXERCISE 18D.2

1



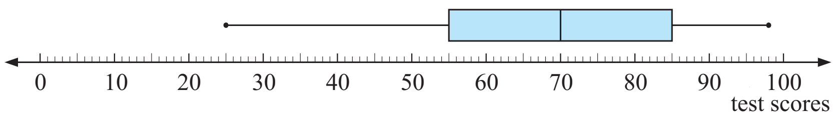
a The boxplot given summarises the goals scored by a basketball team. Locate:

i the median **ii** the maximum value **iii** the minimum value

iv the upper quartile **v** the lower quartile

b Calculate: **i** the range **ii** the interquartile range

2



The boxplot shown summarises the results of a test (out of 100 marks). Copy and complete the following statements about the test results:

a The highest mark scored for the test was ......

b The lowest mark scored for the test was ......

c Half of the class scored a mark greater than or equal to .....

d The top 25% of the class scored at least ..... marks for the test.

e The middle half of the class had scores between .... and .... for this test.

f Find the range of the data set.

g Find the interquartile range of the data set.

3 For the following data sets:

i construct a 5-number summary **ii** draw a boxplot
iii find the range **iv** find the interquartile range

a 3, 5, 5, 7, 10, 9, 4, 7, 8, 6, 6, 5, 8, 6

b 3, 7, 0, 1, 4, 6, 8, 8, 8, 9, 7, 5, 6, 8, 7, 8, 8, 2, 9

c

Stem	*Leaf*
11	7
12	0 3 6 6 8
13	0 1 1 1 3 5 5 7
14	4 7 7 9 9
15	1

11 | 7 represents 117

4 The weight, in grams, of a loaf of a particular brand of bread is stated to be 900 g, however some loaves weigh more than this and some weigh less. A sample of loaves is carefully weighed and their weights are given in the ordered stem-and-leaf plot shown.

Stem	*Leaf*
88	3 5 9
89	2 2 5 7 8 8 9 9 9
90	0 0 1 1 2 3 3 3 5 5 6 6 6
91	0 1 1 1 2 3 3 4 6 7 7 8 9
92	0 1 4 7

88 | 3 represents 883 g

a Locate the median, upper and lower quartiles and maximum and minimum weights for the sample.

b Draw a boxplot for the data.

c Find:

i the interquartile range **ii** the range.

d Copy and complete the following statements about the distribution of weights for the loaves of bread in this sample:

i Half of the loaves of bread weighed at least .... grams.

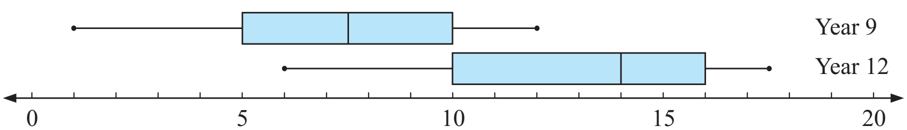
ii ....% of the loaves had a weight less than 900 grams.

iii The weights of the middle 50% of the loaves in this sample were spread over ...... grams.

iv The lightest 25% of the loaves had a weight of ..... grams or less.

e Is the distribution of weights in this sample symmetrical or positively or negatively skewed?

5 The following boxplots compare the time students in years 9 and 12 spend on homework over a one week period.



a Copy and complete:

Statistic	*Year 9*	*Year 12*
min. value		
Q_1		
median		
Q_3		
max. value		

b Determine the: **i** range **ii** interquartile range for each group.

c True or false:

i On average, Year 12 students spend about twice as much time on homework as Year 9 students.

ii Over 25% of Year 9 students spend less time on homework than all Year 12 students.

6 Julie examines a new variety of bean and does a count on the number of beans in 33 pods. Her results were:

5, 8, 10, 4, 2, 12, 6, 5, 7, 7, 5, 5, 5, 13, 9, 3, 4, 4, 7, 8, 9, 5, 5, 4, 3, 6, 6, 6, 6, 9, 8, 7, 6

a Find the median, lower quartile and upper quartile of the data set.

b Find the interquartile range of the data set.

c Draw a boxplot of the data set.

7 Andrew counts the number of bolts in several boxes and tabulates the data as shown below:

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

a Find the five-number summary for this data set.

b Find the **i** range **ii** IQR for this data set.

c Construct a boxplot for the data set.

PERCENTILES

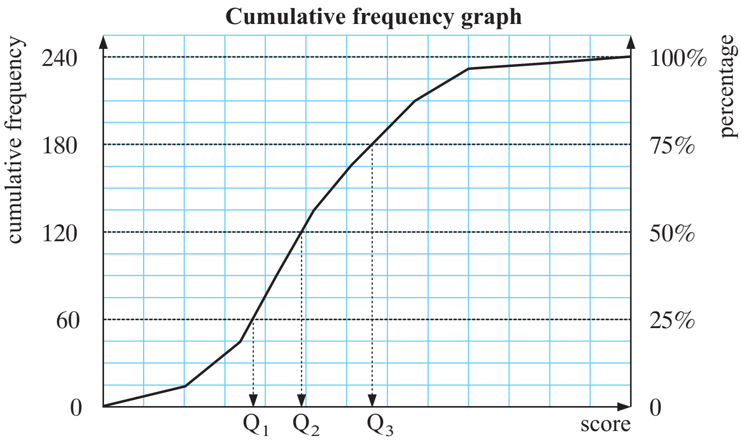
A **percentile** is the score, below which a certain percentage of the data lies.

For example, the 85th percentile is the score below which 85% of the data lies.

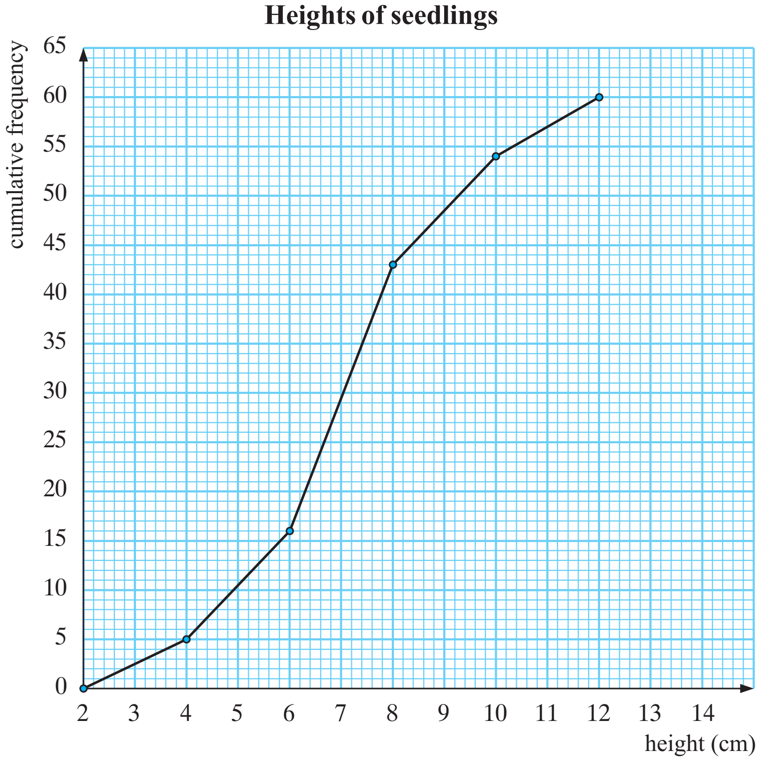
Notice that:
- the **lower quartile** ($\mathbf{Q_1}$) is the 25th percentile
- the **median** ($\mathbf{Q_2}$) is the 50th percentile
- the **upper quartile** ($\mathbf{Q_3}$) is the 75th percentile.

If your score in a test is the 95th percentile, then 95% of the class have scored less than you.

Note:



8 A botanist has measured the heights of 60 seedlings and has presented her findings on the cumulative frequency graph below.



a How many seedlings have heights of 5 cm or less?

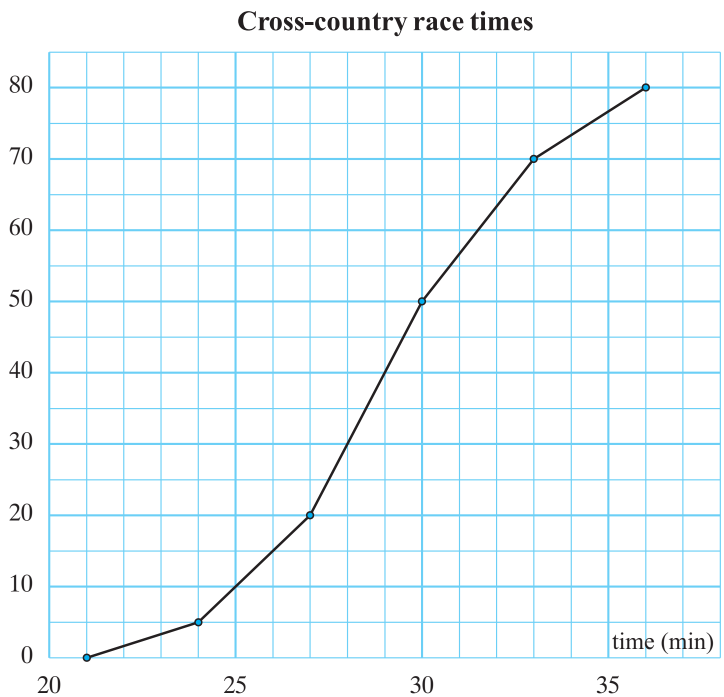
b What percentage of seedlings are taller than 8 cm?

c What is the median height?

d What is the interquartile range for the heights?

e Find the 90th percentile for the data and explain what your answer means.

9 The following cumulative frequency graph displays the performance of 80 competitors in a cross-country race.



Find:

a the lower quartile time

b the median

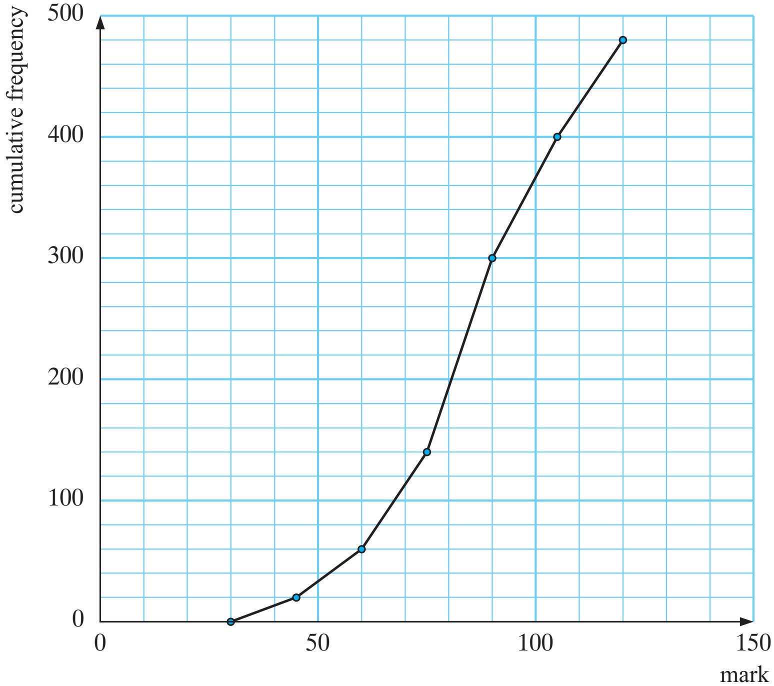
c the upper quartile

d the interquartile range

e an estimate of the 40th percentile.

10 The cumulative frequency graph below displays the marks scored by year 12 students from a cluster of schools in a common trial mathematics exam.

Trial mathematics exam



Find:

a how many students sat for the examination

b the probable maximum possible mark for the exam

c the median mark

d the interquartile range

e an estimate of the 85th percentile.

E STATISTICS USING TECHNOLOGY

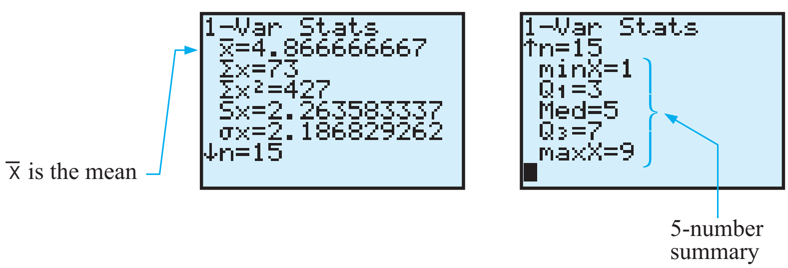
GRAPHICS CALCULATOR

A **graphics calculator** can be used to find descriptive statistics and to draw some types of graphs. (You will need to change the **viewing window** as appropriate.)

Consider the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5

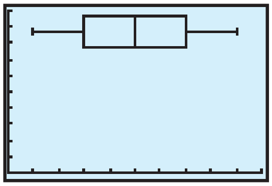
No matter what brand of calculator you use you should be able to:

- Enter the data as a **list**.
- Enter the **statistics calculation** part of the menu and obtain the descriptive statistics like these shown.

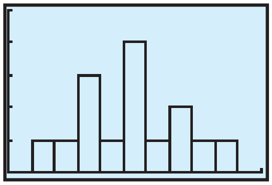


- Obtain a box-and-whisker plot such as:

  (These screen dumps are from a TI-83.)

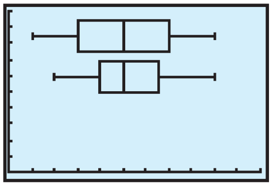


- Obtain a vertical barchart if required.



- Enter a second data set into another list and obtain a side-by-side boxplot for comparison with the first one.

  Use: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4



Now you should be able to create these by yourself.

EXERCISE 18E.1

1 **a** For your calculator enter the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5 and obtain the mean and the 5-number summary. This is the graphics calculator example shown on page **453** and you should check your results from it.

b Obtain the boxplot for part **a**.

c Obtain the vertical bar chart for part **a**.

d Enter this data set: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into a second list. Find the mean and 5-number summary. Now create a side-by-side boxplot for both sets of data.

STATISTICS FROM A COMPUTER PACKAGE

Various **statistical packages** are available for computer use. Many commercial packages are expensive and often not easy to use. Click on the icon to enter the **statistics package** on the CD.

Enter data set 1: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5

Enter data set 2: 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4

Examine the side-by-side column graphs.

Click on the Box-and-Whisker spot to view the side-by-side boxplots.

Click on the Statistics spot to obtain the descriptive statistics.

Click on Print to obtain a print-out of all of these on one sheet of paper.

EXERCISE 18E.2 (Computer package and/or graphics calculator)

1 Shane and Brett play in the same cricket team and are fierce but friendly rivals when it comes to bowling. During a season the number of wickets per innings taken by each bowler was recorded as:

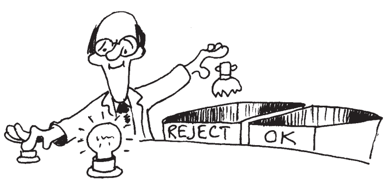
Shane:	1	6	2	0	3	4	1	4	2	3	0	3	2	4	3	4	3	3
	3	4	2	4	3	2	3	3	0	5	3	5	3	2	4	3	4	3
Brett:	7	2	4	8	1	3	4	2	3	0	5	3	5	2	3	1	2	0
	4	3	4	0	3	3	0	2	5	1	1	2	2	5	1	4	0	1

a Is the variable discrete or continuous?
b Enter the data into a graphics calculator or statistics package.
c Produce a vertical column graph for each data set.
d Are there any outliers? Should they be deleted before we start to analyse the data?
e Describe the shape of each distribution.
f Compare the measures of the centre of each distribution.
g Compare the spreads of each distribution.
h Obtain side-by-side boxplots.
i If using the statistics package, print out the graphs, boxplots and relevant statistics.
j What conclusions, if any, can be drawn from the data?

2 A manufacturer of light globes claims that the newly invented type has a life 20% longer than the current globe type. Forty of each globe type are randomly selected and tested. Here are the results to the nearest hour.

Old type:	103	96	113	111	126	100	122	110	84	117
	111	87	90	121	99	114	105	121	93	109
	87	127	117	131	115	116	82	130	113	95
	103	113	104	104	87	118	75	111	108	112
New type:	146	131	132	160	128	119	133	117	139	123
	191	117	132	107	141	136	146	142	123	144
	133	124	153	129	118	130	134	151	145	131
	109	129	109	131	145	125	164	125	133	135

a Is the variable discrete or continuous?
b Enter the data into a graphics calculator or statistics package.
c Are there any outliers? Should they be deleted before we start to analyse the data?
d Compare the measures of centre and spread.
e Obtain side-by-side boxplots.
f Use **e** to describe the shape of each distribution.
g What conclusions, if any, can be drawn from the data?



3 Enter these grouped continuous data sets:

Set 1:

Value	*Frequency*
21.6	1
21.7	3
21.8	9
21.9	28
22.0	18
22.1	9
22.2	7

Set 2:

Value	*Frequency*
21.5	1
21.6	8
21.7	25
21.8	31
21.9	13
22.0	8
22.1	5
22.2	3

Examine the graphs, boxplots and descriptive statistics for each and print the results. Write a brief comparative report.

F VARIANCE AND STANDARD DEVIATION

The problem with the range and the IQR as a measure of spread is that both only use two values in their calculation. Some data sets can have their characteristics hidden when the IQR is quoted. It would be helpful if we could have a measure of spread that used all of the data values in its calculation. One such statistic is the **variance** (s^2).

Variance measures the average of the squared deviations of each data value from the mean. The **deviation** of a data value x from the mean $\overline{x}$ is given by $x - \overline{x}$.

For a **sample**:

- the **variance** is $s^2 = \dfrac{\sum(x-\overline{x})^2}{n}$ where n is the sample size.
- the **standard deviation** is $s = \sqrt{\dfrac{\sum(x-\overline{x})^2}{n}}$.

For a **census**, i.e., we have the whole population:

We have all of the data that is available for a given population and we use a slightly different formula, and symbol. The **variance** is given the symbol σ^2 while the **standard deviation** is given the symbol σ. (σ is the Greek letter *sigma*.)

The formula for the **standard deviation** of a population is $\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$.

Note that in this case we use μ rather than $\overline{x}$ for the mean.

The population standard deviation σ, is generally unknown and the standard deviation of a sample s, is used as an estimation of σ.

Another point to note is that the standard deviation is a **non-resistant** measure of spread. This is due to its dependence on the mean of the data set and that extreme values will give large values for $(x-\mu)^2$ or $(x-\overline{x})^2$. It is only a useful measure if the distribution is approximately symmetrical. It does however have a powerful use when the data from which it came is **normally distributed**. This will be discussed later.

Clearly the IQR and percentiles are more appropriate tools for measuring spread if the distribution is considerably skewed.

Example 16

Find the standard deviations for the apple samples on page **445**.

Wholesaler Redapp

x	$x-\overline{x}$	$(x-\overline{x})^2$
5	-5	25
17	7	49
15	5	25
3	-7	49
9	-1	1
11	1	1
60	*Total*	150

$\therefore \quad \overline{x} = \frac{60}{6} = 10$

$$s = \sqrt{\frac{\sum(x-\overline{x})^2}{n}} = \sqrt{\frac{150}{6}} = 5$$

Wholesaler Pureapp

x	$x-\overline{x}$	$(x-\overline{x})^2$
10	-1.5	2.25
13	1.5	2.25
12	0.5	0.25
11	-0.5	0.25
12	0.5	0.25
11	-0.5	0.25
69	*Total*	5.5

$\therefore \quad \overline{x} = \frac{69}{6} = 11.5$

$$s = \sqrt{\frac{\sum(x-\overline{x})^2}{n}} = \sqrt{\frac{5.5}{6}} \doteqdot 0.957$$

Note: The sum of the deviations should always be 0.

Clearly, Wholesaler Purapp supplied apples with more blemishes on average but with less variability (smaller standard deviation) than for those supplied by Redapp.

EXERCISE 18F

1 Netballers Sally and Joanne compare their goal throwing scores for the last 8 matches.

Goals by Sally	23	17	31	25	25	19	28	32
Goals by Joanne	9	29	41	26	14	44	38	43

a Find the mean and standard deviation for the number of goals thrown by each goal shooter for these matches.

b Which measure is used to determine which of the goal shooters is more consistent?

2 Two test cricketers compare their bowling performances for the ten test matches for 2002. The number of wickets per match was recorded as:

Glen	0	10	1	9	11	0	8	5	6	7
Shane	4	3	4	1	4	11	7	6	12	5

a Show that each bowler has the same mean and range.

b Which performance do you suspect is more variable, Glen's bowling over the period or Shane's?

c Check your answer to **b** by finding the standard deviation for each distribution.

d Does the range or the standard deviation give a better indication of variability?

3 A manufacturer of softdrinks employs a statistician for quality control. Suppose that he needs to check that 375 mL of drink goes into each can. The machine which fills the cans may malfunction or slightly change its delivery due to constant vibration or other factors.

a Would you expect the standard deviation for the whole production run to be the same for one day as it is for one week? Explain.

b If samples of 125 cans are taken each day, what measure would be used to:

i check that 375 mL of drink goes into each can

ii check the variability of the volume of drink going into each can?

c What is the significance of a low standard deviation in this case?

4 The weights, in kg, of a sample of seven footballers are: 79, 64, 59, 71, 68, 68 and 74.

a Find the sample mean and standard deviation.

b Surprisingly, each footballer's weight had increased by exactly 10 kg when measured five years later. Find the new sample mean and standard deviation.

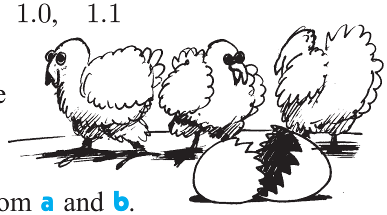
c Comment on your findings from **a** and **b** in general terms.

5 The weights of a sample of ten young turkeys to the nearest 0.1 kg are:

0.8, 1.1, 1.2, 0.9, 1.2, 1.2, 0.9, 0.7, 1.0, 1.1

a Find the sample mean and standard deviation.

b After being fed a special diet for one month, the weights of the turkeys doubled. Find the new sample mean and standard deviation.



c Comment, in general terms, on your findings from **a** and **b**.

6 For the sample: 3, 7, a, 4, b, the mean is 5 and the standard deviation is $\sqrt{2}$.

Find a and b given that $a > b$.

STANDARD DEVIATION FOR GROUPED DATA

For grouped data $s = \sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}}$ where s is the **standard deviation**
x is **any score**
$\bar{x}$ is the **mean**
f is the **frequency** of each score.

Example 17

Find the standard deviation of the distribution:

score	1	2	3	4	5
frequency	1	2	4	2	1

x	f	$f\,x$	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
1	1	1	-2	4	4
2	2	4	-1	1	2
3	4	12	0	0	0
4	2	8	1	1	2
5	1	5	2	4	4
Total	10	30			12

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{30}{10} = 3$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\tfrac{12}{10}} \doteqdot 1.10$$

Example 18

Find the standard deviation of the weights of 40 year 11 students whose weights were measured in kilograms to the nearest kg and then collated as:

Weight (kg)	*Frequency*
50 to 54	2
55 to 59	1
60 to 64	7
65 to 69	18
70 to 74	11
75 to 79	1

Weight class (kg)	*Centre of class (x)*	*Frequency*	$f\,x$	$f(x-\bar{x})^2$
50-54	52	2	104	435.1250
55-59	57	1	57	95.0625
60-64	62	7	434	157.9375
65-69	67	18	1206	1.1250
70-74	72	11	792	303.1875
75-79	77	1	77	105.0625
	Totals	40	2670	1097.5000

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \tfrac{2670}{40} = 66.75$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\tfrac{1097.5}{40}} \doteqdot 5.24 \quad \text{(to 3 s.f.)}$$

7 Find the standard deviation of the family size sample.

Number of children, x	0	1	2	3	4	5	6	7
Frequency, f	14	18	13	5	3	2	2	1

Use $s = \sqrt{\dfrac{\sum f_i(x_i-\bar{x})^2}{\sum f_i}}$.

8 Find the mean and standard deviation of the following distributions:

a

score	0	1	2	3	4	5	6	7	8
frequency	1	0	1	1	2	6	5	3	1

b

score	11	12	13	14	15	16	17	18
frequency	2	1	4	5	6	4	2	1

9 The number of toothpicks in 48 boxes was counted and the results tabulated.

Number of toothpicks	33	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	2

Find the mean and standard deviation of the distribution.

10 The lengths of 30 randomly selected 12-day old babies was measured to the nearest cm and the following data obtained:

Find estimates of the mean length and the standard deviation of the lengths.

Length (cm)	*Frequency*
40-41	1
42-43	1
44-45	3
46-47	7
48-49	11
50-51	5
52-53	2

11 The weekly wages (in dollars) of 200 workers in a steel yard are given below:

Find estimates of the mean wage and the standard deviation of the wages.

Wage (\$)	*Number of Workers*
360-369.99	17
370-379.99	38
380-389.99	47
390-399.99	57
400-409.99	18
410-419.99	10
420-429.99	10
430-439.99	3

Use technology to answer the following questions:

12 Find the mean and standard deviation of:

a 23, 24, 25, 26, 27, 28, 29, 30

b 7, 19, 5, 14, 13, 18, 21, 14, 11, 13, 15, 8

c 161, 156, 172, 183, 166, 184, 177, 162

d -3, -2, -1, 0, 1, 2, 3

13 A company recorded the following weekly petrol usage (in litres) by its salespersons:

62, 40, 52, 48, 64, 55, 44, 75, 40, 68, 60, 42, 70, 49, 56.

Find the mean and standard deviation of the petrol used.

14 Use your calculator to find the mean and standard deviation of the following sets of data.

a

score	*frequency*
31	2
32	5
33	11
34	18
35	9
36	1

b

score	*frequency*
17.1	2
17.2	8
17.3	11
17.4	23
17.5	6
17.6	1
17.8	1

c

class	*frequency*
0-9	2
10-19	16
20-29	23
30-39	83
40-49	41
50-59	11

In **c** do not forget to use the midpoint of the class.

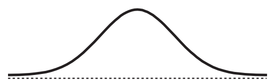
15 The cost of filling the fuel tank of a motor vehicle was studied and the results tabulated as shown.

Determine the mean and standard deviation of this distribution.

Cost (\$)	*Frequency*
0-4.99	21
5.00-9.99	34
10.00-14.99	85
15.00-19.99	76
20.00-24.99	94
25.00-29.99	43
30.00-34.99	22
35.00-39.99	14

G THE SIGNIFICANCE OF STANDARD DEVIATION

If a large sample from a typical bell-shaped data distribution is taken, what percentage of the data values would lie between $\bar{x} - s$ and $\bar{x} + s$?



bell-shaped distribution

Click on the icon and try to answer this question.
Repeat the sampling many times.

Now try to determine the percentage of data values which would lie between $\bar{x} - 2s$ and $\bar{x} + 2s$ and between $\bar{x} - 3s$ and $\bar{x} + 3s$.

It can be shown that for any measured variable from any population that is normally distributed, no matter the values of the mean and standard deviation:

- approximately **68%** of the population will have a measure that falls between **1** standard deviation either side of the mean
- approximately **95%** of the population will have a measure that falls between **2** standard deviations either side of the mean
- approximately **99.7%** of the population will have a measure that falls between **3** standard deviations either side of the mean.

Note: More accurate percentages will be given in **Chapter 29**.

Example 19

A sample of 200 cans of peaches was taken from a warehouse and the contents of each can measured for net weight. The sample mean was 486 g with standard deviation 6.2 g. What proportion of the cans might lie within:

a 1 standard deviation from the mean **b** 3 standard deviations from the mean?

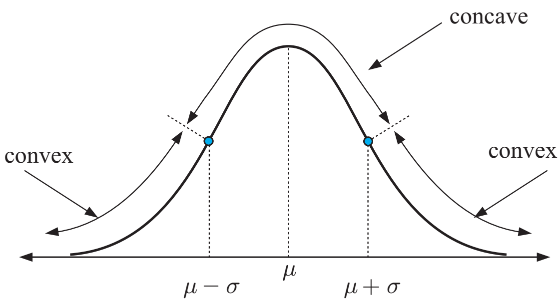
a About 68% of the cans would be expected to have contents between 486 ± 6.2 g i.e., 479.8 g and 492.2 g.

b Nearly all of the cans would be expected to have contents between $486 \pm 3 \times 6.2$ g i.e., 467.4 and 504.6 g.

THE NORMAL CURVE

The smooth curve that models normal population data is asymptotic to the horizontal axis, so in theory the measurement limits within which all the members of the population will fall, do not exist.

Note also that the position of 1 standard deviation either side of the mean corresponds to the point where the normal curve changes from a concave to a convex curve.



EXERCISE 18G

1 The mean height of players in a basketball competition is 184 cm. If the standard deviation is 5 cm, what percentage of them are likely to be:

a taller than 189 cm **b** taller than 179 cm
c between 174 cm and 199 cm **d** over 199 cm tall?

2 The mean average rainfall of Claudona for August is 48 mm with a standard deviation of 6 mm. Over a 20 year period, how many times would you expect there to be less than 42 mm of rainfall during August in Claudona?

3 Two hundred lifesavers competed in a swimming race. The mean time was 10 minutes 30 seconds. The standard deviation was 15 seconds. Find the number of competitors who probably:

a took longer than 11 minutes **b** took less than 10 minutes 15 seconds
c completed the race in a time between 10 min 15 sec and 10 min 45 sec.

4 The weights of babies born at Prince Louis Maternity Hospital last year averaged 3.0 kg with a standard deviation of 200 grams. If there were 545 babies born at this hospital last year, estimate the number that weighed:

a less than 3.2 kg **b** between 2.8 kg and 3.4 kg.

REVIEW SET 18A

1 For each of the following variables,

a country of residence **b** breathing rate **c** height

give: **i** a list of possible levels (categories) if there are only a few, or state that there are many or infinite levels
ii measurement units (where appropriate)
iii the variable type, as categorical, quantitative discrete or quantitative continuous.

2 The data supplied below is the diameter (in cm) of a number of bacteria colonies as measured by a microbiologist 12 hours after seeding.

0.4	2.1	3.4	3.9	4.7	3.7	0.8	3.6	4.1	4.9	2.5	3.1	1.5	2.6	4.0
1.3	3.5	0.9	1.5	4.2	3.5	2.1	3.0	1.7	3.6	2.8	3.7	2.8	3.2	3.3

a Produce a stemplot for this data.
b Find the **i** median **ii** range of the data.
c Comment on the skewness of the data.

3 The back-to-back stemplot alongside represents the times for the 100 metre freestyle recorded by members of a swimming squad.

Girls		*Boys*
	32	1
4	33	0 2 2 7
7 6 3	34	1 3 4 4 8
8 7 4 3 0	35	0 2 4 7 9 9
8 8 3 3	36	7 8 8
7 6 6 6	37	0
6	38	
0	39	
	40	
1	41	

leaf unit: 0.1 seconds

a Copy and complete the following table:

	Girls	*Boys*
shape		
centre (median)		
spread (range)		

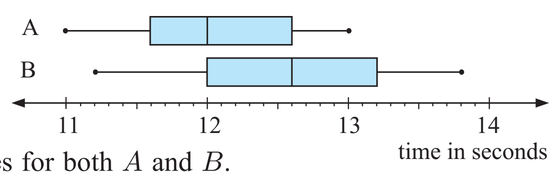
b Write an argument that supports the conclusion you may make about the girls' and boys' swimming times.

4 The data below shows the distance, in metres, Thabiso threw a cricket ball.

71.2	65.1	68.0	71.1	74.6	68.8	83.2	85.0	74.5	87.4
84.3	77.0	82.8	84.4	80.6	75.9	89.7	83.2	97.5	82.9
90.5	85.5	90.7	92.9	95.6	85.5	64.6	73.9	80.0	86.5

a Determine the highest and lowest value for the data set.
b Produce between 6 and 12 groups in which to place all the data values.
c Prepare a frequency distribution table.
d For this data, draw a frequency histogram.
e Determine: **i** the mean **ii** the median.

5 The given parallel boxplots represent the 100-metre sprint times for the members of two athletics squads.



a Determine the 5 number summaries for both A and B.
b Determine the **i** range **ii** interquartile range for each group.

c Copy and complete:
- **i** The members of squad ...... generally ran faster times.
- **ii** The times in squad ...... were more varied.

6 Katja's golf scores for her last 20 rounds were:

90	106	84	103	112	100	105	81	104	98
107	95	104	108	99	101	106	102	98	101

a Find the **i** median **ii** lower quartile **iii** upper quartile

b Find the interquartile range of the data set.

c Find the mean and standard deviation of her scores.

7 Explain why it is feasible that the distribution of each of the following variables is normal:

a the length of nails immediately after manufacture

b the life time of a particular brand of light globe.

8 The number of litres of petrol purchased by motor vehicle drivers was as shown alongside:

Find the mean and standard deviation of the number of litres purchased.

Litres	*Number of Vehicles*
15 – 19	5
20 – 24	13
25 – 29	17
30 – 34	29
35 – 39	27
40 – 44	18
45 – 49	7

9 For the following distribution of continuous grouped data:

Scores	0 to 9.9	10 to 19.9	20 to 29.9	30 to 39.9	40 to 49.9
Frequency	1	13	27	17	2

a Construct a cumulative frequency graph.

b Find the median of the data.

c Find the interquartile range.

d Find the mean and standard deviation.

10 6, x, 9, y, 3, and 11 have a mean of 7 and a standard deviation of $\frac{5}{\sqrt{3}}$.
Find x and y given that $x < y$.

REVIEW SET 18B

1 The winning margin in 100 basketball games was recorded. The results are given alongside:

Draw a histogram to represent this information.

Margin (points)	*Frequency*
1 - 10	13
11 - 20	35
21 - 30	27
31 - 40	18
41 - 50	7

2 Find the mean, mode and median of the following sets of data:

a 2, 5, 4, 0, 7, 3, 4, 2, 4, 6, 4, 3, 4 **b** 52, 63, 42, 58, 47, 61, 58, 63, 67, 49.

3 Find the mean and standard deviation of the number of matches per box when the numbers counted were as given alongside:

Number	47	48	49	50	51	52
Frequency	21	29	35	42	18	31

Does this result justify a claim that the average number of matches per box is 50?

4

No. of customers	*frequency*
250 - 299	14
300 - 349	34
350 - 399	68
400 - 449	72
450 - 499	54
500 - 549	23
550 - 599	7

The table alongside shows the number of customers visiting a supermarket on various days.

Find the mean number of customers per day.

5 Find the range, lower quartile, upper quartile and standard deviation for the following data: 120, 118, 132, 127, 135, 116, 122, 128.

6 Draw box and whisker plots for the following data:

11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 17, 17, 18.

7 The weekly supermarket bill for a number of families was calculated.

Find the mean bill and the standard deviation of the bills.

Bill (\$)	*No. of families*
70 - 79.99	27
80 - 89.99	32
90 - 99.99	48
100 - 109.99	25
110 - 119.99	37
120 - 129.99	21
130 - 139.99	18
140 - 149.99	7

8 The mean and standard deviation of a normal distribution are 150 and 12 respectively. What percentage of values lie between:

a 138 and 162 **b** 126 and 174 **c** 126 and 162 **d** 162 and 174?

9 Find the mean and standard deviation of the following data:

a

Score	*Frequency*
43	7
44	13
45	22
46	28
47	27
48	19
49	3

b

Score	*Frequency*
110 - 119	4
120 - 129	15
130 - 139	42
140 - 149	73
150 - 159	59
160 - 169	42
170 - 179	23

10 A bottle shop sells on average 2500 bottles per day with a standard deviation of 300 bottles. Assuming that the number of bottles is normally distributed, calculate the percentage of days when:

a less than 1900 bottles are sold

b more than 2200 bottles are sold

c between 2200 and 3100 bottles are sold.

11 3, a, 6, b, and 13 have a mean of 6.8 and a standard deviation of $\sqrt{12.56}$. Find a and b given that $a > b$.

Chapter 19

Probability

Contents:

A Experimental probability

Investigation 1: Tossing drawing pins

Investigation 2: Coin tossing experiments

Investigation 3: Dice rolling experiments

B Sample space

C Theoretical probability

D Using grids to find probabilities

E Compound events

Investigation 4: Probabilities of compound events

Investigation 5: Revisiting drawing pins

F Using tree diagrams

G Sampling with and without replacement

Investigation 6: Sampling simulation

Investigation 7: How many should I plant?

H Pascal's triangle revisited

I Sets and Venn diagrams

J Laws of probability

K Independent events revisited

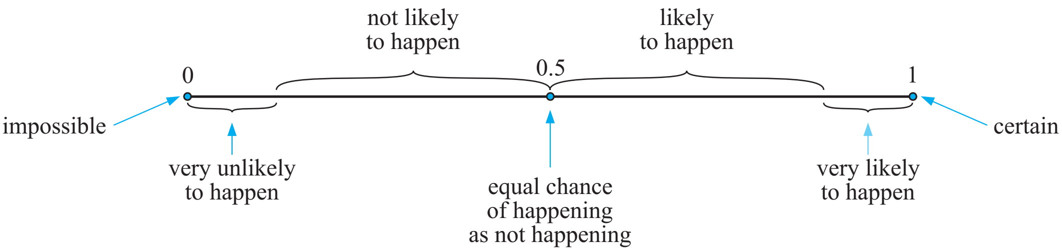
Review set 19A

Review set 19B

In the study of chance, we need a mathematical method to describe the likelihood of an event happening. We do this by carefully assigning a number which lies between 0 and 1 (inclusive).

> An event which has a 0% chance of happening (i.e., is impossible) is assigned a probability of 0.
>
> An event which has a 100% chance of happening (i.e., is certain) is assigned a probability of 1.
>
> All other events can then be assigned a probability between 0 and 1.

The number line below shows how we could interpret different probabilities:



The assigning of probabilities is usually based on either:

- observing the results of an experiment (experimental probability), or
- using arguments of symmetry (theoretical probability).

> **Probability theory** is the study of the *chance* (or likelihood) of events happening.

The study of the theory of chance has vitally important applications in physical and biological sciences, economics, politics, sport, life insurance, quality control, production planning in industry and a host of other areas.

HISTORICAL NOTE

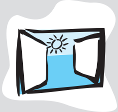
The development of modern probability theory began in 1653 when gambler Chevalier de Mere contacted mathematician **Blaise Pascal** with a problem on how to divide the stakes when a gambling game is interrupted during play. Pascal involved **Pierre de Fermat**, a lawyer and amateur mathematician, and together they solved the problem. While thinking about it they laid the foundations upon which the laws of probability were formed.


Blaise Pascal


Pierre de Fermat

In the late 17th century, English mathematicians compiled and analysed mortality tables. These tables showed how many people died at different ages. From these tables they could estimate the probability that a person would be alive at a future date. This led to the establishment of the first life-insurance company in 1699.

OPENING PROBLEM



Life Insurance Companies use statistics on **life expectancy** and **death rates** in order to work out the premiums to charge people who insure with them.

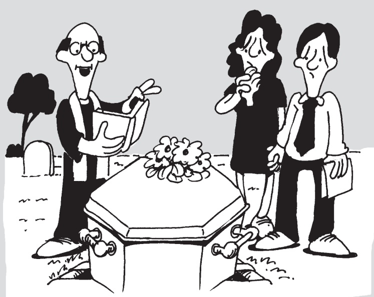
The table shows *expected numbers surviving at a given age* and the *expected remaining life at a given age*.

Life table					
Male			Female		
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	73.03	0	100 000	79.46
5	98 809	68.90	5	99 307	75.15
10	98 698	63.97	10	99 125	70.22
15	98 555	59.06	15	98 956	65.27
20	98 052	54.35	20	98 758	60.40
25	97 325	49.74	25	98 516	55.54
30	96 688	45.05	30	98 278	50.67
35	96 080	40.32	35	98 002	45.80
40	95 366	35.60	40	97 615	40.97
45	94 323	30.95	45	96 997	36.22
50	92 709	26.45	50	95 945	31.59
55	89 891	22.20	55	94 285	27.10
60	85 198	18.27	60	91 774	22.76
65	78 123	14.69	65	87 923	18.64
70	67 798	11.52	70	81 924	14.81
75	53 942	8.82	75	72 656	11.36
80	37 532	6.56	80	58 966	8.38
85	20 998	4.79	85	40 842	5.97
90	8416	3.49	90	21 404	4.12
95	2098	2.68	95	7004	3.00
99	482	2.23	99	1953	2.36

Notice that out of 100 000 births, 98 052 males are expected to survive to the age of 20 and at that age the survivors are expected to live a further 54.35 years.

Things to think about:

- Can you use the life table to estimate how many years you can expect to live?
- What is the estimated probability of a new born boy (or girl) reaching the age of 15?
- Can the table be used to estimate the probability that
  a 15 year old boy *will* reach the age of 75
  a 15 year old girl *will not* reach the age of 75?
- An insurance company sells policies to people to insure them against death over a 30-year period. If the person dies during this period the beneficiaries receive the agreed payout figure. Why are such policies cheaper to take out for say a 20 year old, than a 50 year old?
- How many of your classmates would you expect to be alive and able to attend a 30 year class reunion?



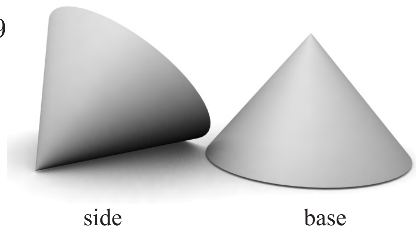
A EXPERIMENTAL PROBABILITY

In experiments involving chance we agree to use appropriate language to accurately describe what we are doing and the results we are obtaining.

- The number of **trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

When a small plastic cone was tossed into the air 279 times it fell on its *side* 183 times and on its *base* 96 times.

The relative frequencies of *side* and *base* are $\frac{183}{279} \doteqdot 0.656$ and $\frac{96}{279} \doteqdot 0.344$ respectively.



In the absence of any further data we say that the relative frequency of each event is our best estimate of the probability of each event occurring.

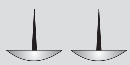
That is, Experimental probability = relative frequency.

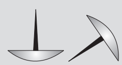
We write Experimental P(side) = 0.656, Experimental P(base) = 0.344

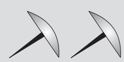
INVESTIGATION 1 TOSSING DRAWING PINS

If a drawing pin finishes  we say it has finished on its *back* and if  we say it has finished on its *side*.

If two drawing pins are tossed simultaneously the possible results are:


two backs


back and side


two sides

What to do:

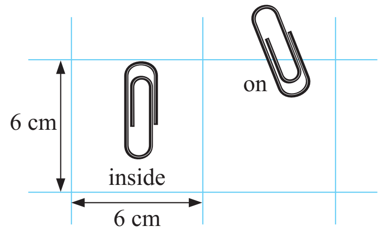
1. Obtain two drawing pins of the same shape and size. Toss the pair 80 times and record the outcomes in a table.
2. Obtain relative frequencies (experimental probabilities) for each of the three events.
3. Pool your results with four other people and so obtain experimental probabilities from 400 tosses. **Note:** The others must have pins from the same batch, i.e., the same shape.
4. Which gives the more reliable estimates, your results or the groups'? Why?
5. Keep your results as they may be useful later in this chapter.

Note: In some cases, such as in the investigation above, experimentation is the only way of obtaining probabilities.

EXERCISE 19A

1 When a batch of 145 paper clips were dropped onto 6 cm by 6 cm squared paper it was observed that 113 fell completely inside squares and 32 finished up on the grid lines. Find, to 2 decimal places, the estimated probability of a clip falling:

a inside a square **b** on a line.



2

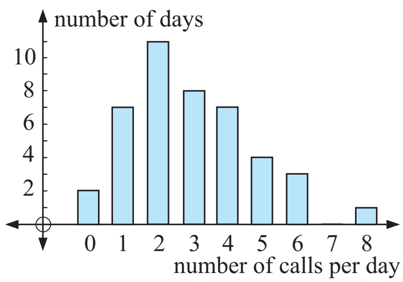
Length	*Frequency*
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

Jose surveyed the length of TV commercials (in seconds). Find to 3 decimal places the estimated probability that a randomly chosen TV commercial will last:

a 20 to 39 seconds **b** more than a minute
c between 20 and 59 seconds (inclusive)

3 Betul keeps records of the number of phone calls she receives over a period of consecutive days.

a For how many days did the survey last?
b Estimate Betul's chance of receiving:
  i no phone calls on one day
  ii 5 or more phone calls on a day
  iii less than 3 phone calls on a day.



4 Pat does a lot of travelling in her car and she keeps records on how often she fills her car with petrol. The table alongside shows the frequencies of the *number of days between refills*. Estimate the likelihood that:

a there is a four day gap between refills
b there is at least a four day gap between refills.

Days between refills	*Frequency*
1	37
2	81
3	48
4	17
5	6
6	1

INVESTIGATION 2 COIN TOSSING EXPERIMENTS

In this investigation we will find experimental results when tossing:

- one coin 40 times
- two coins 60 times
- three coins 80 times

The coins do not have to be all the same type.

What to do:

1 Toss *one coin* 40 times. Record the number of heads resulting in a table.

Result	*Tally*	*Frequency*	*Relative frequency*
1 head			
0 head			

2 Toss *two coins* 60 times. Record the number of heads resulting in a table.

Result	*Tally*	*Frequency*	*Relative frequency*
2 heads			
1 head			
0 head			

3 Toss *three coins* 80 times. Record the number of heads resulting in a table.

Result	*Tally*	*Frequency*	*Relative frequency*
3 heads			
2 heads			
1 head			
0 head			

4 Share your results to **1**, **2** and **3** with several others. Comment on any similarities and differences.

5 Pool your results and find new relative frequencies for tossing one coin, two coins, tossing three coins.

6 Click on the icon to examine a coin tossing simulation.

COIN TOSSING

In Number of coins type 1 .

In Number of flips type 10 000 .

Click START and click START 9 more times, each time recording the % Frequency for each possible result. Comment on these results. Do your results agree with what you expected?

7 Repeat **6** but this time with *two coins* and then repeat **6** but this time with *three coins*.

From the previous investigation you should have observed that there are roughly twice as many 'one head' results as there are 'no heads' or 'two heads'.

The explanation for this is best seen using two different coins where you could get:

two heads one head one head no heads

This shows that we should expect two heads : one head : no heads to be 1 : 2 : 1. However, due to chance, there will be variations from this when we look at experimental results.

INVESTIGATION 3 DICE ROLLING EXPERIMENTS

You will need: At least one normal six-sided die with numbers 1 to 6 on its faces. Several dice would be useful to speed up the experimentation.

WORKSHEET

What to do:

1 Examine a die. List the possible outcomes for the uppermost face when the die is rolled.

2 Consider the possible outcomes when the die is rolled 60 times.

Copy and complete the following table of your **expected results**:

Outcomes	*Expected frequency*	*Expected relative frequency*
⋮		

3 Roll the die 60 times and record the result on the uppermost face in a table like the one following:

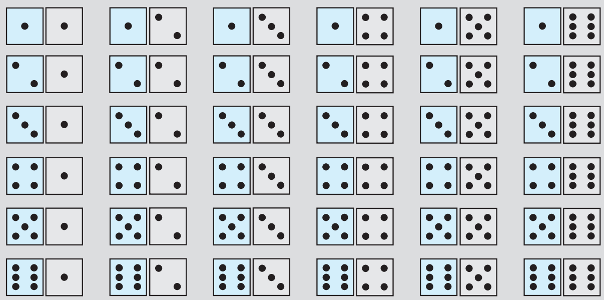
Outcome	*Tally*	*Frequency*	*Relative frequency*
1			
2			
⋮			
6			
	Total	60	

4 Pool as much data as you can with other students.
- Look at similarities and differences from one set to another.
- Look at the overall pooled data added into one table.

5 How close to your expectation were your results?

6 Use the die rolling simulation from the computer package on the CD to roll the die 10 000 times and repeat this 10 times. On each occasion, record your results in a table like that in **3**. Do your results further confirm your expected results?

7 These are the different possible results when a pair of dice is rolled.



The illustration given shows that when two dice are rolled there are 36 possible outcomes. Of these, $\{1, 3\}$, $\{2, 2\}$ and $\{3, 1\}$ give a sum of 4.

Using the illustration above, copy and complete the following table of expected (theoretical) results:

Sum	2	3	4	5	$\cdots$	12
Fraction of total			$\frac{3}{36}$			
Fraction as decimal			0.083			

8 If a pair of dice is rolled 360 times, how many of each result (2, 3, 4, ..., 12) would you expect to get? Extend your table of **7** by adding another row and write your **expected frequencies** within it.

9 Toss two dice 360 times and record in a table the *sum of the two numbers* for each toss.

Sum	*Tally*	*Frequency*	*Rel. Frequency*
2			
3			
4			
⋮			
12			
	Total	360	1

WORKSHEET

10 Pool as much data as you can with other students and find the overall relative frequency of each *sum*.

SIMULATION

11 Use the two dice simulation from the computer package on the CD to roll the pair of dice 10 000 times. Repeat this 10 times and on each occasion record your results in a table like that of **9**. Are your results consistent with your expectations?

B SAMPLE SPACE

A **sample space** is the set of all possible outcomes of an experiment.

There are a variety of ways of representing or illustrating sample spaces.

LISTING OUTCOMES

Example 1

List the sample space of possible outcomes for:

a tossing a coin

b rolling a die.

a When a coin is tossed, there are two possible outcomes.
$\therefore$ sample space = {H, T}

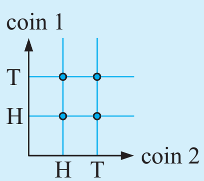
b When a die is rolled, there are 6 possible outcomes.
$\therefore$ sample space = {1, 2, 3, 4, 5, 6}

2-DIMENSIONAL GRIDS

When an experiment involves more than one operation we can still use listing to illustrate the sample space. However, a grid can often be a better way of achieving this.

Example 2

Illustrate the possible outcomes when 2 coins are tossed by using a 2-dimensional grid.



Each of the points on the grid represents one of the possible outcomes:
{HH, HT, TH, TT}

TREE DIAGRAMS

The sample space in **Example 2** could also be represented by a tree diagram. The advantage of tree diagrams is that they can be used when more than two operations are involved.

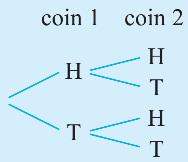
Example 3

Illustrate, using a tree diagram, the possible outcomes when:

a tossing two coins

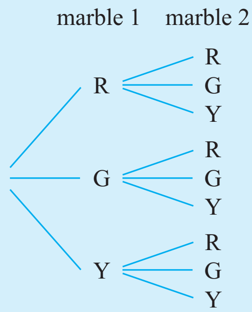
b drawing two marbles from a bag containing a number of red, green and yellow marbles.

a



Each "branch" gives a different outcome and the sample space can be seen to be {HH, HT, TH, TT}.

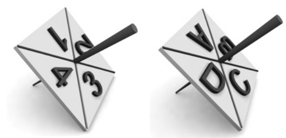
b



EXERCISE 19B

1 List the sample space for the following:

- **a** twirling a square spinner labelled A, B, C, D
- **b** the sexes of a 2-child family
- **c** the order in which 4 blocks A, B, C and D can be lined up
- **d** the 8 different 3-child families.



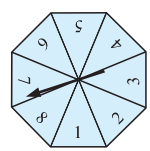
2 Illustrate on a 2-dimensional grid the sample space for:

- **a** rolling a die and tossing a coin simultaneously
- **b** rolling two dice
- **c** rolling a die and spinning a spinner with sides A, B, C, D
- **d** twirling two square spinners; one labelled A, B, C, D and the other 1, 2, 3, 4.

3 Illustrate on a tree diagram the sample space for:

- **a** tossing a 5-cent and 10-cent coin simultaneously
- **b** tossing a coin and twirling an equilateral triangular spinner labelled A, B and C
- **c** twirling two equilateral triangular spinners labelled 1, 2 and 3 and X, Y and Z
- **d** drawing two tickets from a hat containing a number of pink, blue and white tickets.

C THEORETICAL PROBABILITY



Consider the **octagonal spinner** alongside.

Since the spinner is symmetrical, when it is spun the arrowed marker could finish with **equal likelihood** on each of the sections marked 1 to 8.

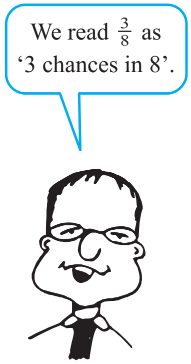
Therefore, we would say that the likelihood of obtaining a particular number, for example, 4, would be

$$1 \text{ chance in } 8, \quad \tfrac{1}{8}, \quad 12\tfrac{1}{2}\% \quad \text{or} \quad 0.125$$

This is a **mathematical** (or **theoretical**) probability and is based on what we theoretically expect to occur.

The **theoretical probability** of a particular event is a measure of the chance of that event occurring in any trial of the experiment.

If we are interested in the event of getting a result of *6 or more* from one spin of the octagonal spinner, there are three favourable results (6, 7 or 8) out of the eight possible results, and each of these is equally likely to occur.



So, the probability of a result of 6 or more is $\frac{3}{8}$,

i.e., $\text{P(6 or more)} = \frac{3}{8}$

In general, for an event E containing **equally likely** possible results:

$$\mathbf{P(E)} = \frac{\textbf{the number of members of the event E}}{\textbf{the total number of possible outcomes}}.$$

Example 4

A ticket is *randomly selected* from a basket containing 3 green, 4 yellow and 5 blue tickets. Determine the probability of getting:

a a green ticket
b a green or yellow ticket
c an orange ticket
d a green, yellow or blue ticket

The sample space is $\{G, G, G, Y, Y, Y, Y, B, B, B, B, B\}$

which has $3 + 4 + 5 = 12$ outcomes.

a P(green)
$= \frac{3}{12}$
$= \frac{1}{4}$

b P(a green or a yellow)
$= \frac{3+4}{12}$
$= \frac{7}{12}$

c P(orange)
$= \frac{0}{12}$
$= 0$

d P(green, yellow or blue)
$= \frac{3+4+5}{12}$
$= \frac{12}{12}$
$= 1$

In **Example 4** notice that in **c** an orange result cannot occur and the calculated probability is 0, which fits the fact that it has *no chance* of occurring.

Also notice in **d**, a green, yellow or blue result is certain to occur. It is 100% likely which is perfectly described using a 1.

The two events of *no chance of occurring* with probability 0 and
certain to occur with probability 1 are two extremes.

Consequently, for any event E, $\mathbf{0 \leqslant P(E) \leqslant 1}$.

COMPLEMENTARY EVENTS

Example 5

An ordinary 6-sided die is rolled once. Determine the chance of:

a getting a 6 **b** not getting a 6
c getting a 1 or 2 **d** not getting a 1 or 2

The sample space of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$

a P(6)
$= \frac{1}{6}$

b P(not getting a 6)
= P(1, 2, 3, 4 or 5)
$= \frac{5}{6}$

c P(1 or 2)
$= \frac{2}{6}$

d P(not getting a 1 or 2)
= P(3, 4, 5, or 6)
$= \frac{4}{6}$

In **Example 5**, did you notice that P(6) + P(not getting a 6) = 1 and that
P(1 or 2) + P(not getting a 1 or 2) = 1?

This is no surprise as *getting a* 6 and *not getting a* 6 are **complementary events** where one of them **must occur**.

NOTATION

If E is an event, then E′ is the **complementary event** of E.

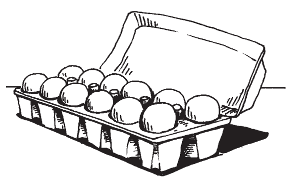
So, **P(E) + P(E′) = 1**

A useful rearrangement is: P(E **not** occurring) = 1 − P(E occurring)

EXERCISE 19C

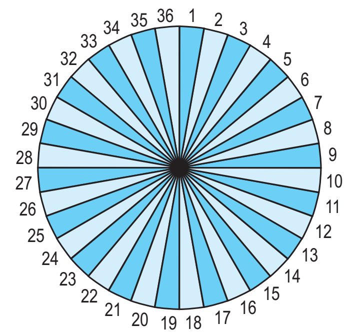
1 A marble is randomly selected from a box containing 5 green, 3 red and 7 blue marbles. Determine the probability that the marble is:
a red **b** green **c** blue
d not red **e** neither green nor blue **f** green or red

2 A carton of a dozen eggs contains eight brown eggs. The rest are white.
a How many white eggs are there in the carton?
b What is the probability that an egg selected at random is: **i** brown **ii** white?



3 In a class of 32 students, eight have one first name, nineteen have two first names and five have three first names. A student is selected at random. Determine the probability that the student has:
a no first name **b** one first name **c** two first names **d** three first names.

4 A dart board has 36 sectors, labelled 1 to 36. Determine the probability that a dart thrown at the board hits:



- **a** a multiple of 4
- **b** a number between 6 and 9 inclusive
- **c** a number greater than 20
- **d** 9
- **e** a multiple of 13
- **f** an odd number that is a multiple of 3.

5 What is the probability that a randomly chosen person has his/her next birthday on:
a a Tuesday **b** a week-end **c** in July **d** in January or February?

6 List the six different orders in which Antti, Kai and Neda may sit in a row. If the three of them sit randomly in a row, determine the probability that:

- **a** Antti sits in the middle
- **b** Antti sits at the left end
- **c** Antti sits at the right end
- **d** Kai and Neda are seated together

7 **a** List the 8 possible 3-child families, according to the gender of the children. For example, GGB means *"the first is a girl, the second is a girl, and the third is a boy"*.

b Assuming that each of these is equally likely to occur, determine the probability that a randomly selected 3-child family consists of:

- **i** all boys
- **ii** all girls
- **iii** boy, then girl, then girl
- **iv** two girls and a boy
- **v** a girl for the eldest
- **vi** at least one boy.

8 **a** List, in systematic order, the 24 different orders in which four people A, B, C and D may sit in a row.

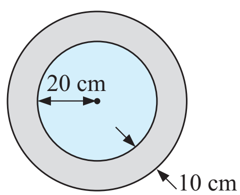
b Hence, determine the probability that when the four people sit at random in a row:

- **i** A sits on one end
- **ii** B sits on one of the two middle seats
- **iii** A and B are seated together
- **iv** A, B and C are seated together, not necessarily in that order.

9 List the possible outcomes when four coins are tossed simultaneously. Hence determine the probability of getting:

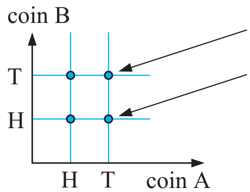
- **a** all heads
- **b** two heads and 2 tails
- **c** more tails than heads
- **d** at least one tail
- **e** exactly one head.

10 Abdul hits the target with one shot from his rifle. What is the probability that he hits the bulls-eye (centre circle)?



D USING GRIDS TO FIND PROBABILITIES

Two dimensional grids give us excellent visual displays of sample spaces. From these we can count favourable outcomes and so calculate probabilities.

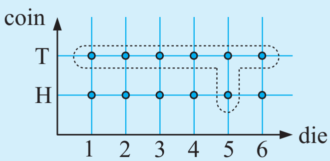


This point represents 'a tail from coin A' and 'a tail from coin B'.
This point represents 'a tail from coin A' and 'a head from coin B'.
There are four members of the sample space.

Example 6

Use a two-dimensional grid to illustrate the sample space for tossing a coin and rolling a die simultaneously. From this grid determine the probability of:

a tossing a head **b** getting a tail and a 5 **c** getting a tail or a 5



There are 12 members in the sample space.

a P(head) $= \frac{6}{12} = \frac{1}{2}$ **b** P(tail and a '5') $= \frac{1}{12}$

c P(tail or a '5') $= \frac{7}{12}$ {the enclosed points}

EXERCISE 19D

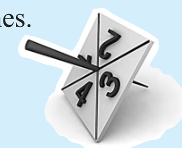
1 Draw the grid of the sample space when a 5-cent and a 10-cent coin are tossed simultaneously. Hence determine the probability of getting:

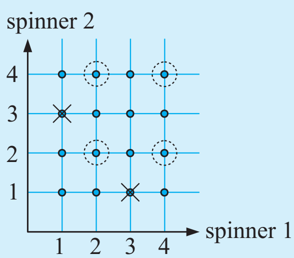
a two heads **b** two tails
c exactly one head **d** at least one head

Example 7

Two square spinners, each with 1, 2, 3 and 4 on their edges, are twirled simultaneously. Draw a two-dimensional grid of the possible outcomes.
Use your grid to determine the probability of getting:

a a 3 with each spinner **b** a 3 and a 1
c an even result for each spinner





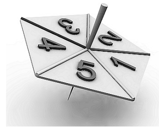
The sample space has 16 members.

a P(a 3 with each spinner) $= \frac{1}{16}$

b P(a 3 and a 1) $= \frac{2}{16}$ {crossed points}
$= \frac{1}{8}$

c P(an even result for each spinner)
$= \frac{4}{16}$ {circled points}
$= \frac{1}{4}$

2 A coin and a pentagonal spinner with sectors 1, 2, 3, 4 and 5 are rolled and spun respectively.



a Draw a grid to illustrate the sample space of possible outcomes.

b How many outcomes are possible?

c Use your grid to determine the chance of getting:

i a tail and a 3 **ii** a head and an even number

iii an odd number **iv** a head or a 5

3 A pair of dice is rolled. The 36 different possible 'pair of dice' results are illustrated below on a 2-dimensional grid.

Use the 2-dimensional grid of the 36 possible outcomes to determine the probability of getting:

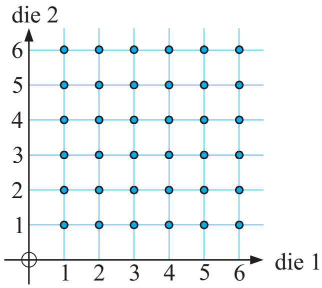
a two 3's **b** a 5 and a 6

c a 5 or a 6 **d** at least one 6

e exactly one 6 **f** no sixes

g a sum of 7 **h** a sum greater than 8

i a sum of 7 or 11 **j** a sum of no more than 8.



DISCUSSION

Read and discuss:
Three children have been experimenting with a coin, tossing it in the air and recording the outcomes. They have done this 10 times and have recorded 10 tails. Before the next toss they make the following statements:

Jackson: "It's got to be a head next time!"

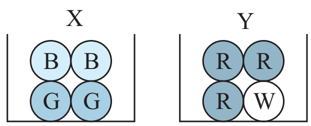
Sally: "No, it always has an equal chance of being a head or a tail. The coin cannot remember what the outcomes have been."

Amy: "Actually , I think it will probably be a tail again, because I think the coin must be biased - it might be weighted somehow so that it is more likely to give a tail."

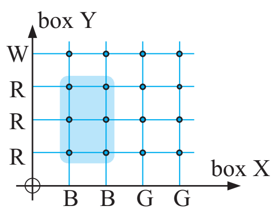
E COMPOUND EVENTS

Consider the following problem:

Box X contains 2 blue and 2 green balls and Box Y contains 3 red and 1 white ball. A ball is randomly selected from each of the boxes. Determine the probability of getting "a blue ball from X and a red ball from Y."



By illustrating the sample space on a two-dimensional grid as shown alongside, it can be seen that as 6 of the 16 possibilities are blue from X and red from Y and each outcome is equally likely,



P(blue from X **and** red from Y) $= \frac{6}{16}$

The question arises, "Is there a quicker, easier way to find this probability?"

INVESTIGATION 4 PROBABILITIES OF COMPOUND EVENTS

The purpose of this investigation is to find, if possible, a rule for finding P(A *and* B) for events A and B.

A coin is tossed and at the same time, a die is rolled. The result of the coin toss will be called outcome A, and likewise for the die, outcome B.

What to do:

a Copy and complete:

P(A *and* B)	P(A)	P(B)
P(a head and a 4)		
P(a head and an odd number)		
P(a tail and a number larger than 1)		
P(a tail and a number less than 3)		

b What is the connection between P(A and B) and P(A), P(B)?

INVESTIGATION 5 REVISITING DRAWING PINS

Since we cannot find by theoretical argument the probability that a drawing pin will land on its back  , the question arises for tossing two drawing pins, does

$$\text{P(back } \textit{and} \text{ back)} = \text{P(back)} \times \text{P(back)?}$$

What to do:

1 From **Investigation 1** on page **470**, what is your estimate of P(back *and* back)?

2 **a** Count the number of drawing pins in a full packet. They must be identical to each other and the same ones that you used in **Investigation 1**.

b Drop the whole packet onto a solid surface and count the number of *backs* and *sides*. Repeat this several times. Pool results with others and finally estimate P(back).

3 Find P(back) $\times$ P(back) using **2 b**.

4 Is P(back *and* back) $\doteqdot$ P(back) $\times$ P(back)?

From **Investigation 4 and 5**, it seems that:

If A and B are two events, where the occurrence of one of them does not affect the occurrence of the other, then

$$\text{P(A } \textbf{and} \text{ B)} = \text{P(A)} \times \text{P(B)}.$$

Before we can formulate a rule, we need to distinguish between **independent** and **dependent** events.

INDEPENDENT EVENTS

Independent events are events where the occurrence of one of the events **does not** affect the occurrence of the other event.

Consider again the example on the previous page. Suppose we happen to choose a blue ball from box X. This in no way affects the outcome when we choose a ball from box Y. The two events "a blue ball from X" and "a red ball from Y" are **independent events**.

In general: If A and B are **independent events** then P(A **and** B) = P(A) $\times$ P(B).

This rule can be extended for any number of independent events.

For example: If A, B and C are all **independent events**, then P(A **and** B **and** C) = P(A) $\times$ P(B) $\times$ P(C).

Example 8

A coin and a die are tossed simultaneously. Determine the probability of getting a head and a 3 without using a grid.

$$\begin{aligned} \text{P(head and 3)} &= \text{P(H)} \times \text{P(3)} \qquad \{\text{as events are clearly physically independent}\} \\ &= \tfrac{1}{2} \times \tfrac{1}{6} \\ &= \tfrac{1}{12} \end{aligned}$$

EXERCISE 19E.1

1 At a mountain village in New Guinea it rains on average 6 days a week. Determine the probability that it rains on:
- **a** any one day
- **b** two successive days
- **c** three successive days.

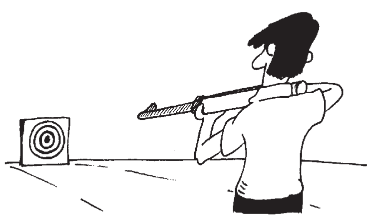
2 A coin is tossed 3 times. Determine the probability of getting the following sequences of results:
- **a** head, then head, then head
- **b** tail, then head, then tail

3 A school has two photocopiers. On any one day, machine A has an 8% chance of malfunctioning and machine B has a 12% chance of malfunctioning.

Determine the probability that on any one day both machines will:
- **a** malfunction
- **b** work effectively.

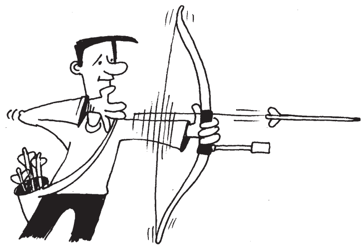
4 A couple decide that they want 4 children, none of whom will be adopted. They will be disappointed if the children are not born in the order boy, girl, boy, girl. Determine the probability that:
- **a** they will be happy with the order of arrival
- **b** they will be unhappy with the order of arrival.

5 Two marksmen fire at a target simultaneously. Jiri hits the target 70% of the time and Benita hits it 80% of the time. Determine the probability that:
- **a** they both hit the target
- **b** they both miss the target
- **c** Jiri hits it but Benita misses
- **d** Benita hits it but Jiri misses.



6 An archer always hits a circular target with each arrow shot, and hits the bullseye 2 out of every 5 shots on average. If 3 arrows are shot at the target, determine the probability that the bullseye is hit:

a every time
b the first two times, but not on the third shot
c on no occasion.



DEPENDENT EVENTS

Suppose a hat contains 5 red and 3 blue tickets. One ticket is randomly chosen, its colour is noted and it is thrown in a bin. A second ticket is randomly selected. What is the chance that it is red?

If the first ticket was red, P(second is red) $= \frac{4}{7}$ ← 4 reds remaining, ← 7 to choose from

If the first ticket was blue, P(second is red) $= \frac{5}{7}$ ← 5 reds remaining, ← 7 to choose from

So, the probability of the second ticket being red **depends** on what colour the first ticket was.

Here we have **dependent events**.

> Two or more events are **dependent** if they are **not independent**.
> **Dependent** events are events where the occurrence of one of the events *does affect* the occurrence of the other event.

For compound events which are dependent, a similar product rule applies as to that for independent events:

> If A and B are dependent events then
> P(A then B) = P(A) × P(B given that A has occurred).

Example 9

A box contains 4 red and 2 yellow tickets. Two tickets are randomly selected, one by one from the box, *without* replacement. Find the probability that:

a both are red **b** the first is red and the second is yellow.

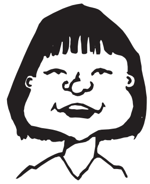
a P(both red)
= P(first selected is red *and* second is red)
= P(first selected is red) × P(second is red given that the first is red)
$= \frac{4}{6} \times \frac{3}{5}$ ← 3 reds remain out of a total of 5 after a red first draw
$= \frac{2}{5}$ ← 4 reds out of a total of 6 tickets

b P(first is red *and* second is yellow)
= P(first is red) × P(second is yellow given that the first is red)
$= \frac{4}{6} \times \frac{2}{5}$ ← 2 yellows remain out of a total of 5 after a red first draw
$= \frac{4}{15}$ ← 4 reds out of a total of 6 tickets

EXERCISE 19E.2

1 A box contains 7 red and 3 green balls. Two balls are randomly selected from the box without replacement. Determine the probability that:

- **a** both are red
- **b** the first is green and the second is red
- **c** a green and a red are obtained.



Example 10

A hat contains tickets with numbers 1, 2, 3, ..., 19, 20 printed on them. If 3 tickets are drawn from the hat, without replacement, determine the probability that all are prime numbers.

There are 20 numbers of which 8 are primes:

$\{2, 3, 5, 7, 11, 13, 17, 19\}$ are primes.

$\therefore$ P(3 primes)

$=$ P(1st drawn is prime *and* 2nd is prime *and* 3rd is prime)

$= \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18}$

8 primes out of 20 numbers

7 primes out of 19 numbers after a successful first draw

6 primes out of 18 numbers after two successful draws

$\doteqdot 0.049\,12$

2 A bin contains 12 identically shaped chocolates of which 8 are strawberry-creams. If 3 chocolates are selected at random from the bin, determine the probability that:

- **a** they are all strawberry-creams
- **b** none of them are strawberry-creams.

3 A lottery has 100 tickets which are placed in a barrel. Three tickets are drawn at random from the barrel to decide 3 prizes. If John has 3 tickets in the lottery, determine his probability of winning:

- **a** first prize
- **b** first and second prize
- **c** all 3 prizes
- **d** none of the prizes

4 A hat contains 7 names of players in a tennis squad including the captain and the vice captain. If a team of 3 is chosen at random by drawing the names from the hat, determine the probability that it:

- **a** contain the captain
- **b** contain the captain or the vice captain.

F USING TREE DIAGRAMS

Tree diagrams can be used to illustrate sample spaces provided that the alternatives are not too numerous. Once the sample space is illustrated, the tree diagram can be used for determining probabilities.

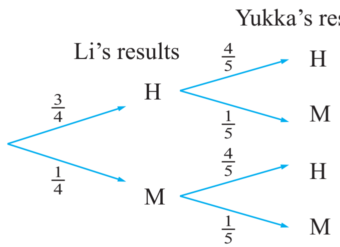
Consider two archers:

Li with probability $\frac{3}{4}$ of hitting a target and Yukka with probability $\frac{4}{5}$.

They both shoot simultaneously.

The tree diagram for this information is:

H = hit
M = miss



outcome	probability
H and H	$\frac{3}{4} \times \frac{4}{5} = \frac{12}{20}$
H and M	$\frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$
M and H	$\frac{1}{4} \times \frac{4}{5} = \frac{4}{20}$
M and M	$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$
	total 1

Notice that:

- The probabilities for hitting and missing are marked on the branches.
- There are *four* alternative paths and each branch shows a particular outcome.
- All outcomes are represented and the probabilities are obtained by **multiplying**.

Example 11

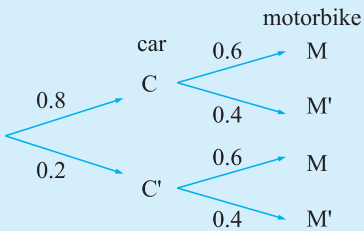
Carl is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time.

a Draw a tree diagram to illustrate this situation.

b Use the tree diagram to determine the chance that:

i both will start **ii** Carl has no choice but to use his car.

a C = car starts M = motorbike starts



outcome	probability
C and M	$0.8 \times 0.6 = 0.48$
C and M'	$0.8 \times 0.4 = 0.32$
C' and M	$0.2 \times 0.6 = 0.12$
C' and M'	$0.2 \times 0.4 = 0.08$
	total 1.00

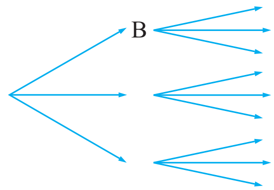
b **i** P(both start)
$= \text{P(C and M)}$
$= 0.8 \times 0.6$
$= 0.48$

ii P(car starts, but motorbike does not)
$= \text{P(C and M}')$
$= 0.8 \times 0.4$
$= 0.32$

EXERCISE 19F

1 Suppose this spinner is spun twice.

a Copy and complete the branches on the tree diagram shown.



b What is the probability that black appears on both spins?

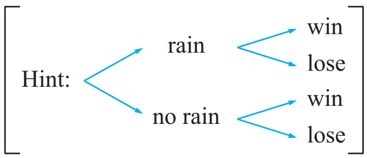
c What is the probability that yellow appears on both spins?

d What is the probability that different colours appear on both spins?

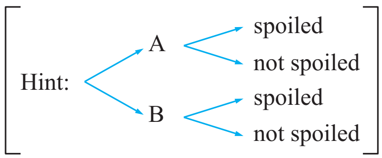
e What is the probability that black appears on either spin?

2 The probability of rain tomorrow is estimated to be $\frac{1}{5}$.

If it does rain, Mudlark will start favourite with probability $\frac{1}{2}$ of winning. If it is fine he has a 1 in 20 chance of winning. Display the sample space of possible results of the horse race on a tree diagram. Determine the probability that Mudlark will win tomorrow.

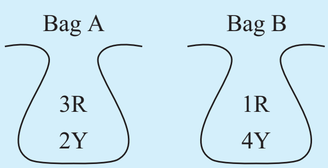


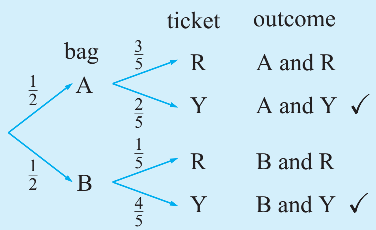
3 Machine A makes 40% of the bottles produced at a factory. Machine B makes the rest. Machine A spoils 5% of its product, while Machine B spoils only 2%. Determine the probability that the next bottle inspected at this factory is spoiled.



Example 12

Bag A contains 3 red and 2 yellow tickets. Bag B contains 1 red and 4 yellow tickets. A bag is randomly selected by tossing a coin and one ticket is removed from it. Determine the probability that it is yellow.

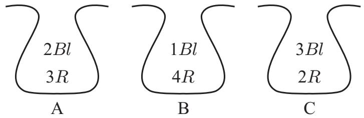




$$
\begin{aligned}
\text{P(yellow)} &= \text{P(A and Y)} + \text{P(B and Y)} \\
&= \tfrac{1}{2} \times \tfrac{2}{5} + \tfrac{1}{2} \times \tfrac{4}{5} \qquad \{\text{branches marked with a } \checkmark\} \\
&= \tfrac{6}{10} \\
&= \tfrac{3}{5}
\end{aligned}
$$

4 Jar A contains 2 white and 3 red discs and Jar B contains 3 white and 1 red disc. A jar is chosen at random (by the flip of a coin) and one disc is taken at random from it. Determine the probability that the disc is red.

5 Three bags contain different numbers of blue and red marbles. A bag is selected using a die which has three A faces and two B faces and one C face.



One marble is then selected randomly from the bag. Determine the probability that it is:

a blue **b** red.

G SAMPLING WITH AND WITHOUT REPLACEMENT

SAMPLING

Sampling is the process of selecting an object from a large group of objects and inspecting it, noting some feature(s). The object is then either **put back** (sampling **with replacement**) or **put to one side** (sampling **without replacement**).

Sometimes the inspection process makes it impossible to return the object to the large group.

Such processes include:

- Is the chocolate hard- or soft-centred? Bite it or squeeze it to see.
- Does the egg contain one or two yolks? Break it open and see.
- Is the object correctly made? Pull it apart to see.

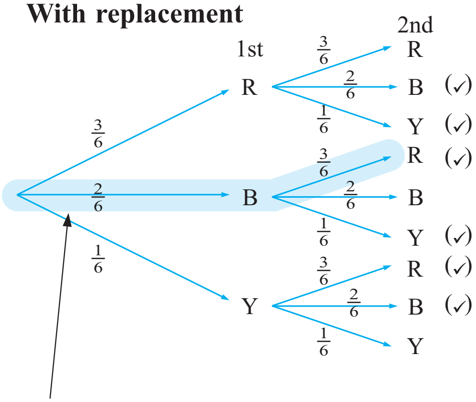
This sampling process is used to maintain Quality Control in industrial processes.

Consider a box containing 3 red, 2 blue and 1 yellow marble. Suppose we wish to sample two marbles:

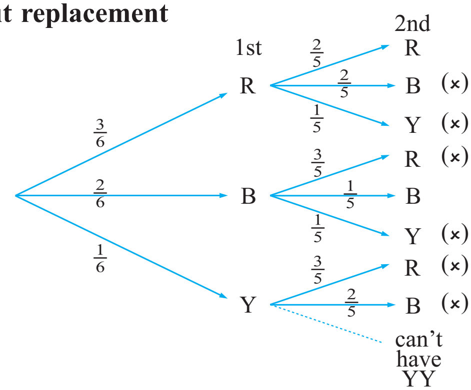
- by **replacement** of the first before the second is drawn
- by **not replacing** the first before the second is drawn.

Examine how the tree diagrams differ:

With replacement



Without replacement



This branch represents Blue with the 1st draw and Red with the second draw and this is written BR.

Notice that:

- with replacement
  P(two reds) $= \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$
- without replacement
  P(two reds) $= \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$

Example 13

For the example of the box containing 3 red, 2 blue and 1 yellow marble find the probability of getting two different colours:

a if replacement occurs **b** if replacement does not occur.

a P (two different colours)

$= $ P (RB or RY or BR or BY or YR or YB) {ticked ones}

$= \frac{3}{6} \times \frac{2}{6} + \frac{3}{6} \times \frac{1}{6} + \frac{2}{6} \times \frac{3}{6} + \frac{2}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{3}{6} + \frac{1}{6} \times \frac{2}{6}$

$= \frac{22}{36}$ which is $\frac{11}{18}$

b P (two different colours)

$= $ P(RB or RY or BR or BY or YR or YB) {crossed ones}

$= \frac{3}{6} \times \frac{2}{5} + \frac{3}{6} \times \frac{1}{5} + \frac{2}{6} \times \frac{3}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5} + \frac{1}{6} \times \frac{2}{5}$

$= \frac{22}{30}$ which is $\frac{11}{15}$

EXERCISE 19G

Use a tree diagram to help answer the following:

1 Two marbles are drawn in succession from a box containing 2 purple and 5 green marbles. Determine the probability that the two marbles are different colours if:

a the first is replaced **b** the first is *not* replaced.

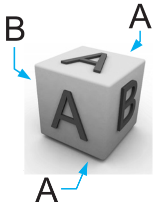
2 5 tickets numbered 1, 2, 3, 4 and 5, are placed in a bag. Two are taken from the bag without replacement. Determine the probability of getting:

a both odd **b** both even **c** one odd and the other even.

3 Jar A contains 3 red and 2 green tickets. Jar B contains 3 red and 7 green tickets. A die has 4 faces with A's and 2 faces with B's, and when rolled it is used to select either jar A or jar B.

When a jar has been selected, two tickets are randomly selected without replacement from it. Determine the probability that:

a both are green **b** they are different in colour.



4 Marie has a bag of sweets which are all identical in shape. The bag contains 6 orange drops and 4 lemon drops. She selects one sweet at random, eats it and then takes another, also at random. Determine the probability that:

a both sweets were orange drops

b both sweets were lemon drops

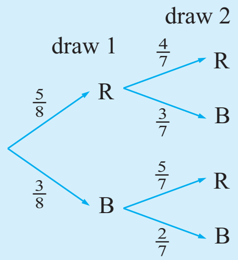
c the first was an orange drop and the second was a lemon drop

d the first was a lemon drop and the second was an orange drop

Add your answers to **a**, **b**, **c** and **d**. Explain why the answer must be 1.

Example 14

A bag contains 5 red and 3 blue marbles. Two marbles are drawn simultaneously from the bag. Determine the probability that at least one is red.



$$\begin{aligned}\text{P(at least one red)} &= \text{P(RR or RB or BR)}\\ &= \tfrac{5}{8}\times\tfrac{4}{7} + \tfrac{5}{8}\times\tfrac{3}{7} + \tfrac{3}{8}\times\tfrac{5}{7}\\ &= \frac{20+15+15}{56}\\ &= \frac{50}{56}\\ &= \frac{25}{28}\end{aligned}$$

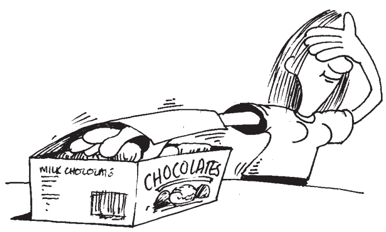
Note: *Alternatively,* P(at least one red) $= 1 -$ P(no reds) {complementary events}
$= 1 -$ P(BB), etc

5 A cook selects an egg at random from a carton containing 6 ordinary eggs and 3 double-yolk eggs. She cracks the egg into a bowl and sees whether it has two yolks or not. She then selects another egg at random from the carton and checks it.

Let S represent "a single yolk egg" and D represent "a double yolk egg".

a Draw a tree diagram to illustrate this sampling process.
b What is the probability that both eggs had two yolks?
c What is the probability that both eggs had only one yolk?

6



Petra selects a chocolate at random from a box containing 10 hard-centred and 15 soft-centred chocolates. She bites it to see whether it is hard-centred or not. She then selects another chocolate at random from the box and checks it. (She eats both chocolates.)

Let H represent "a hard-centred chocolate" and S represent "a soft-centred chocolate".

a Draw a tree diagram to illustrate this sampling process.
b What is the probability that both chocolates had hard centres?
c What is the probability that both chocolates had soft centres?

7 A bag contains four red and two blue marbles. Three marbles are selected simultaneously. Determine the probablity that:

a all are red **b** only two are red **c** at least two are red.

8 Bag A contains 3 red and 2 white marbles. Bag B contains 4 red and 3 white marbles. One marble is randomly selected from A and its colour noted. If it is red, 2 reds are added to B. If it is white, 2 whites are added to B. A marble is then selected from B.

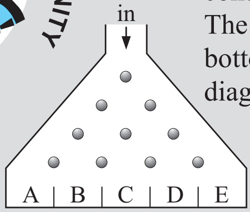
What are the chances that the marble selected from B is white?

9 A man holds two tickets in a 100-ticket lottery in which there are two winning tickets. If no replacement occurs, determine the probability that he will win:

a both prizes **b** neither prize **c** at least one prize.

INVESTIGATION 6 SAMPLING SIMULATION

PORTFOLIO OPPORTUNITY



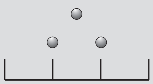
When balls enter the 'sorting' chamber they hit a metal rod and may go left or right with *equal chance*. This movement continues as the balls fall from one level of rods to the next. The balls finally come to rest in collection chambers at the bottom of the sorter. This sorter looks very much like a tree diagram rotated through 90^o.

Click on the icon to open the simulation. Notice that we can use the sliding bar to alter the probabilities of balls going to the left or right at each rod.

What to do:

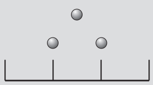
1 To simulate the results of tossing *two coins*, set the bar to 50%

SIMULATION

and the sorter to show 

Run the simulation 200 times and repeat this four more times. Record each set of results.

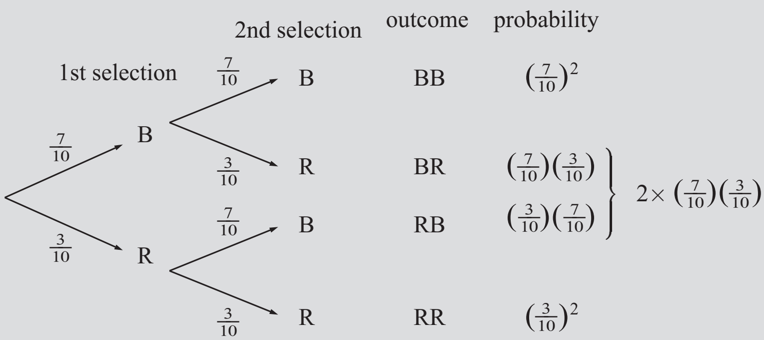
2 A bag contains 7 blue and 3 red marbles and *two marbles* are randomly selected from it, the first being *replaced* before the second is drawn.

The sorter should show  and set the bar to 70%

as $\text{P(blue)} = \frac{7}{10} = 0.7 = 70\%$.

Run the simulation a large number of times and use the results to estimate the probabilities of getting: **a** two blues **b** one blue **c** no blues.

3 The tree diagram representation of the marble selection in **2** is:

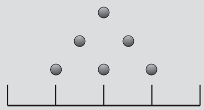


a The tree diagram gives us expected, theoretical probabilities for the different outcomes. Do they agree with the experimental results obtained in **2**?

b Write down the algebraic expansion of $(a+b)^2$.

c Substitute $a = \frac{7}{10}$ and $b = \frac{3}{10}$ in the $(a+b)^2$ expansion. What do you notice?

4 From the bag of 7 blue and 3 red marbles, *three* marbles are randomly selected *with replacement*.

Set the sorter to  and the bar to 70%.

Run the simulation a large number of times to obtain experimental estimates of the probabilities of getting:

a three blues **b** two blues **c** one blue **d** no blues.

5 **a** Use a tree diagram showing 1st selection, 2nd selection and 3rd selection to find theoretical probabilities of getting the results of **4**.

b Show that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and use this expansion with $a = \frac{7}{10}$ and $b = \frac{3}{10}$ to also check the results of **4** and **5 a**.

INVESTIGATION 7 HOW MANY SHOULD I PLANT?

You own a seedling business and supply garden centres with punnets containing six tomato seedlings. You need to plant each punnet with seeds where each seed has an 85% chance of germinating and growing to a saleable size. How many seeds should you plant in the punnet (box) to be 95% sure of getting at least six healthy seedlings? The garden centres will not purchase punnets from you with less than six healthy seedlings.

What to do:

1 Click on the simulation icon. Set the number of rows of rods in the sorting chamber to 6, and the number of balls to 5000. Switch ball speed **off**. Move the sliding bar to 85%.

2 Click $\boxed{\text{START}}$. A typical result is:

0	2	32	211	894	2006	1855
0.0%	0.0%	0.6%	4.2%	17.9%	40.1%	37.1%
↑	↑	↑	↑	↑	↑	↑
no seedlings	1 seedling	2 seedlings	3 seedlings	4 seedlings	5 seedlings	6 seedlings

Clearly, *at least six* seedlings is only six seedlings in this case and the estimated chance is only 37.1%.

3 Now set the number of rows of rods to 7. A typical result is:

0	0	5	65	339	1021	1972	1598
0.0%	0.0%	0.1%	1.3%	68%	20.4%	39.4%	32.0%
						6 or 7 seedlings	

Notice that for 7 seeds, *at least six* grow successfully in $39.4\% + 32.0\% = 71.4\%$ of the occasions.

4 Now try 8 rows (corresponding to 8 seeds). Will this show P(6 or more growing successfully) $\geqslant 0.95$? If not try 9 rows, etc.

H PASCAL'S TRIANGLE REVISITED

COMBINATION COUNTING

When tossing two coins the possible outcomes are:

HH HT TH TT

1 outcome is 'two heads'
2 outcomes are 'one head'
1 outcome is 'no heads'

When tossing four coins the possible outcomes are:

HHHH	THHH	HHTT	HTTT	TTTT
	HTHH	HTHT	THTT	
	HHTH	HTTH	TTHT	
	HHHT	TTHH	TTTH	
		THTH		
		THHT		

1 outcome is '4 heads'
4 outcomes are '3 heads'
6 outcomes are '2 heads'
4 outcomes are '1 head'
1 outcome is '0 heads'

Because we would not like to list and then count them, the question arises:

"When tossing 25 coins how many possible outcomes will consist of:
'25 heads', '24 heads', '23 heads', ... '0 heads'?"

Fortunately the combinations key, $\boxed{C^n_r}$ of a calculator provides us with a quick way of counting these possibilities.

For example, when tossing 4 coins, the total number of possible outcomes consisting of '2 heads' is $C^4_2 = 6$. This result is confirmed above.

To find C^4_2 press: 4 $\boxed{C^n_r}$ 2 $\boxed{=}$.

In general, $\boxed{C^n_r}$ is the number of possible ways of getting r successes ('heads' in this case) and $n - r$ failures ('tails' in this case) from n repetitions.

The number of different ways of getting '6 heads' (and 19 'tails') when tossing 25 coins is C^{25}_6, and to find it we press 25 $\boxed{C^n_r}$ 6 $\boxed{=}$. So there are 177 100 possible ways of getting '6 heads' (and 19 'tails') when tossing 25 coins.
Fortunately we do not have to list them to get this result.

The following triangles of numbers, called **Pascal's triangle**, display C^n_r values.

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

is really

$$\begin{array}{ccccccccccccc}
 & & & & & C^1_0 & & C^1_1 & & & & & \\
 & & & & C^2_0 & & C^2_1 & & C^2_2 & & & & \\
 & & & C^3_0 & & C^3_1 & & C^3_2 & & C^3_3 & & & \\
 & & C^4_0 & & C^4_1 & & C^4_2 & & C^4_3 & & C^4_4 & & \\
 & C^5_0 & & C^5_1 & & C^5_2 & & C^5_3 & & C^5_4 & & C^5_5 & \\
C^6_0 & & C^6_1 & & C^6_2 & & C^6_3 & & C^6_4 & & C^6_5 & & C^6_6
\end{array}$$

EXERCISE 19H

1 **a** Find the rule for getting the next row of Pascal's triangle from the previous row.
b Predict the seventh row of Pascal's triangle.
c Use your calculator to check the seventh row by finding C_0^7, C_1^7, C_2^7, ..., C_7^7.

2 Draw Pascal's Triangle down to row 8. Use the triangle to find the number of ways of getting:
a 2 successes in 3 tosses **b** 5 successes in 7 tosses **c** 2 successes in 5 tosses
d 1 success in 8 tosses **e** 0 successes in 4 tosses **f** 6 successes in 6 tosses
Verify each using your calculator. (A success is getting a 'head'.)

3 Use a calculator to find the number of ways of getting:
a 5 successes in 9 trials **b** 3 successes in 14 trials
c 1 success in 40 trials **d** 2 successes in 3 trials
e 10 successes in 40 trials **f** 20 successes in 40 trials

4 **a** When tossing five coins, what are the possible outcomes? List all 32 of them:
b In **a**, how many outcomes consist of:
i '5 heads' **ii** '4 heads' **iii** '3 heads'
iv '2 heads' **v** '1 head' **vi** '0 heads'?

5 **a** When tossing 18 coins, how many different ways can you get 10 heads and 8 tails?
b Over her lifetime, Jessie the cow had 23 calves; 14 male and 9 female. In how many different ways could she have had her calves according to their sex? (**Note:** MMFMFFFMFMMMFMFMMFMFMMM is one such way.)

6 $(p+q)^2 = p^2 + 2pq + q^2$.
a Find $(p+q)^3$ using $(p+q)(p+q)^2$. List the coefficients of each term.
b Likewise find the expansion of $(p+q)^4$ and list its coefficients.
c Without using algebraic expansion, predict the expansions of:
i $(p+q)^5$ **ii** $(p+q)^6$
d If $p+q=1$, what is the value of $C_0^4\,p^4 + C_1^4\,p^3q + C_2^4\,p^2q^2 + C_3^4\,pq^3 + C_4^4\,q^4$?

7 **a** Expand $(p+q)^4$.
b If a coin is tossed *four* times, what is the probability of getting 3 heads?

8 **a** Expand $(p+q)^5$.
b If *five* coins are tossed simultaneously, what is the probability of getting:
i 4 heads and 1 tail **ii** 2 heads and 3 tails?

9 **a** Expand $(\frac{2}{3} + \frac{1}{3})^4$.
b Four chocolates are randomly taken (with replacement) from a box containing strawberry creams and almond centres in the ratio $2:1$. What is the probability of getting:
i all strawberry creams **ii** two of each type
iii at least 2 strawberry creams?

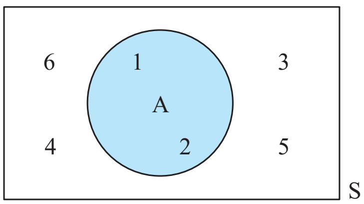
10 **a** Expand $(\frac{3}{4}+\frac{1}{4})^5$.

b In New Zealand in 1946, coins of value two shillings were of two types: normal kiwis and 'flat back' kiwis in the ratio 3 : 1. From a batch of 1946 two shilling coins, five were selected at random with replacement. What is the probability that:

i two were 'flat backs'   **ii** at least 3 were 'flat backs'?

I SETS AND VENN DIAGRAMS

A **Venn Diagram** usually consists of a rectangle which represents the sample space, and circles within it representing particular events.

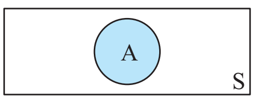


Venn diagrams are particularly useful for solving certain types of probability questions.

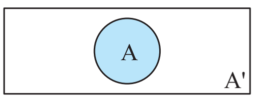
A number of probability laws can also be established using Venn diagrams.

This Venn diagram shows the event A = $\{1, 2\}$ when rolling a die. The sample space S = $\{1, 2, 3, 4, 5, 6\}$.

SET NOTATION

- 

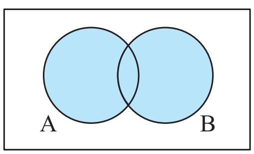
  S, the **sample space** is represented by a rectangle and A, an **event**, is represented by a circle.

- 

  A$'$ is the **complement** of A.
  It represents the non-occurrence of A.
  Note: P(A) + P(A$'$) = 1

  If S = $\{1, 2, 3, 4, 5, 6, 7\}$ and A = $\{2, 4, 6\}$ then A$'$ = $\{1, 3, 5, 7\}$

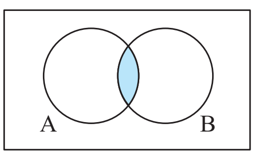
- $x \in$ A reads 'x is in A' i.e., x is an element of set A.
- n(A) reads 'the number of elements in set A'.
- A $\cup$ B denotes the **union** of sets A and B. This set contains all elements belonging to A **or** B **or both** A and B.



A $\cup$ B is shaded.

Note: A $\cup$ B = $\{x\text{:}\ \ x \in \text{A}$ **or** $x \in \text{B}\}$

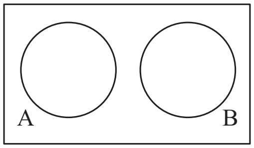
- A $\cap$ B denotes the **intersection** of sets A and B. This is the set of all elements common to both sets.



A $\cap$ B is shaded.

Note: A $\cap$ B = $\{x\text{:}\ \ x \in \text{A}$ **and** $x \in \text{B}\}$

- **Disjoint sets** are sets which do not have elements in common.



These two sets are disjoint.

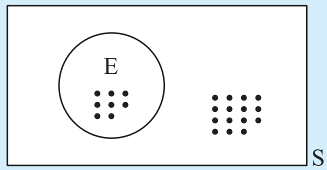
$A \cap B = \phi$ where ϕ represents an **empty set**.

A and B are said to be **mutually exclusive.**

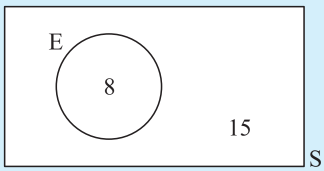
Note: It is not possible to represent independent events on a Venn diagram.

Example 15

The Venn diagram alongside represents a sample space, S, of all children in a class. Each dot represents a student. The event, A, shows all those students with blue eyes. Determine the probability that a randomly selected child:



a has blue eyes **b** does not have blue eyes.



$n(\text{S}) = 23, \quad n(\text{A}) = 8$

a $\text{P(blue eyes)} = \dfrac{n(\text{A})}{n(\text{S})} = \frac{8}{23}$

b $\text{P(not blue eyes)} = \frac{15}{23}$

{as 15 of the 23 are not in A}

or $\text{P(not blue)} = 1 - \text{P(blue eyes)}$ {complementary events}

$= 1 - \frac{8}{23}$

$= \frac{15}{23}$

Example 16

If A is the set of all factors of 36 and B is the set of all factors of 54, find: **a** $A \cup B$ **b** $A \cap B$

$A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ and $B = \{1, 2, 3, 6, 9, 18, 27, 54\}$

a $A \cup B$ = the set of factors of 36 **or** 54

$= \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54\}$

b $A \cap B$ = the set of factors of both 36 **and** 54 $= \{1, 2, 3, 6, 9, 18\}$

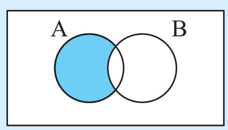
Example 17

On separate Venn diagrams, using two events A and B that intersect, shade the region representing:

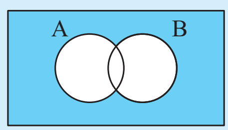
a in A but not in B

b neither in A nor B.

a



b



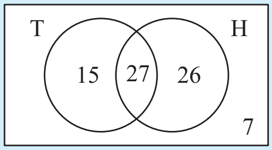
EXERCISE 19I

1 On separate Venn diagrams, using two events A and B that intersect, shade the region representing:

a in A **b** in B **c** in both A and B
d in A or B **e** in B but not in A **f** in exactly one of A or B

Example 18

If the Venn diagram alongside illustrates the number of people in a sporting club who play tennis (T) and hockey (H), determine the number of people:



a in the club
b who play hockey
c who play both sports
d who play neither sport
e who play at least one sport

a Number in the club
$= 15 + 27 + 26 + 7 = 75$

b Number who play hockey
$= 27 + 26 = 53$

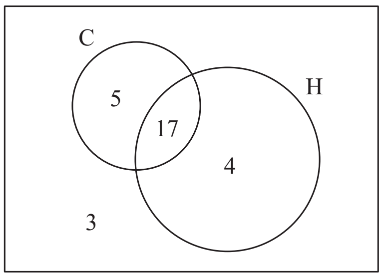
c Number who play both sports $= 27$

d Number who play neither sport
$= 7$

e Number who play at least one sport
$= 15 + 27 + 26 = 68$

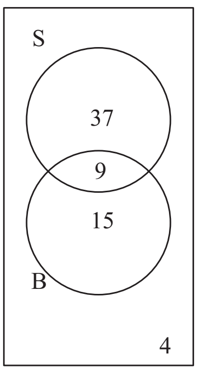
2 The Venn diagram alongside illustrates the number of students in a particular class who study Chemistry (C) and History (H). Determine the number of students:

a in the class
b who study both subjects
c who study at least one of the subjects
d who only study Chemistry.



3 In a survey at an alpine resort, people were asked whether they liked skiing (S) or snowboarding (B). Use the Venn diagram to determine the number of people:

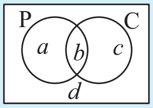
a in the survey
b who liked both activities
c who liked neither activity
d who liked exactly one of the activities.

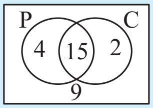


Example 19

In a class of 30 students, 19 study Physics, 17 study Chemistry and 15 study both of these subjects. Display this information on a Venn diagram and hence determine the probability that a randomly selected class member studies:

a both subjects
b at least one of the subjects
c Physics, but not Chemistry
d exactly one of the subjects
e neither subject
f Chemistry if it is known that the student studies Physics





Let P represent the event of 'studying Physics', and C represent the event of 'studying Chemistry'.

Now $a+b=19$ {as 19 study Physics}
$b+c=17$ {as 17 study Chemistry}
$b=15$ {as 15 study both}
$a+b+c+d=30$ {as there are 30 in the class}

$\therefore\ b=15,\quad a=4,\quad c=2,\quad d=9.$

a P(studies both)
$=\frac{15}{30}$ or $\frac{1}{2}$

b P(studies at least one subject)
$=\frac{4+15+2}{30}$
$=\frac{21}{30}$
$=\frac{7}{10}$

c P(studies P, but not C)
$=\frac{4}{30}$
$=\frac{2}{15}$

d P(studies exactly one)
$=\frac{4+2}{30}$
$=\frac{6}{30}$
$=\frac{1}{5}$

e P(studies neither)
$=\frac{9}{30}$
$=\frac{3}{10}$

f P(studies C if it is known studies P)
$=\frac{15}{15+4}$
$=\frac{15}{19}$

4 In a class of 40 students, 19 play tennis, 20 play netball and 8 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:

a plays tennis
b does not play netball
c plays at least one of the sports
d plays one and only one of the sports
e plays netball, but not tennis
f plays tennis knowing he/she plays netball.

5 50 married men were asked whether they gave their wife flowers or chocolates for their last birthday. The results were: 31 gave chocolates, 12 gave flowers and 5 gave both chocolates and flowers. If one of the married men was chosen at random, determine the probability that he gave his wife:

a chocolates or flowers
b chocolates but not flowers
c neither chocolates nor flowers
d flowers if it is known that he did not give her chocolates.

6 The medical records for a class of 30 children showed whether they had previously had measles or mumps. The records showed 24 had had measles, 12 had had measles and mumps, and 26 had had measles or mumps. If one child from the class is selected randomly from the group, determine the probability that he/she has had:

a mumps **b** mumps but not measles **c** neither mumps nor measles

d measles if it is known that the child has had mumps.

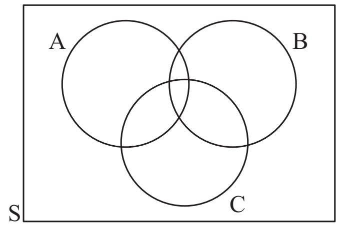
7 If A and B are two non-disjoint sets, shade the region of a Venn diagram representing

a A' **b** $A' \cap B$ **c** $A \cup B'$ **d** $A' \cap B'$

8 The diagram alongside is the most general case for three events in the same sample space S.
On separate Venn diagram sketches, shade:

a A **b** B' **c** $B \cap C$

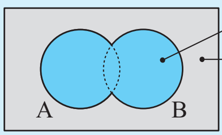
d $A \cup C$ **e** $A \cap B \cap C$ **f** $(A \cup B) \cap C$?



Set identities can be verified using Venn diagrams.

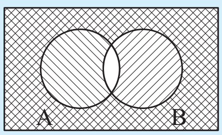
Example 20

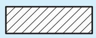
Verify that $(A \cup B)' = A' \cap B'$.

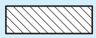


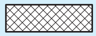
this shaded region is $(A \cup B)$.

$\therefore$ this shaded region is $(A \cup B)'$



 represents A'

 represents B'

 represents $A' \cap B'$

Thus $(A \cup B)'$ and $A' \cap B'$ are represented by the same regions, verifying that $(A \cup B)' = A' \cap B'$.

9 Verify that

a $(A \cap B)' = A' \cup B'$

b $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

c $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

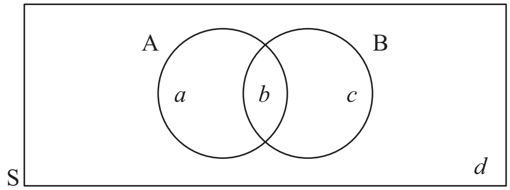
10 Suppose $S = \{x\colon x \text{ is a positive integer} < 100\}$.
Let $A = \{\text{multiples of 7 in S}\}$ and $B = \{\text{multiples of 5 in S}\}$.

a How many elements are there in:

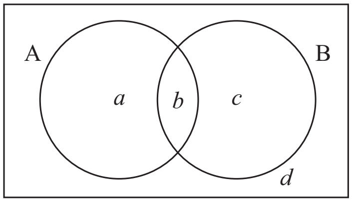
i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$?

b If $n(E)$ represents the number of elements in set E, verify that
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

11 Use the figure alongside to establish that $n(\text{A} \cup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})$ for all sets A and B in a universal set S.



12



From the Venn diagram $\text{P(A)} = \dfrac{a+b}{a+b+c+d}$

a Use the Venn diagram to find:

i P(B) **ii** P(A and B) **iii** P(A or B) **iv** P(A) + P(B) − P(A and B)

b What is the connection between P(A or B) and P(A) + P(B) − P(A and B)?

J LAWS OF PROBABILITY

THE ADDITION LAW

From question **12** of the previous exercise we showed that

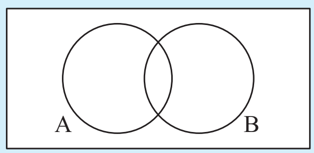
for two events A and B, $\text{P}(\text{A} \cup \text{B}) = \text{P(A)} + \text{P(B)} - \text{P}(\text{A} \cap \text{B})$.

This is known as the **addition law of probability**, and can be written as

P(**either** A **or** B) = P(A) + P(B) − P(**both** A **and** B)

Example 21

If $\text{P(A)} = 0.6$, $\text{P}(\text{A} \cup \text{B}) = 0.7$ and $\text{P}(\text{A} \cap \text{B}) = 0.3$, find P(B).

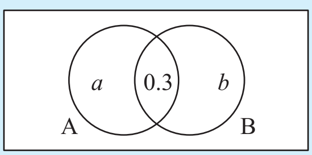


Since $\text{P}(\text{A} \cup \text{B}) = \text{P(A)} + \text{P(B)} - \text{P}(\text{A} \cap \text{B})$,

then $0.7 = 0.6 + \text{P(B)} - 0.3$

$\therefore \text{P(B)} = 0.4$

or



Using a Venn diagram with the probabilities on it,

$a + 0.3 = 0.6 \qquad \therefore a = 0.3$

$a + b + 0.3 = 0.7$

$\therefore a + b = 0.4$

$\therefore 0.3 + b = 0.4$

$\therefore b = 0.1$

$\therefore \text{P(B)} = 0.3 + b = 0.4$

MUTUALLY EXCLUSIVE EVENTS

If A and B are **mutually exclusive** events then $P(A \cap B) = 0$
and so the addition law becomes $P(A \cup B) = P(A) + P(B)$.

Example 22

A box of chocolates contains 6 with hard centres (H) and 12 with soft centres (S).

a Are the events H and S mutually exclusive?

b Find **i** P(H) **ii** P(S) **iii** $P(H \cap S)$ **iv** $P(H \cup S)$.

a Chocolates cannot have both a hard and a soft centre.
$\therefore$ H and S are mutually exclusive.

b **i** $P(H) = \frac{6}{18} = \frac{1}{3}$ **ii** $P(S) = \frac{12}{18} = \frac{2}{3}$ **iii** $P(H \cap S) = 0$ **iv** $P(H \cup S) = \frac{18}{18} = 1$

CONDITIONAL PROBABILITY

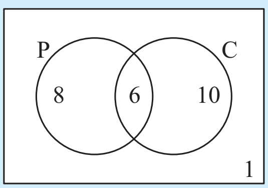
Suppose we have two events A and B, then

A / B is used to represent that 'A occurs knowing that B has occurred'.

Example 23

In a class of 25 students, 14 like Pizza and 16 like coffee. One student likes neither and 6 students like both. One student is randomly selected from the class.
What is the probability that the student:

a likes Pizza **b** likes Pizza given that he/she likes coffee?



The Venn diagram of the situation is shown.

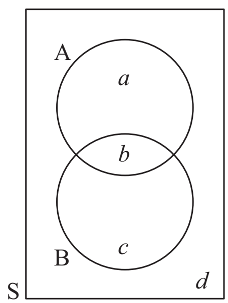
a $P(\text{Pizza}) = \frac{14}{25}$ {of the 25 students 14 like Pizza}

b $P(\text{Pizza/coffee}) = \frac{6}{16}$
{of the 16 who like coffee, 6 like Pizza}

If A and B are events then

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

Proof:



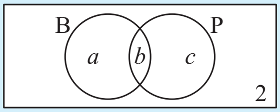
$$P(A/B) = \frac{b}{b+c} \quad \text{\{Venn diagram\}}$$
$$= \frac{b/(a+b+c+d)}{(b+c)/(a+b+c+d)}$$
$$= \frac{P(A \cap B)}{P(B)}$$

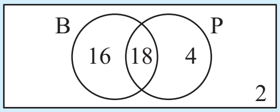
It follows that $P(A \cap B) = P(A/B)P(B)$ or $P(A \cap B) = P(B/A)P(A)$

Example 24

In a class of 40, 34 like bananas, 22 like pineapples and 2 dislike both fruits. If a student is randomly selected, find the probability that the student:

a likes both fruits
b likes at least one fruit
c likes bananas given that he/she likes pineapples
d dislikes pineapples given that he/she likes bananas.





B represents students who like bananas
P represents students who like pineapples

We are given that
$$a + b = 34$$
$$b + c = 22$$
$$a + b + c = 38$$

$\therefore$ $c = 38 - 34 = 4$ and so $b = 18$ and $a = 16$

a P(likes both) $= \frac{18}{40} = \frac{9}{20}$

b P(likes at least one) $= \frac{38}{40} = \frac{19}{20}$

c P(B/P) $= \frac{18}{22} = \frac{9}{11}$

d P(P′/B) $= \frac{16}{34} = \frac{8}{17}$

EXERCISE 19J

1 In a group of 50 students, 40 study Mathematics, 32 study Physics and each student studies at least one of these subjects.

a From a Venn diagram find how many students study both subjects.
b If a student from this group is randomly selected, find the probability that he/she:
  i studies Mathematics but not Physics
  ii studies Physics given that he/she studies Mathematics.

2 In a class of 40 students, 23 have dark hair, 18 have brown eyes, and 26 have dark hair, brown eyes or both. A child is selected at random. Determine the probability that the child has:

a dark hair and brown eyes
b neither dark hair nor brown eyes
c dark hair but not brown eyes
d brown eyes given that the child has dark hair.

3 50 students go bushwalking. 23 get sunburnt, 22 get bitten by ants and 5 are both sunburnt and bitten by ants. Determine the probability that a randomly selected student:

a escaped being bitten
b was either bitten or sunburnt
c was neither bitten nor sunburnt
d was bitten, given that the student was sunburnt
e was sunburnt, given that the student was not bitten.

4 30 students sit for an examination in both French and English. 25 pass French, 24 pass English and 3 fail both. Determine the probability that a student who:

a passed French, also passed English
b failed English, passed in French.

5 400 families were surveyed. It was found that 90% had a TV set and 60% had a computer. Every family had at least one of these items. If one of these families is randomly selected, find the probability it has a TV set given that it has a computer.

6 In a certain town 3 newspapers are published. 20% of the population read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C and 2% read all 3 newspapers. A person is selected at random.

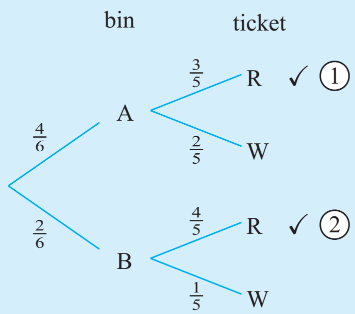
Determine the probability that the person reads:

- **a** none of the papers
- **b** at least one of the papers
- **c** exactly one of the papers
- **d** either A or B
- **e** A, given that the person reads at least one paper
- **f** C, given that the person reads either A or B or both.

Example 25

Bin A contains 3 red and 2 white tickets. Bin B contains 4 red and 1 white. A die with 4 faces marked A and two faces marked B is rolled and used to select bin A or B. A ticket is then selected from this bin. Determine the probability that:

a the ticket is red **b** the ticket was chosen from B given it is red.



a P(R)

$= \frac{4}{6} \times \frac{3}{5} + \frac{2}{6} \times \frac{4}{5}$ {the ✓ paths}

$= \frac{20}{30}$

$= \frac{2}{3}$

b P(B/R)

$= \dfrac{P(B \cap R)}{P(R)}$

$= \dfrac{\frac{2}{6} \times \frac{4}{5}}{\frac{2}{3}}$ ← path 2

$= \frac{2}{5}$

7 Urn A contains 2 red and 3 blue marbles, and urn B contains 4 red and 1 blue marble. Peter selects an urn by tossing a coin, and takes a marble from that urn.

- **a** Determine the probability that it is red.
- **b** Given that the marble is red, what is the probability it came from B?

8 The probability that Greta's mother takes her shopping is $\frac{2}{5}$. When Greta goes shopping with her mother she gets an icecream 70% of the time. When Greta does not go shopping with her mother she gets an icecream 30% of the time.

Determine the probability that

- **a** Greta's mother buys her an icecream when shopping.
- **b** Greta went shopping with her mother, given that her mother buys her an icecream.

9 On a given day, photocopier A has a 10% chance of a malfunction and machine B has a 7% chance of the same. Given that at least one of the machines has malfunctioned, what is the chance that it was machine B?

10 On any day, the probability that a boy eats his prepared lunch is 0.5 and the probability that his sister does likewise is 0.6. The probability that the girl eats her lunch given that the boy eats his is 0.9. Determine the probability that:

a both eat their lunch

b the boy eats his lunch given that the girl eats hers

c at least one of them eats lunch.

11 The probability that a randomly selected person has cancer is 0.02. The probability that he reacts positively to a test which detects cancer is 0.95 if he has cancer, and 0.03 if he does not. Determine the probability that a randomly selected person when tested: **a** reacts positively **b** has cancer given that he reacts positively.

12 A double-headed, a double-tailed and an ordinary coin are placed in a tin can. One of the coins is randomly chosen without identifying it. The coin is tossed and falls "heads". Determine the probability that the coin is the "double-header".

K INDEPENDENT EVENTS REVISITED

A and B are **independent events** if the occurrence (or non-occurrence) of one event does not affect the occurrence of the other,

i.e., $P(A/B) = P(A)$ and $P(B/A) = P(B)$.

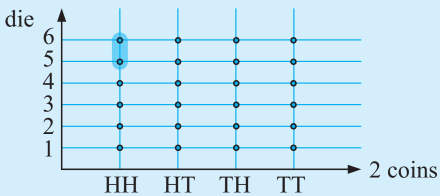
So, as $P(A \cap B) = P(A/B)P(B)$,

A and B are **independent events** $\Leftrightarrow$ $P(A \cap B) = P(A)\,P(B)$.

Example 26

When two coins are tossed, A is the event of getting 2 heads. When a die is rolled, B is the event of getting a 5 or 6. Prove that A and B are independent events.

$P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{6}$. Therefore, $P(A)\,P(B) = \frac{1}{4} \times \frac{2}{6} = \frac{1}{12}$



$P(A \cap B)$

$= P(\text{2 heads } \textbf{and} \text{ a 5 or a 6})$

$= \frac{2}{24}$

$= \frac{1}{12}$

So, as $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

Example 27

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = p$. Find p if:

a A and B are mutually exclusive

b A and B are independent.

a If A and B are mutually exclusive, $A \cap B = \phi$

and so $P(A \cap B) = 0$

But $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad p = \frac{1}{2} + \frac{1}{3} - 0$

i.e., $p = \frac{5}{6}$

b If A and B are independent,
$$P(A \cap B) = P(A)\,P(B) = \tfrac{1}{2} \times \tfrac{1}{3} = \tfrac{1}{6}$$

$\therefore \quad P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$

i.e., $p = \frac{2}{3}$

EXERCISE 19K

1 If $P(R) = 0.4$, $P(S) = 0.5$ and $P(R \cup S) = 0.7$, are R and S independent events?

2 If $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{1}{2}$, find:

a $P(A \cap B)$ **b** $P(B/A)$ **c** $P(A/B)$

Are A and B independent events?

3 If $P(X) = 0.5$, $P(Y) = 0.7$, and X and Y are independent events, determine the probability of the occurrence of:

a both X and Y **b** X or Y **c** neither X nor Y

d X but not Y **e** X given that Y occurs.

4 The probability that A, B and C solve a particular problem is $\frac{3}{5}$, $\frac{2}{3}$, $\frac{1}{2}$, respectively.

If they all try, determine the probability that the group solves the problem.

5 **a** Determine the probability of getting at least one six when an ordinary die is rolled 3 times.

b If a die is rolled n times, find the smallest n such that
P(at least one 6 in n throws) $> 99\%$.

6 A and B are independent events. Prove that A and B' are also independent events.

REVIEW SET 19A

1 List the different orderings in which 4 people A, B, C and D could line up. If they line up at random, determine the probability that:

a A is next to C **b** there is exactly one person between A and C.

2 A coin is tossed and a square spinner, labelled A, B, C, D, is twirled. Determine the probability of obtaining:

a a head and consonant **b** a tail and C **c** a tail or a vowel.

3 A class contains 25 students. 13 play tennis, 14 play volleyball and 1 plays neither of these two sports. A student is randomly selected from the class. Determine the probability that the student:

a plays both tennis and volleyball **b** plays at least one of these two sports.

c plays volleyball given that he/she does not play tennis.

4 Niklas and Rolf play tennis and the winner is the first to win two sets. Niklas has a 40% chance of beating Rolf in any set. Draw a tree diagram showing the possible outcomes and hence determine the probability that Niklas will win the match.

5 The probability that a man will be alive in 25 years is $\frac{3}{5}$, and the probability that his wife will be alive is $\frac{2}{3}$. Determine the probability that in 25 years:

a both will be alive **b** at least one will be alive

c only the wife will be alive.

6 Each time Mae and Ravi play chess, Mae has a probability of $\frac{4}{5}$ that she wins. If they play 5 games, determine the probability that:

a Mae wins 3 games **b** Mae wins either 4 or 5 of the games.

7 If I buy 4 tickets in a 500 ticket lottery, determine the probability that I win:

a the first 3 prizes **b** at least one of the first 3 prizes.

8 A school photocopier has a 95% chance of working on any particular day. Draw a tree diagram showing the possibilities for the working or otherwise of the photocopier over 2 consecutive days. Hence determine the likelihood that it will be working on at least one of these days.

9 Box A contains 7 Strawberry creams and 3 Turkish delights,
Box B contains 6 Strawberry creams and 4 Turkish delights,
Box C contains 5 Strawberry creams and 5 Turkish delights.
A bin is chosen using a die with 3 of its six faces marked A, 2 marked B and one marked C and one chocolate is chosen at random from the box.

a Determine the probability that it is a Strawberry cream

b Given that it is a Strawberry cream, find the probability it came from box B.

REVIEW SET 19B

1 Systematically list the sexes of 4-child families. Hence determine the probability that a randomly selected 4-child family consists of two children of each sex.

2 A pair of dice is rolled. Graph the sample space of all possible outcomes. Hence determine the probability that:

a a sum of 7 or 11 results **b** a sum of at least 8 results.

3 In a group of 40 students, 22 study Economics, 25 study Law and 3 study neither of these subjects. Determine the probability that a randomly chosen student studies:

a both Economics and Law **b** at least one of these subjects

c Economics given that he/she studies Law.

4 An integer lies between 0 and 100. Determine the likelihood that is divisible by 6 or 8.

5 A bag contains 3 red, 4 yellow and 5 blue marbles. Two marbles are randomly selected from the bag (without replacement). What is the probability that:

a both are blue **b** both are the same colour

c at least one is red **d** exactly one is yellow?

6 What is meant by:

a independent events **b** disjoint events?

7 On any one day it could rain with 25% chance and be windy with 36% chance. Draw a tree diagram showing the possibilities with regard to wind and rain on a particular day. Hence determine the probability that on a particular day there will be:

a rain and wind **b** rain or wind.

8 A, B and C have 10%, 20% and 30% chance of independently solving a certain maths problem. If they all try independently of one another, what is the probability that this group will solve the problem?

9 Jon goes cycling on three mornings of each week (at random). When he goes cycling he has eggs for breakfast 70% of the time. When he does not go cycling he has eggs for breakfast 25% of the time. Determine the probability that he

a has eggs for breakfast

b goes cycling given that he has eggs for breakfast.

10 **a** Expand $\left(\frac{3}{5}+\frac{2}{5}\right)^4$.

b Four pens were randomly selected (with replacement) from a tin containing 20 pens of which 12 had blue ink. What is the probability that:

i two of them had blue ink **ii** at most two had blue ink?

Chapter 20

Introduction to calculus

Contents:

Investigation 1: The speed of falling objects

A Rate of change

B Instantaneous rates of change

Investigation 2: Instantaneous speed

Review set 20

INTRODUCTION

Calculus is a branch of mathematics which deals with **rate of change**.

Speed is a rate of change; it is the rate of change in distance per unit of time.

We say that $$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

An average speed of 60 km/h means that in 1 hour, 60 km are travelled.

Calculus developed from these questions:

- What is speed?
- How is instantaneous speed calculated?

Calculus has been used in many branches of science and engineering since 1600 AD.

INVESTIGATION 1 THE SPEED OF FALLING OBJECTS

Secret agent James Speed has fallen from a plane. The problem he faces is to calculate his speed at any particular moment. If only he had brought his personal speedometer!

He knows that the distance fallen (d metres) by a falling object (from rest) is given by $d(t) = 5t^2$ and the average speed in the time interval $t_1 \leqslant t \leqslant t_2$ is given by $\dfrac{d(t_2) - d(t_1)}{t_2 - t_1}$ metres/second.

What to do:

a Copy and complete:

Time interval	*Average speed in this interval*
$1 \leqslant t \leqslant 2$	
$1 \leqslant t \leqslant 1.5$	
$1 \leqslant t \leqslant 1.1$	
$1 \leqslant t \leqslant 1.01$	
$1 \leqslant t \leqslant 1.001$	

b Using **a**, predict James' speed at the precise moment when $t = 1$. Explain your reasoning.

c Find the average speed between 1 and $1 + h$ seconds where h represents a small increment of time (which can be made as small as we like). Show how this answer supports your answer to **b**.

d Repeat steps **a**, **b** and **c** in order to find James' instantaneous speed at:
i $t = 2$ seconds **ii** $t = 3$ seconds **iii** $t = 4$ seconds

e Copy and complete:

t	*Instantaneous speed at this time*
1	
2	
3	
4	

f Use **e** to predict a formula for instantaneous speed when $d = 5t^2$
i.e., complete: "if $d = 5t^2$, then instantaneous speed = ..."

g Calculate James' average speed between t and $t + h$ seconds and use this result to explain why your answer in **f** is correct.

A RATE OF CHANGE

Often we judge sporting performances by using rates. For example:

- Sir Donald Bradman's batting rate at Test cricket level was 99.94 *runs per innings*
- Dasha's basketball goal scoring rate was 17.25 *goals per game*

Other rates are commonly quoted. For example: "Jodie's typing speed is 63 *words per minute* with an error rate of 2.3 *errors per page*.

We see that:

A **rate** is a comparison between two quantities of different kinds.

FINDING RATES

Rates are invariably used to compare performance.

Example 1

Joost typed 213 words in 3 minutes and made 6 errors, whereas Irma typed 260 words in 5 minutes and made 7 errors. Compare their performances using rates.

$$\text{Joost's typing rate} = \frac{213 \text{ words}}{3 \text{ minutes}} = 71 \text{ words/min}$$

$$\text{Irma's typing rate} = \frac{260 \text{ words}}{4 \text{ minutes}} = 65 \text{ words/minute}$$

$$\text{Joost's error rate} = \frac{6 \text{ errors}}{213 \text{ words}} \doteqdot 0.0282 \text{ errors/word}$$

$$\text{Irma's error rate} = \frac{7 \text{ errors}}{260 \text{ words}} \doteqdot 0.0269 \text{ errors/word}$$

$\therefore$ Joost typed at a faster rate but Irma typed with greater accuracy.

EXERCISE 20A.1

1 Ozair's pulse rate was measured at 67 beats/minute.
- **a** Explain exactly what this rate means.
- **b** How many heart beats would Ozair expect to have each hour?

2 Marjut typed a 14 page document and made eight errors. If an average page of typing has 380 words, find Marjut's error rate in:
- **a** errors/word
- **b** errors/100 words.

3 Paul worked 12 hours for \$148.20 whereas Marita worked 13 hours for \$157.95. Who worked for the better hourly rate of pay?

4 New tyres have a tread depth of 8 mm. After driving for 32 178 km the tread depth was reduced to 2.3 mm. What was the wearing rate of the tyres in:
- **a** mm per km travelled
- **b** mm per 10 000 km travelled?

5 We left Adelaide at 11.43 am and travelled to Victor Harbor, a distance of 71 km. We arrived there are 12.39 pm. What was our average speed in:
- **a** km/h
- **b** m/s?

AVERAGE RATE OF CHANGE

Consider a trip from Adelaide to Melbourne.

The following table gives towns along the way, distances travelled and time taken.

We plot the *distance travelled* against the *time taken* to obtain a graph of the situation.

Even though there would be variable speed between each place we will join points with straight line segments.

Place	*Time taken (min)*	*Distance travelled (km)*
Adelaide tollgate	0	0
Tailem Bend	63	98
Bordertown	157	237
Nhill	204	324
Horsham	261	431
Ararat	317	527
Midland H/W Junction	386	616
Melbourne	534	729

We can find the average speed between any two places.

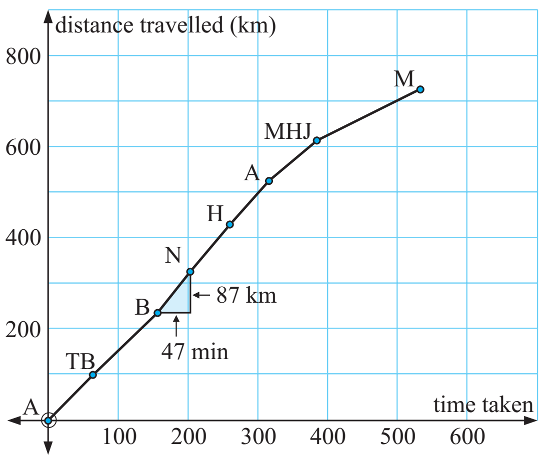
For example, the average speed from Bordertown to Nhill is:

$$\frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(324 - 237)\text{ km}}{(204 - 157)\text{ min}}$$

$$= \frac{87\text{ km}}{\frac{47}{60}\text{ h}}$$

$$\doteq 111\text{ km/h}$$



We notice that the average speed is the $\dfrac{y\text{-step}}{x\text{-step}}$ on the graph.

So, the average speed is the **slope of the line segment** joining the two points, which means that the faster the trip between two places, the greater the slope of the graph.

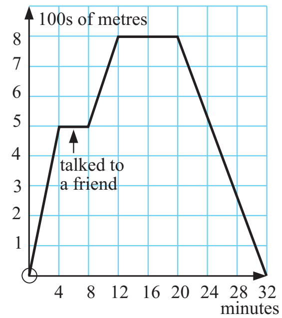
If $s(t)$ is the distance travelled function then the **average speed** over the time interval from $t = t_1$ to $t = t_2$ is given by:

$$\textbf{Average speed} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

EXERCISE 20A.2

1 For the Adelaide to Melbourne data on page **510**, find the average speed from:
- **a** Tailem Bend to Nhill
- **b** Horsham to Melbourne.

2 Tian walks to the newsagent to get a paper each morning. The travel graph alongside shows Tian's distance from home. Use the graph to answer the following questions.



- **a** How far is the newsagent from Tian's house?
- **b** What is the slope of the line segment for the first 4 minutes of the walk?
- **c** What was Tian's average walking speed for the first 4 minutes (m/min)?
- **d** What is the physical representation of the slope in this problem?
- **e** How many minutes did Tian stay at the newsagent's store?
- **f** What was Tian's average speed on the return journey?
- **g** What total distance did Tian walk?

3 During December Mount Bold reservoir was losing water at a constant rate due to usage and evaporation. On December 12 the estimated water in the reservoir was 53.8 million kL and on December 23 the estimate was 48.2 million kL. No water entered the reservoir. What was the average rate of water loss during this period?

4 Kolya's water consumption invoices for a one year period show:

Period	*Consumption*
First quarter (Jan 01 to Mar 31)	106.8 kL
Second quarter (Apr 01 to Jun 30)	79.4 kL
Third quarter (Jul 01 to Sep 30)	81.8 kL
Fourth quarter (Oct 01 to Dec 31)	115.8 kL

Find the average rate of water consumption per day for:
- **a** the first quarter
- **b** the first six months
- **c** the whole year.

CONSTANT AND VARIABLE RATES OF CHANGE

Once again let us visit the water filling demonstration. Water flows from a tap at a constant rate into vessels of various shapes. We see the water level change over time. A corresponding *height* against *time* graph shows what is happening to the height as time increases. Click on the icon to investigate the rate of change in height over time for the different shaped vessels given.

DEMO

DISCUSSION

- In which vessel was there a constant rate of change in height over time?
- What is the nature of the graph when there is a constant rate of change?
- What graphical features indicate
  - an increasing rate of change
  - a decreasing rate of change?

AVERAGE RATES FROM CURVED GRAPHS

Consider the following example where we are to find average rates over a given time interval.

Example 2

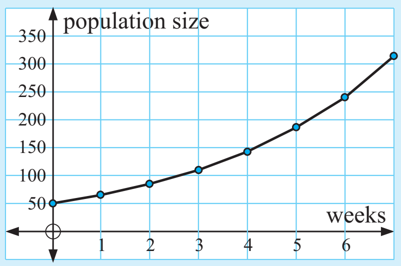
The number of mice in a colony was recorded on a weekly basis.

a Estimate the average rate of increase in population for:

 i the period from week 3 to week 6

 ii the seven week period.

b What is the overall trend with regard to population increase over this period?



a **i** population rate

$$= \frac{\text{increase in population}}{\text{increase in time}}$$

$$= \frac{(240 - 110) \text{ mice}}{(6 - 3) \text{ weeks}}$$

$$\doteqdot 43 \quad \text{mice/week}$$

ii population rate

$$= \frac{(315 - 50) \text{ mice}}{(7 - 0) \text{ weeks}}$$

$$\doteqdot 38 \quad \text{mice/week}$$

b The population rate is increasing over the seven week period as shown by the increasing y-steps on the graph for equal x-steps.

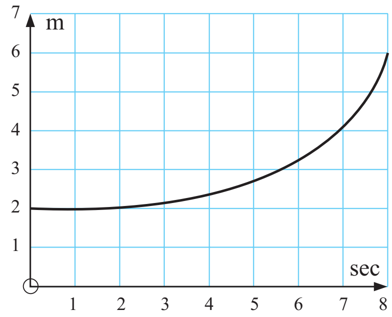
You should notice that:

> The **average rate of change** between two points on the graph is the **slope of the chord** (or **secant**) connecting these two points.

EXERCISE 20A.3

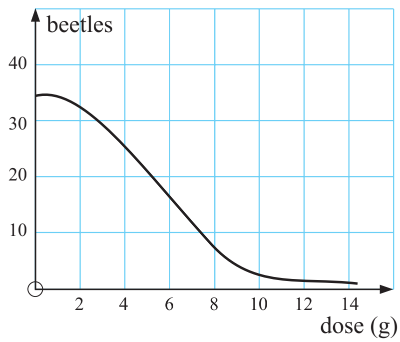
1 For the travel graph given alongside find estimates of the average speed:

- **a** in the first 4 seconds
- **b** in the last 4 seconds
- **c** in the 8 second interval.



2 The number of lawn beetles per m^2 surviving in a lawn for various doses of poison is given in the graph.

- **a** Estimate the rate of beetle decrease when:
  - **i** the dose increases from 0 to 10 g
  - **ii** the dose increases from 4 to 8 g.
- **b** Describe the effect on the rate of beetle decline as the dose goes from 0 to 14 g.



B INSTANTANEOUS RATES OF CHANGE

A moving object such as a motor car, an aeroplane or a runner has variable speed.

At a particular instant in time, the speed of the object is called its **instantaneous speed**.

To examine this concept in greater detail consider the following investigation.

INVESTIGATION 2 INSTANTANEOUS SPEED

Earlier we noticed that:

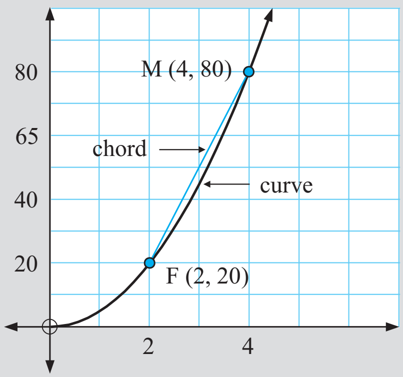
"The average rate of change between two points on a graph is the slope of the chord connecting them."

But, what happens if these two points are extremely close together or in fact coincide?

To discover what will happen consider the following problem:

A ball bearing is dropped from the top of a tall building. The distance fallen after t seconds is recorded and the following graph of distance against time is obtained.

The question is, "What is the speed of the ball bearing at $t = 2$ seconds?



Notice that the average speed in the time interval $2 \leqslant t \leqslant 4$ is

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80-20)\text{ m}}{(4-2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m/s}$$

$$= 30\text{ m/s}$$

DEMO

What to do:

1. Click on the icon to start the demonstration. The slope box gives the starting slope of chord MF where F is the fixed point at which we require the speed of the ball bearing. Check that the slope of MF is that which is given in the slope box when the moving point M is at (4, 80).
2. Click on M and drag it slowly towards F. Write down the slope of the chord when M is at the point where t is:
   a 3 **b** 2.5 **c** 2.1 **d** 2.01
3. When M reaches F observe and record what happens. Why is this so?
4. What do you suspect is the speed of the ball bearing at $t = 2$? This speed is the *instantaneous speed* of the ball bearing at this instant.
5. Move M to the origin and then move it towards F from the other direction. Do you get the same result?

From the investigation you should have discovered that:

The **instantaneous rate of change** (speed in this case) at a particular point is given by the **slope of the tangent** to the graph at that point.

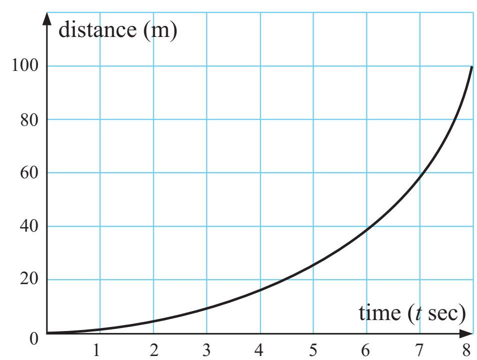
VARIABLE RATES OF CHANGE

THE GRAPHICAL METHOD

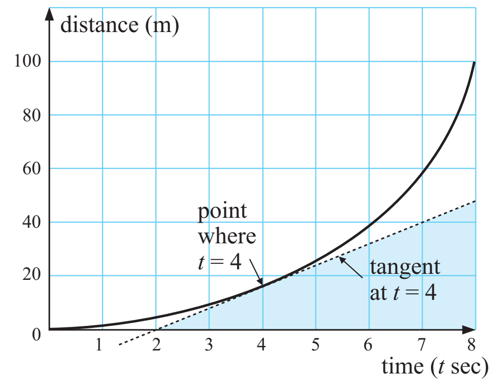
Consider a cyclist who is stationary at an intersection. The graph alongside shows how the cyclist accelerates away from the intersection.

Notice that the average speed over the first 8 seconds is $\dfrac{100\text{ m}}{8\text{ sec}} = 12.5$ m/s.

Notice also that the cyclist's early speed is quite small, but is increasing as time goes by.



To find the instantaneous speed at any time instant, for example, $t = 4$ we simply draw the tangent at that point and find its slope.



Notice that the tangent passes through (2, 0) and (7, 40)

$\therefore$ instantaneous speed at $t = 4$

$= $ slope of tangent

$= \dfrac{(40 - 0)\ \text{m}}{(7 - 2)\ \text{s}}$

$= \frac{40}{5}$ m/s

$= 8$ m/s

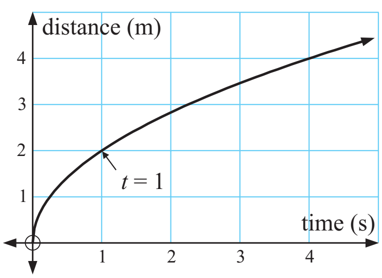
At any point in time we can use this method to find the speed of the cyclist at that instant.

EXERCISE 20B.1

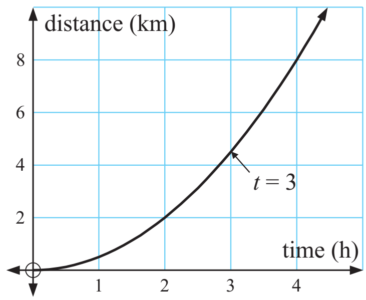
1 For each of the following graphs, find the approximate rate of change at the point shown by the arrow. Make sure your answer contains the correct units.

PRINTABLE GRAPHS

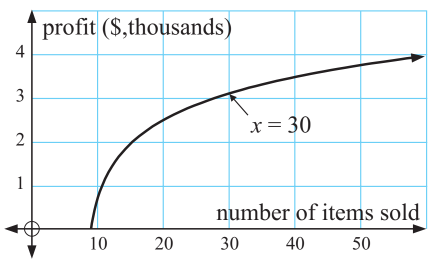
a



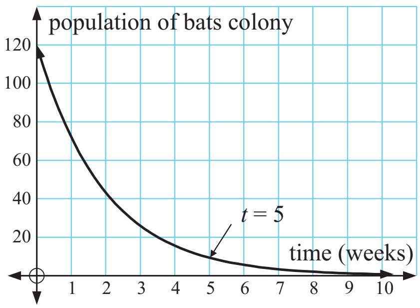
b



c



d



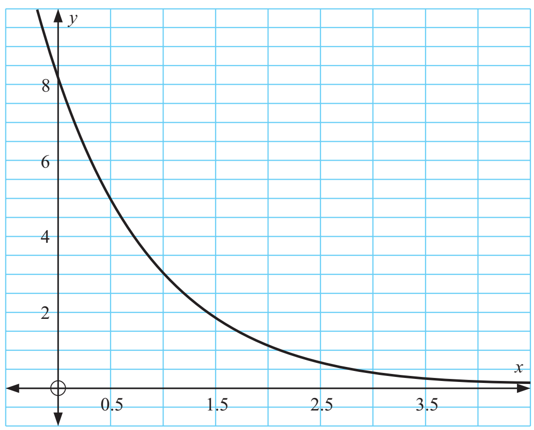
2 Water is leaking from a tank. The amount of water left in the tank (measured in thousands of litres) after x hours is given in the graph alongside.

a How much water was in the tank originally?

b How much water was in the tank after 1 hour?

c How quickly was the tank losing water initially?

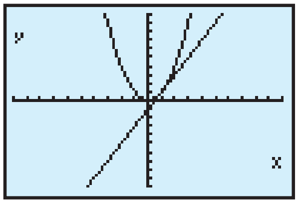
d How quickly was the tank losing water after 1 hour?

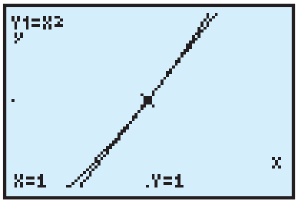


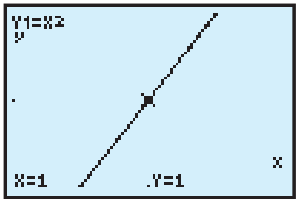
THE 'ZOOMING-IN' METHOD

Consider the graph of $y = x^2$ and its tangent at (1, 1).

Screen dumps, showing the zooming-in technique, are given below.







Each successive dump shows zooming in to the point (1, 1).

The more we zoom-in, the closer the graph gets to a straight line. In fact the graph of $y = x^2$ and the tangent become indistinguishable over successively smaller x-intervals.

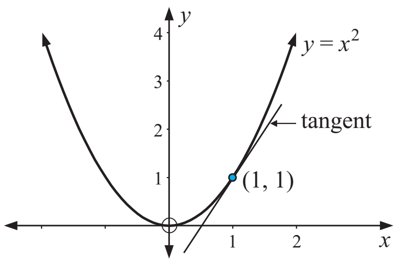
So we could choose two points on $y = x^2$ which are extremely close to (1, 1), perhaps one on each side of (1, 1), find the slope between these points and use this slope to estimate the slope of the tangent.

THE TABLE METHOD

We should have noticed in earlier exercises that drawing an accurate tangent to a curve at a given point is difficult. Three different people may produce three different results. So we need better methods for performing this procedure.

Consider the curve $y = x^2$ and the tangent at the point (1, 1).

A table of values could be used to find the slope of the tangent at (1, 1). We consider a point not at (1, 1), find the slope to (1, 1), and do the same for points closer and closer to (1, 1).



Consider this table:

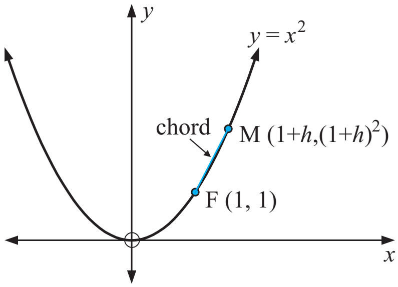
x-coordinate	*y-coordinate*	*slope of chord*
2	4	$\frac{4-1}{2-1} = \frac{3}{1} = 3$
1.5	2.25	$\frac{2.25-1}{1.5-1} = \frac{1.25}{0.5} = 2.5$
1.1	1.21	$\frac{1.21-1}{1.1-1} = \frac{0.21}{0.1} = 2.1$
1.01	1.0201	$\frac{1.0201-1}{1.01-1} = \frac{0.201}{0.01} = 2.01$
1.001	1.002 001	$\frac{1.002\,001-1}{1.001-1} = \frac{0.002\,001}{0.001} = 2.001$

It is fairly clear that:

- the slope of the tangent at (1, 1) would be exactly 2
- the table method is tedious, but it does help to understand the ideas behind finding slopes at a given point.

THE ALGEBRAIC METHOD

To illustrate the algebraic method once again we will consider the curve $y = x^2$ and the tangent at F(1, 1).



Let the moving point have x-coordinate $1+h$ where h is small.

$\therefore$ the y-coordinate of $M = (1+h)^2$. {as $y = x^2$}

So, M is $(1+h,\ (1+h)^2)$.

Now the slope of chord MF is
$$\begin{aligned} &\frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(1+h)^2 - 1}{1+h-1} \\ &= \frac{1+2h+h^2-1}{h} \\ &= \frac{2h+h^2}{h} \\ &= \frac{h(2+h)}{h} \\ &= 2+h \qquad \{\text{if } h \neq 0\} \end{aligned}$$

Now as M approaches F, h approaches 0.

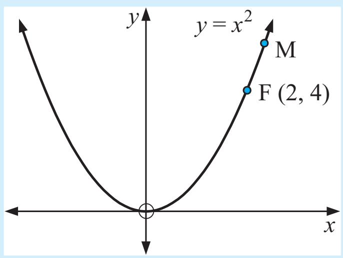
Consequently, $2+h$ approaches 2.

So, we conclude that the tangent at (1, 1) has slope 2.

Example 3

Use the algebraic method to find the slope of the tangent to $y = x^2$ at the point where $x = 2$.

Let $M(2+h,\ (2+h)^2)$ be a point on $y = x^2$ which is close to F(2, 4).



$$\begin{aligned} \text{Slope of MF} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(2+h)^2 - 4}{2+h-2} \\ &= \frac{4+4h+h^2-4}{h} \qquad \{\text{using } (a+b)^2 = a^2+2ab+b^2\} \\ &= \frac{h(4+h)}{h} \\ &= 4+h \qquad \{\text{as } h \neq 0\} \end{aligned}$$

Now as M approaches F, h approaches 0, $4+h$ approaches 4,

$\therefore$ the tangent at (2, 4) has slope 4.

EXERCISE 20B.2

1 Use the algebraic method to find the slope of the tangent to $y = x^2$ at the point where $x = 1.5$.

2 **a** Using $(x+h)^3 = (x+h)^2(x+h)$, show that $(x+h)^3 = x^3+3x^2h+3xh^2+h^3$.

b Find $(1+h)^3$ in expanded form using **a**.

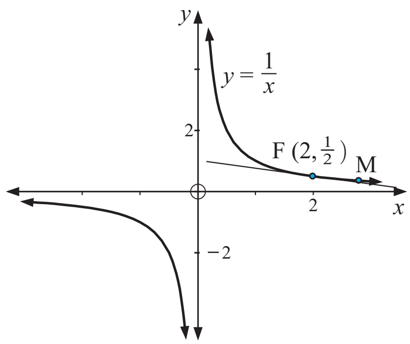
c Consider finding the slope of the tangent to $y = x^3$ at the point F(1, 1). If the x-coordinate of a moving point M, which is close to F, has value $1+h$, state the coordinates of M.

d Find the slope of the chord MF in simplest form $(h \neq 0)$.

e What is the slope of the tangent to $y = x^3$ at the point (1, 1)?

3 Repeat **2** but this time find the slope of the tangent to $y = x^3$ at the point (2, 8).

4 Consider the graph of $y = \dfrac{1}{x}$.



a Write $\dfrac{1}{x+h} - \dfrac{1}{x}$ as a single fraction.

b If point M has x-coordinate $2+h$ state its y-coordinate and find the slope of MF in terms of h.

c What is the slope of the tangent at the point where $x = 2$?

d What is the slope of the tangent to $y = \dfrac{1}{x}$ at the point where $x = 3$?

5 **a** Show that $\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2} = \dfrac{-2xh - h^2}{x^2(x+h)^2}$.

b What does the identity in **a** become when $x = 2$?

c Sketch the graph of $y = \dfrac{1}{x^2}$ for $x > 0$ only.

d Find the slope of the tangent to $y = \dfrac{1}{x^2}$ at the point where $x = 2$.

6 **a** Copy and complete:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \ldots\ldots$$

b What identity results when $x = 9$ in part **a**?

c Sketch the graph of $y = \sqrt{x}$.

d Find the slope of the tangent to $y = \sqrt{x}$ at the point where $x = 9$.

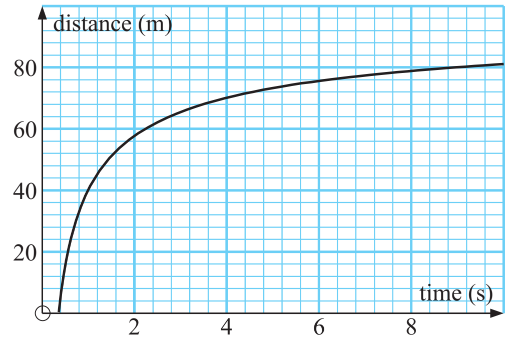
e The number of insects (in thousands) in a colony at time t days is modelled by $N = \sqrt{t}$ for $t \geqslant 4$. At what rate is the colony increasing in size after 9 days?

REVIEW SET 20

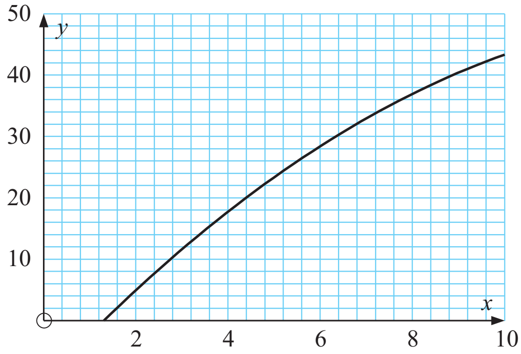
1 For the travel graph alongside, estimate the average speed for:

a the time interval $1 \leqslant t \leqslant 4$ seconds

b the time interval $1 \leqslant t \leqslant 10$ seconds.



2



a Carefully draw a tangent to touch the curve at the point where $x = 4$.

b Hence, estimate the slope of $y = f(x)$ at $x = 4$.

3 Algebraically, find the slope of the tangent to $y = x^2$ at the point when $x = 4$.

4 Consider $f(x) = (2x + 3)^2$.

a Write $f(x+h)$ in simplest form.

b Find, in simplest form $\dfrac{f(x+h) - f(x)}{h}$.

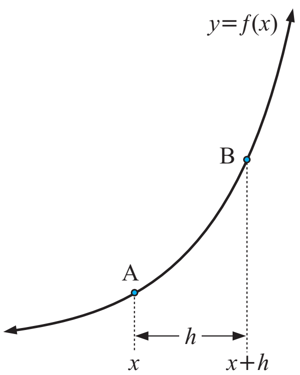
c Interpret $\dfrac{f(x+h) - f(x)}{h}$ (see the diagram)

d Now let h approach 0. What does

i $\dfrac{f(x+h) - f(x)}{h}$ approach

ii $\dfrac{f(1+h) - f(1)}{h}$ approach?

e What information is obtained in **d i** and **d ii**?



5 A particle moves in a straight line with displacement from O given by $s(t) = t^2 + 4t$ metres at time t seconds, $t \geqslant 0$. Find:

a the average velocity in the time interval $2 \leqslant t \leqslant 5$ seconds

b the average velocity in the time interval $2 \leqslant t \leqslant 2 + h$ seconds

c Find the number that $\dfrac{s(2+h) - s(2)}{h}$ approaches as h approaches 0.

Comment on the significance of this value.

Note: If as h approaches 0, $\dfrac{s(2+h) - s(2)}{h}$ approaches some number A, we write $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h} = A$ where $\lim\limits_{h \to 0}$ is read "the limit as h approaches 0 of"

Chapter 21

Differential calculus

Contents:

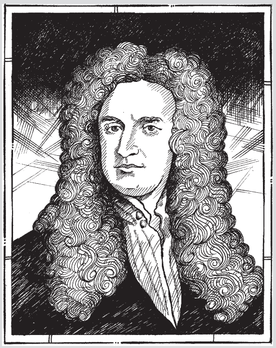
A The idea of a limit
Investigation 1: The slope of a tangent
B Derivatives at a given x-value
C The derivative function
Investigation 2: Finding slopes of functions with technology
D Simple rules of differentiation
Investigation 3: Simple rules of differentiation
E The chain rule
Investigation 4: Differentiating composites
F Product and quotient rules
G Tangents and normals
H The second derivative

Review set 21A
Review set 21B
Review set 21C

HISTORICAL NOTE

Differential Calculus is a branch of Mathematics which originated in the 17th Century. **Sir Isaac Newton** and **Gottfried Wilhelm Leibnitz** are credited with the vital breakthrough in thinking necessary for the development of calculus. Both mathematicians were attempting to find an algebraic method for solving problems dealing with

- **slopes of tangents** to curves at any point on the curve, and
- finding the **rate of change** in one variable with respect to another.



Isaac Newton 1642 – 1727

Gottfried Leibnitz 1646 – 1716

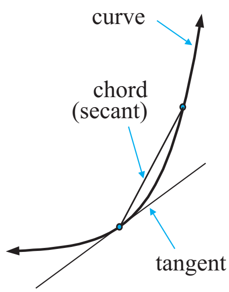
Note:

Calculus is a Latin word meaning pebble. Ancient Romans used stones to count with.

Calculus has applications in a wide variety of fields including engineering, biology, chemistry, physics, economics and geography.

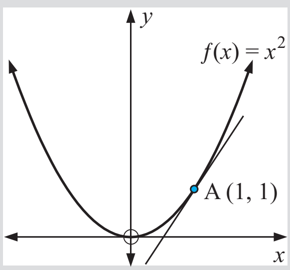
A THE IDEA OF A LIMIT

We now investigate the slopes of chords (secants) from a fixed point on a curve over successively smaller intervals.



Note:
- A chord (secant) of a curve is a straight line segment which joins any two points on the curve.
- A tangent is a straight line which touches a curve at a point.

INVESTIGATION 1 THE SLOPE OF A TANGENT

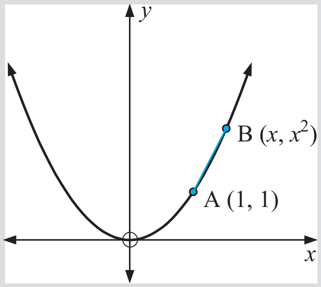


Given a curve, how can we find the slope of a tangent at any point on it?

For example, the point A(1, 1) lies on the curve $f(x) = x^2$. What is the slope of the tangent at A?

DEMO

What to do:

1 

Suppose B lies on $f(x) = x^2$ and B has coordinates (x, x^2).

a Show that the chord AB has slope $\dfrac{f(x) - f(1)}{x - 1}$ or $\dfrac{x^2 - 1}{x - 1}$.

b Copy and complete:

x	Point B	Slope of AB
5	(5, 25)	6
3		
2		
1.5		
1.1		
1.01		
1.001		

2 Comment on the slope of AB as x gets closer to 1.

3 Repeat the process as x gets closer to 1, but from the left of A.

4 Click on the icon to view a demonstration of the process.

5 What do you suspect is the slope of the tangent at A?

The above investigation shows us that as x approaches 1, the slope of the chord approaches the slope of the tangent at $x = 1$.

Notation: We use a horizontal arrow, $\rightarrow$, to represent the word '*approaches*' or the phrase '*tends to*'.

So, $x \rightarrow 1$ is read as 'x approaches 1' or 'x tends to 1'.

In the investigation we noticed that the slope of AB approached a limiting value of 2 as x approached 1, from either side of 1.

Consequently we can write, as $x \rightarrow 1$, $\dfrac{x^2 - 1}{x - 1} \rightarrow 2$.

This idea is written simply as $\displaystyle\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$

and is read as: the limit as x approaches 1 of $\dfrac{x^2 - 1}{x - 1}$ is 2.

In general,

if $\dfrac{f(x) - f(a)}{x - a}$ can be made as close as we like to some real number L by making x sufficiently close to a, we say that $\dfrac{f(x) - f(a)}{x - a}$ approaches a limit of L as x approaches a and write

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = L.$$

ALGEBRAIC/GEOMETRIC APPROACH

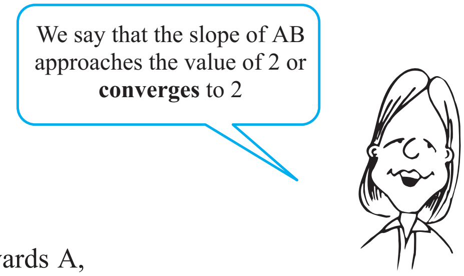
Fortunately we do not have to go through the graphical/table of values method (as illustrated in the investigation) each time we wish to find the slope of a tangent.

Recall that the slope of AB $= \dfrac{x^2-1}{x-1}$

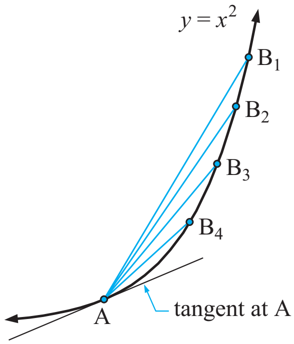
$\therefore$ slope of AB $= \dfrac{(x+1)(x-1)}{x-1} = x+1$ provided that $x \neq 1$

Now as B approaches A, $x \to 1$

$\therefore$ slope of AB $\to 2$ ...... (1)



From a geometric point of view:



as B moves towards A,

the slope of AB $\to$ the slope of the tangent at A .... (2)

Thus, from (1) and (2), we conclude that as both limits must be the same, the slope of the tangent at A is 2.

LIMIT RULES

The following are useful limit rules:

- $\lim\limits_{x\to a} c = c$ c is a constant
- $\lim\limits_{x\to a} c \times u(x) = c \times \lim\limits_{x\to a} u(x)$ c is a constant, $u(x)$ is a function of x
- $\lim\limits_{x\to a} [u(x)+v(x)] = \lim\limits_{x\to a} u(x) + \lim\limits_{x\to a} v(x)$ $u(x)$ and $v(x)$ are functions of x
- $\lim\limits_{x\to a} [u(x)v(x)] = \left[\lim\limits_{x\to a} u(x)\right]\left[\lim\limits_{x\to a} v(x)\right]$ $u(x)$ and $v(x)$ are functions of x

We make no attempt to prove these rules at this stage. However, all can be readily verified.

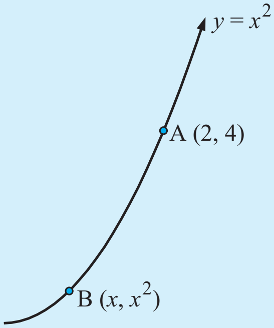
For example: as $x \to 2$, $x^2 \to 4$ and $5x \to 10$ and $x^2+5x \to 14$ clearly verifies the third rule.

Before proceeding to a more formal method we will reinforce the algebraic/geometric method of finding slopes of tangents.

Example 1

Use the algebraic/geometric method to find the slope of the tangent to $y = x^2$ at the point (2, 4).

Let B be (x, x^2)



$\therefore$ slope of AB $= \dfrac{x^2 - 4}{x - 2}$

$= \dfrac{(x+2)(x-2)}{(x-2)}$

$= x + 2$ provided that $x \neq 2$

$\therefore \lim\limits_{x\to 2}(\text{slope of AB}) = 4$ ...... (1)

But, as B $\to$ A, i.e., $x \to 2$

$\lim\limits_{x\to 2}(\text{slope of AB}) =$ slope of tangent at A ...... (2)

$\therefore$ slope of tangent at A $= 4$ {from (1) and (2)}

EXERCISE 21A

1 Use the algebraic/geometric method to find the slope of the tangent to:

a $y = x^2$ at the point where $x = 3$, i.e., (3, 9)

b $y = \dfrac{1}{x}$ at the point where $x = 2$.

2 **a** Show that $(x - a)(x^2 + ax + a^2) = x^3 - a^3$.

b Use the algebraic/geometric method and **a** to find the slope of the tangent to $y = x^3$ at the point where $x = 2$.

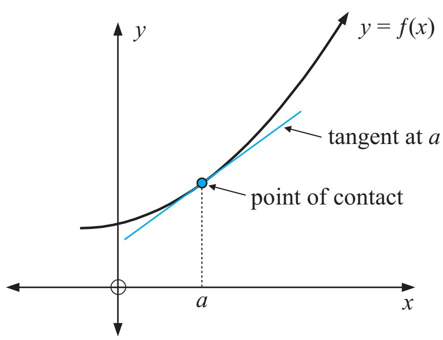
3 **a** Show that $\dfrac{\sqrt{x} - \sqrt{a}}{x - a} = \dfrac{1}{\sqrt{x} + \sqrt{a}}$.

b Use the algebraic/geometric method and **a** to find the slope of the tangent to $y = \sqrt{x}$ at the point where $x = 9$.

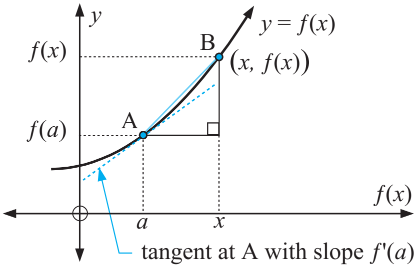
B DERIVATIVES AT A GIVEN x-VALUE

We are now at the stage where we can find slopes of tangents at any point on a simple curve using a limit method.

Notation: The slope of the tangent to a curve $y = f(x)$ at $x = a$ is $f'(a)$, read as '*eff dashed* a'.



Consider a general function $y = f(x)$, a fixed point A(a, $f(a)$) and a variable point B(x, $f(x)$).



The slope of chord AB $= \dfrac{f(x) - f(a)}{x - a}$.

Now as B $\to$ A, $x \to a$

and the slope of chord AB $\to$ slope of tangent at A

So, $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Thus $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ is the slope of the tangent at $x = a$ and is called **the derivative** at $x = a$.

Note:
- The slope of the tangent at $x = a$ is defined as the **slope of the curve** at the point where $x = a$, and is the instantaneous rate of change in y with respect to x at that point.
- Finding the slope using the limit method is said to be using **first principles**.

Example 2

Find, from first principles, the slope of the tangent to:

a $y = 2x^2 + 3$ at $x = 2$ **b** $y = 3 - x - x^2$ at $x = -1$

a $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ where $f(2) = 2(2)^2 + 3 = 11$

$\therefore f'(2) = \lim\limits_{x \to 2} \dfrac{2x^2 + 3 - 11}{x - 2}$

$= \lim\limits_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$

$= \lim\limits_{x \to 2} \dfrac{2(x+2)\cancel{(x-2)}^{1}}{\cancel{x-2}_{1}}$ {as $x \neq 2$}

$= 2 \times 4$

$= 8$

b $f'(-1) = \lim\limits_{x \to -1} \dfrac{f(x) - f(-1)}{x - (-1)}$ where $f(-1) = 3 - (-1) - (-1)^2$
$= 3 + 1 - 1$
$= 3$

$= \lim\limits_{x \to -1} \dfrac{\cancel{3} - x - x^2 - \cancel{3}}{x + 1}$

Note: $-f(-1) \neq f(1)$

$= \lim\limits_{x \to -1} \dfrac{-x - x^2}{x + 1}$

$= \lim\limits_{x \to -1} \dfrac{-x\cancel{(1+x)}^{1}}{\cancel{x+1}_{1}}$ {as $x \neq -1$}

$= 1$

EXERCISE 21B

1 Find, from first principles, the slope of the tangent to:

a $f(x) = 1 - x^2$ at $x = 2$

b $f(x) = 2x^2 + 5x$ at $x = -1$

c $f(x) = 5 - 2x^2$ at $x = 3$

d $f(x) = 3x + 5$ at $x = -2$

Example 3

Find, from first principles, the derivative of:

a $f(x) = \frac{9}{x}$ at $x = 2$ **b** $f(x) = \frac{2x-1}{x+3}$ at $x = -1$

a
$$\begin{aligned}
f'(2) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \\
&= \lim_{x \to 2} \left(\frac{\frac{9}{x} - \frac{9}{2}}{x - 2} \right) \\
&= \lim_{x \to 2} \left(\frac{\frac{9}{x} - \frac{9}{2}}{x - 2} \right) \frac{2x}{2x} && \{2x \text{ is the LCD of } \tfrac{9}{x} \text{ and } \tfrac{9}{2}\} \\
&= \lim_{x \to 2} \frac{18 - 9x}{2x(x-2)} && \{\text{Do not 'multiply out' the denominator.}\} \\
&= \lim_{x \to 2} \frac{-9\cancel{(x-2)}^{1}}{2x\cancel{(x-2)}_{1}} && \{\text{as } x \neq 2\} \\
&= -\tfrac{9}{4}
\end{aligned}$$

b
$$\begin{aligned}
f'(-1) &= \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} && \text{where } f(-1) = \frac{2(-1) - 1}{(-1) + 3} \\
&= \lim_{x \to -1} \left(\frac{\frac{2x-1}{x+3} + \frac{3}{2}}{x + 1} \right) && = -\tfrac{3}{2} \\
&= \lim_{x \to -1} \left(\frac{\frac{2x-1}{x+3} + \frac{3}{2}}{x + 1} \right) \times \frac{2(x+3)}{2(x+3)} \\
&= \lim_{x \to -1} \frac{2(2x-1) + 3(x+3)}{2(x+1)(x+3)} \\
&= \lim_{x \to -1} \frac{4x - 2 + 3x + 9}{2(x+1)(x+3)} \\
&= \lim_{x \to -1} \frac{7x + 7}{2(x+1)(x+3)} \\
&= \lim_{x \to -1} \frac{7\cancel{(x+1)}^{1}}{2\cancel{(x+1)}_{1}(x+3)} \\
&= \frac{7}{2(2)} \\
&= \tfrac{7}{4}
\end{aligned}$$

Note: There should always be cancelling of the original divisor at this step. Why?

2 Find, from first principles, the derivative of:

a $f(x) = \dfrac{4}{x}$ at $x = 2$

b $f(x) = -\dfrac{3}{x}$ at $x = -2$

c $f(x) = \dfrac{1}{x^2}$ at $x = 4$

d $f(x) = \dfrac{4x}{x-3}$ at $x = 2$

e $f(x) = \dfrac{4x+1}{x-2}$ at $x = 5$

f $f(x) = \dfrac{3x}{x^2+1}$ at $x = -4$

Example 4

Find, using first principles, the instantaneous rate of change in $y = \sqrt{x}$ at $x = 9$.

$f(x) = \sqrt{x}$ and $f(9) = \sqrt{9} = 3$

$$\begin{aligned}
\text{Now} \quad f'(9) &= \lim_{x\to 9} \frac{f(x) - f(9)}{x - 9} \\
&= \lim_{x\to 9} \frac{\sqrt{x} - 3}{x - 9} \\
&= \lim_{x\to 9} \frac{\cancel{\sqrt{x} - 3}^{\,1}}{(\sqrt{x} + 3)\cancel{(\sqrt{x} - 3)}_{\,1}} \quad \{\text{treating } x - 9 \text{ as the difference of two squares, } x \neq 9\} \\
&= \frac{1}{\sqrt{9} + 3} \\
&= \tfrac{1}{6}
\end{aligned}$$

3 Find, from first principles, the instantaneous rate of change in:

a $\sqrt{x}$ at $x = 4$

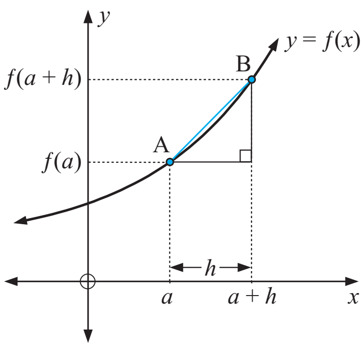
b $\sqrt{x}$ at $x = \frac{1}{4}$

c $\dfrac{2}{\sqrt{x}}$ at $x = 9$

d $\sqrt{x - 6}$ at $x = 10$

An alternative formula for finding $f'(a)$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$



slope of AB $= \dfrac{f(a+h) - f(a)}{h}$

Note that as $B \to A$, $h \to 0$

and $f'(a) = \lim\limits_{h\to 0}$ (slope of AB)

which justifies the alternative formula.

Example 5

Use the first principles formula $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ to find:

a the slope of the tangent to $f(x)=x^2+2x$ at $x=5$

b the instantaneous rate of change of $f(x)=\dfrac{4}{x}$ at $x=-3$

a $f'(5)=\lim\limits_{h\to 0}\dfrac{f(5+h)-f(5)}{h}$ where $f(5)=5^2+2(5)=35$

$$\begin{aligned}
&=\lim_{h\to 0}\frac{(5+h)^2+2(5+h)-35}{h}\\
&=\lim_{h\to 0}\frac{\cancel{25}+10h+h^2+\cancel{10}+2h-\cancel{35}}{h}\\
&=\lim_{h\to 0}\frac{h^2+12h}{h}\\
&=\lim_{h\to 0}\frac{\cancel{h}^{1}(h+12)}{\cancel{h}_{1}} \qquad \{\text{as } h\neq 0\}\\
&=12
\end{aligned}$$

and so the slope of the tangent at $x=5$ is 12.

b $f'(-3)=\lim\limits_{h\to 0}\dfrac{f(-3+h)-f(-3)}{h}$ where $f(-3)=\frac{4}{-3}=-\frac{4}{3}$

$$\begin{aligned}
&=\lim_{h\to 0}\left(\frac{\frac{4}{-3+h}+\frac{4}{3}}{h}\right)\\
&=\lim_{h\to 0}\left(\frac{\frac{4}{h-3}+\frac{4}{3}}{h}\right)\times\frac{3(h-3)}{3(h-3)}\\
&=\lim_{h\to 0}\frac{12+4(h-3)}{3h(h-3)}\\
&=\lim_{h\to 0}\frac{4\cancel{h}^{1}}{3\cancel{h}_{1}(h-3)} \qquad \{\text{as } h\neq 0\}\\
&=-\tfrac{4}{9}
\end{aligned}$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x=-3$ is $-\frac{4}{9}$.

4 Use the first principles formula $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ to find:

a the slope of the tangent to $f(x)=x^2+3x-4$ at $x=3$

b the slope of the tangent to $f(x)=5-2x-3x^2$ at $x=-2$

c the instantaneous rate of change in $f(x)=\dfrac{1}{2x-1}$ at $x=-2$

d the slope of the tangent to $f(x) = \dfrac{1}{x^2}$ at $x = 3$

e the instantaneous rate of change in $f(x) = \sqrt{x}$ at $x = 4$

f the instantaneous rate of change in $f(x) = \dfrac{1}{\sqrt{x}}$ at $x = 1$

5 Using $f'(a) = \lim\limits_{h \to a} \dfrac{f(a+h) - f(a)}{h}$ find:

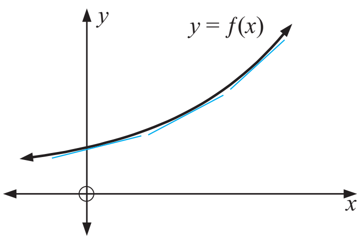
a $f'(2)$ for $f(x) = x^3$ **b** $f'(3)$ for $f(x) = x^4$

Reminder: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

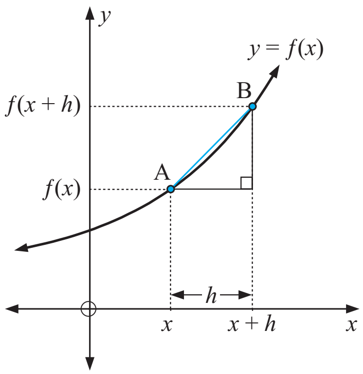
C THE DERIVATIVE FUNCTION

For a non-linear function with equation $y = f(x)$, slopes of tangents at various points continually change.

Our task is to determine a **slope function** so that when we replace x by a, say, we will be able to find the slope of the tangent at $x = a$.



Consider a general function $y = f(x)$ where A is $(x, f(x))$ and B is $(x+h, f(x+h))$.



The chord AB has slope $= \dfrac{f(x+h) - f(x)}{x + h - x}$

$= \dfrac{f(x+h) - f(x)}{h}$.

If we now let B move closer to A, the slope of AB approaches the slope of the tangent at A.

So, the slope of the tangent at the variable point $(x, f(x))$ is the limiting value of

$\dfrac{f(x+h) - f(x)}{h}$ as h approaches 0.

Since this slope contains the variable x it is called a **slope function.**

DERIVATIVE FUNCTION

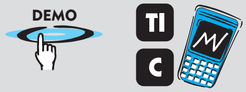
The **slope function**, also known as the **derived function**, or **derivative function** or simply the **derivative** is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

[**Note:** $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ is the shorthand way of writing
"the limiting value of $\dfrac{f(x+h)-f(x)}{h}$ as h gets as close as we like to zero."]

INVESTIGATION 2 FINDING SLOPES OF FUNCTIONS WITH TECHNOLOGY

This investigation can be done by **graphics calculator** or by clicking on the icon to open the **demonstration**. The idea is to find slopes at various points on a simple curve in order to find and table x-coordinates of points and the slopes of the tangents at those points. From this table you should be able to predict or find the slope function for the curve.



What to do:

1 By using a graphical argument only, explain why:

a for $f(x)=c$ where c is a constant, $f'(x)=0$

b for $f(x)=mx+c$ where m and c are constants, $f'(x)=m$.

2 Consider $f(x)=x^2$. Find $f'(x)$ for $x=1, 2, 3, 4, 5, 6$ using technology. Predict $f'(x)$ from your results.

3 Use technology and modelling techniques to find $f'(x)$ for:

a $f(x)=x^3$ **b** $f(x)=x^4$ **c** $f(x)=x^5$

d $f(x)=\dfrac{1}{x}$ **e** $f(x)=\dfrac{1}{x^2}$ **f** $f(x)=\sqrt{x}=x^{\frac{1}{2}}$

4 Use the results of **3** to complete the following:

"if $f(x)=x^n$, then $f'(x)=$ ......"

Unfortunately the way of finding slope functions by the method shown in the investigation is insufficient for more complicated functions. Consequently, we need to use the **slope function** definition, but even this method is limited to relatively simple functions.

Example 6

Find, from first principles, the slope function of $f(x)=x^2$.

If $f(x)=x^2$,
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} \qquad \{\text{as } h\neq 0\} \\ &= 2x \end{aligned}$$

EXERCISE 21C

1 Find, from first principles, the slope function of $f(x)$ where $f(x)$ is:

a x **b** 5 **c** x^3 **d** x^4

[**Reminder:** $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$]

2 Find, from first principles, $f'(x)$ given that $f(x)$ is:

a $2x+5$ **b** $x^2 - 3x$ **c** $x^3 - 2x^2 + 3$

Example 7

Find, from first principles, $f'(x)$ if $f(x) = \dfrac{1}{x}$.

If $f(x) = \dfrac{1}{x}$,
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{\frac{1}{x+h} - \frac{1}{x}}{h}\right] \times \frac{(x+h)x}{(x+h)x} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{\cancel{-h}^{\,-1}}{\cancel{h}_{1}\,x(x+h)} \qquad \{\text{as } h \neq 0\} \\ &= -\frac{1}{x^2} \qquad \{\text{as } h \to 0,\ x+h \to x\} \end{aligned}$$

3 Find, from first principles, the derivative of $f(x)$ when $f(x)$ is:

a $\dfrac{1}{x+2}$ **b** $\dfrac{1}{2x-1}$ **c** $\dfrac{1}{x^2}$ **d** $\dfrac{1}{x^3}$

Example 8

Find, from first principles, the slope function of $f(x) = \sqrt{x}$.

If $f(x) = \sqrt{x}$,
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{\cancel{h}^{\,1}}{\cancel{h}_{1}(\sqrt{x+h} + \sqrt{x})} \qquad \{\text{as } h \neq 0\} \end{aligned}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

4 Find, from first principles, the derivative of $f(x)$ equal to:

a $\sqrt{x+2}$ **b** $\dfrac{1}{\sqrt{x}}$ **c** $\sqrt{2x+1}$

5 Using the results of derivatives in this exercise, copy and complete:

Function	*Derivative (in form* kx^n*)*
x	
x^2	$2x = 2x^1$
x^3	
x^4	
x^{-1}	
x^{-2}	
x^{-3}	
$x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$
$x^{-\frac{1}{2}}$	

Use your table to predict a formula for $f'(x)$ given that $f(x) = x^n$ where n is rational.

D SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding the derivative (i.e., slope function).

Notation: If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

However, if we are given y in terms of x then y' or $\dfrac{dy}{dx}$ are commonly used to represent the derivative.

Note:
- $\dfrac{dy}{dx}$ reads "dee y by dee x", or " the derivative of y with respect to x".
- $\dfrac{dy}{dx}$ is **not a fraction**.
- $\dfrac{d(.....)}{dx}$ reads "the derivative of (....) with respect to x.

From question **5** of the previous exercise you should have discovered that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Are there other rules like this one which can be used to differentiate more complicated functions without having to resort to the tedious limit method? In the following investigation we may discover some additional rules.

INVESTIGATION 3 SIMPLE RULES OF DIFFERENTIATION

In this investigation we attempt to differentiate functions of the form cx^n where c is a constant, and functions which are a sum (or difference) of terms of the form cx^n.

What to do:

1 Find, from first principles, the derivatives of:

a $4x^2$ **b** $2x^3$ **c** $5\sqrt{x}$

2 Compare your results with the derivatives of x^2, x^3 and $\sqrt{x}$ obtained earlier.

Copy and complete: "If $f(x) = cx^n$, then $f'(x) = ...$"

3 Use first principles to find $f'(x)$ for:

a $f(x) = x^2 + 3x$ **b** $f(x) = x^3 - 2x^2$

4 Use **3** to copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = ...$"

You should have discovered the following rules for differentiating functions.

Rules

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$cu'(x)$	**constant times a function**
$u(x) + v(x)$	$u'(x) + v'(x)$	**sum rule**

Each of these rules can be proved using the first principles definition of $f'(x)$.

The following proofs are worth examining.

- If $f(x) = cu(x)$ where c is a constant then $f'(x) = cu'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{cu(x+h) - cu(x)}{h} \\ &= \lim_{h \to 0} c\left[\frac{u(x+h) - u(x)}{h}\right] \\ &= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{u(x+h) + v(x+h) - [u(x) + v(x)]}{h}\right) \end{aligned}$$

$$= \lim_{h \to 0} \left(\frac{u(x+h) - u(x) + v(x+h) - v(x)}{h} \right)$$
$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h}$$
$$= u'(x) + v'(x)$$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$f'(x) = 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0$$
$$= 12x^3 + 6x^2 - 10x + 7$$

Example 9

Find $f'(x)$ for $f(x)$ equal to: **a** $5x^3 + 6x^2 - 3x + 2$ **b** $7x - \dfrac{4}{x} + \dfrac{3}{x^3}$

a
$$f(x) = 5x^3 + 6x^2 - 3x + 2$$
$$\therefore \quad f'(x) = 5(3x^2) + 6(2x) - 3(1) + 0$$
$$= 15x^2 + 12x - 3$$

b
$$f(x) = 7x - \frac{4}{x} + \frac{3}{x^3}$$
$$= 7x - 4x^{-1} + 3x^{-3} \qquad \{\text{each term is in the form } cx^n\}$$
$$\therefore \quad f'(x) = 7(1) - 4(-1x^{-2}) + 3(-3x^{-4})$$
$$= 7 + 4x^{-2} - 9x^{-4}$$
$$= 7 + \frac{4}{x^2} - \frac{9}{x^4}$$

EXERCISE 21D

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$

d $x^2 + x$ **e** $4 - 2x^2$ **f** $x^2 + 3x - 5$

g $x^3 + 3x^2 + 4x - 1$ **h** $5x^4 - 6x^2$ **i** $\dfrac{3x - 6}{x}$

j $\dfrac{2x - 3}{x^2}$ **k** $\dfrac{x^3 + 5}{x}$ **l** $\dfrac{x^3 + x - 3}{x}$

2 Find $f'(x)$ given that $f(x)$ is:

a $\frac{1}{4}x^4$ **b** $x + \dfrac{1}{x}$ **c** $\dfrac{x + 1}{x}$

d $\dfrac{x^2 + 5}{x^3}$ **e** $(x + 1)(x - 2)$ **f** $\dfrac{1}{x^2} + 6\sqrt{x}$

g $\dfrac{1}{\sqrt{x}}$ **h** $(2x - 1)^2$ **i** $(x + 2)^3$

3 Find $\dfrac{dy}{dx}$ for:

a $y = 2x^3 - 7x^2 - 1$ **b** $y = \pi x^2$ **c** $y = \dfrac{1}{5x^2}$

d $y = 100x$ **e** $y = 10(x+1)$ **f** $y = 4\pi x^3$

4 Differentiate with respect to x:

a $6x + 2$ **b** $x\sqrt{x}$ **c** $(5-x)^2$

d $\dfrac{6x^2 - 9x^4}{3x}$ **e** $4x - \dfrac{1}{4x}$ **f** $x(x+1)(2x-5)$

Example 10

Find the slope function of $f(x) = x^2 - \dfrac{4}{x}$ and hence find the slope of the tangent to the function at the point where $x = 2$.

$f(x) = x^2 - \dfrac{4}{x} = x^2 - 4x^{-1}$

$$\begin{aligned}\therefore\quad f'(x) &= 2x - 4(-1x^{-2}) \\ &= 2x + 4x^{-2} \\ &= 2x + \frac{4}{x^2}\end{aligned}$$

Substituting $x = 2$ into the slope function gives the slope of the tangent at $x = 2$.
So, as $f'(2) = 4 + 1 = 5$, the tangent has slope of 5.

5 Find the slope of the tangent to:

a $y = x^2$ at $x = 2$ **b** $y = \dfrac{8}{x^2}$ at $x = 9$

c $y = 2x^2 - 3x + 7$ at $x = -1$ **d** $y = \dfrac{2x^2 - 5}{x}$ at $x = 2$

e $y = \dfrac{x^2 - 4}{x^2}$ at $x = 4$ **f** $y = \dfrac{x^3 - 4x - 8}{x^2}$ at $x = -1$

Example 11

Find the slope function of $f(x)$ where $f(x)$ is:

a $3\sqrt{x} + \dfrac{2}{x}$ **b** $x^2 - \dfrac{4}{\sqrt{x}}$ **c** $1 - x\sqrt{x}$

a
$$\begin{aligned} f(x) &= 3\sqrt{x} + \frac{2}{x} = 3x^{\frac{1}{2}} + 2x^{-1} \\ \therefore\quad f'(x) &= 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2}) \\ &= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2} \\ &= \frac{3}{2\sqrt{x}} - \frac{2}{x^2}\end{aligned}$$

b $f(x) = x^2 - \dfrac{4}{\sqrt{x}} = x^2 - 4x^{-\frac{1}{2}}$

$\therefore\ f'(x) = 2x - 4(-\frac{1}{2}x^{-\frac{3}{2}})$

$= 2x + 2x^{-\frac{3}{2}}$

$= 2x + \dfrac{2}{x\sqrt{x}}$

c $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$

$\therefore\ f'(x) = 0 - \frac{3}{2}x^{\frac{1}{2}}$

$= -\frac{3}{2}\sqrt{x}$

6 Find the slope function of $f(x)$ where $f(x)$ is:

a $4\sqrt{x} + x$ **b** $\sqrt[3]{x}$ **c** $-\dfrac{2}{\sqrt{x}}$ **d** $2x - \sqrt{x}$

e $\dfrac{4}{\sqrt{x}} - 5$ **f** $3x^2 - x\sqrt{x}$ **g** $\dfrac{5}{x^2\sqrt{x}}$ **h** $2x - \dfrac{3}{x\sqrt{x}}$

Example 12

If $y = 3x^2 - 4x$, find $\dfrac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\dfrac{dy}{dx} = 6x - 4$.

$\dfrac{dy}{dx}$ is
- the slope function or derivative of $y = 3x^2 - 4x$ from which the slope at any point can be found
- the instantaneous rate of change in y as x changes.

7 **a** If $y = 4x - \dfrac{3}{x}$, find $\dfrac{dy}{dx}$ and interpret its meaning.

b The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds. Find $\dfrac{dS}{dt}$ and interpret its meaning.

c The cost of producing and selling x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ dollars. Find $\dfrac{dC}{dx}$ and interpret its meaning.

E THE CHAIN RULE

Composite functions are functions like $(x^2 + 3x)^4$, $\sqrt{2 - 3x}$ or $\dfrac{1}{x - x^2}$.

These functions are made up of two simpler functions.

- $y = (x^2 + 3x)^4$ is $y = u^4$ where $u = x^2 + 3x$
- $y = \sqrt{2 - 3x}$ is $y = \sqrt{u}$ where $u = 2 - 3x$
- $y = \dfrac{1}{x - x^2}$ is $y = \dfrac{1}{u}$ where $u = x - x^2$

Notice that in the example $(x^2+3x)^4$, if $f(x) = x^4$ and $g(x) = x^2 + 3x$ then

$$\begin{aligned} f(g(x)) &= f(x^2+3x) \\ &= (x^2+3x)^4 \end{aligned}$$

All of these functions can be made up in this way where we compose a function of a function.

Consequently, these functions are called **composite functions**.

Example 13

a If $f(x) = 3x^2$ and $g(x) = 3x + 7$, find $f(g(x))$.

b If $f(g(x)) = \sqrt{3 - x^2}$, find $f(x)$ and $g(x)$.

a If $f(x) = 3x^2$ and $g(x) = 3x + 7$ then

$$\begin{aligned} f(g(x)) &= f(3x+7) && \{\text{replacing } g(x) \text{ by } 3x+7\} \\ &= 3(3x+7)^2 && \{\text{replacing } x \text{ in the } f \text{ function by } (3x+7)\} \end{aligned}$$

b If $f(g(x)) = \sqrt{3-x^2}$ then $f(x) = \sqrt{x}$ and $g(x) = 3 - x^2$.

EXERCISE 21E.1

1 Find $f(g(x))$ if:

a $f(x) = x^2$ and $g(x) = 2x + 7$

b $f(x) = 2x + 7$ and $g(x) = x^2$

c $f(x) = \sqrt{x}$ and $g(x) = 3 - 4x$

d $f(x) = 3 - 4x$ and $g(x) = \sqrt{x}$

e $f(x) = \dfrac{2}{x}$ and $g(x) = x^2 + 3$

f $f(x) = x^2 + 3$ and $g(x) = \dfrac{2}{x}$

g $f(x) = 2^x$ and $g(x) = 3x + 4$

h $f(x) = 3x + 4$ and $g(x) = 2^x$

2 Find $f(x)$ and $g(x)$ given that $f(g(x))$ is:

a $(3x+10)^3$

b $\dfrac{1}{2x+4}$

c $\sqrt{x^2-3x}$

d $\dfrac{1}{\sqrt{5-2x}}$

e $(x^2+5x-1)^4$

f $\dfrac{10}{(3x-x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

INVESTIGATION 4 DIFFERENTIATING COMPOSITES

The purpose of this investigation is to gain insight into how we can differentiate composite functions.

We might suspect that if $y = (2x+1)^2$ then $\dfrac{dy}{dx} = 2(2x+1)^1 = 2(2x+1)$

based on our previous rule "if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$". But is this so?

What to do:

1. Consider $y=(2x+1)^2$. Expand the brackets and then find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx}=2(2x+1)$?
2. Consider $y=(3x+1)^2$. Expand the brackets and then find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx}=2(3x+1)^1$?
3. Consider $y=(ax+1)^2$. Expand the brackets and find $\dfrac{dy}{dx}$. Is $\dfrac{dy}{dx}=2(ax+1)^1$?
4. If $y=u^2$ where u is a function of x, what do you suspect $\dfrac{dy}{dx}$ will be equal to?
5. Consider $y=(x^2+3x)^2$. Expand it and find $\dfrac{dy}{dx}$.

   Does your answer agree with your suspected rule in **4**?

From the previous investigation you probably formulated the rule that:

$$\text{If } y=u^2 \text{ then } \frac{dy}{dx}=2u\times\frac{du}{dx}=\frac{dy}{du}\frac{du}{dx}.$$

Now consider $y=(2x+1)^3$ which is really $y=u^3$ where $u=2x+1$.

Expanding we have
$$\begin{aligned} y &= (2x+1)^3 \\ &= (2x)^3+3(2x)^2 1+3(2x)1^2+1^3 \quad \{\text{binomial expansion}\} \\ &= 8x^3+12x^2+6x+1 \\ \therefore \frac{dy}{dx} &= 24x^2+24x+6 \\ &= 6(4x^2+4x+1) \\ &= 6(2x+1)^2 \\ &= 3(2x+1)^2\times 2 \\ &= 3u^2\times\frac{du}{dx} \quad \text{which is again } \frac{dy}{du}\frac{du}{dx}. \end{aligned}$$

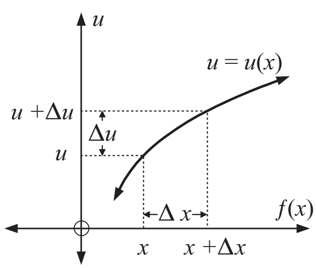
From the investigation and from the above example we formulate the **chain rule**.

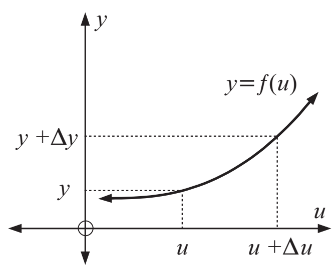
$$\text{If } y=f(u) \text{ where } u=u(x) \text{ then } \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}.$$

A non-examinable proof of this rule is included for completeness.

Proof: Consider $y=f(u)$ where $u=u(x)$.

For a small change of Δx in x, there is a small change of $u(x+h)-u(x)=\Delta u$ in u and a small change of Δy in y.





Now $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \times \dfrac{\Delta u}{\Delta x}$ {fraction multiplication}

Now as $\Delta x \to 0$, $\Delta u \to 0$ also.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \{\text{limit rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

{If in $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, we replace h by Δx and $f(x+h) - f(x)$ by Δy, we have $f'(x) = \dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.}

Example 14

Find $\dfrac{dy}{dx}$ if: **a** $y = (x^2 - 2x)^4$ **b** $y = \dfrac{4}{\sqrt{1-2x}}$

a

$$y = (x^2 - 2x)^4$$

$$\therefore y = u^4 \quad \text{where} \quad u = x^2 - 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$= 4u^3(2x - 2)$$

$$= 4(x^2 - 2x)^3(2x - 2)$$

b

$$y = \frac{4}{\sqrt{1-2x}}$$

$$\therefore y = \frac{4}{\sqrt{u}} \quad \text{where} \quad u = 1 - 2x$$

$$\text{i.e., } y = 4u^{-\frac{1}{2}} \quad \text{where} \quad u = 1 - 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$= 4 \times (-\tfrac{1}{2}u^{-\frac{3}{2}}) \times (-2)$$

$$= 4u^{-\frac{3}{2}} = 4(1-2x)^{-\frac{3}{2}}$$

Note: If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

EXERCISE 21E.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$ **b** $\sqrt{x^2 - 3x}$ **c** $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3 - x^2}$ **e** $\dfrac{4}{(3-x)^3}$ **f** $\dfrac{10}{x^2-3}$

2 Find the slope function $\dfrac{dy}{dx}$ for:

a $y=(4x-5)^2$ **b** $y=\dfrac{1}{5-2x}$ **c** $y=\sqrt{3x-x^2}$

d $y=(1-3x)^4$ **e** $y=6(5-x)^3$ **f** $y=\sqrt[3]{2x^3-x^2}$

g $y=\dfrac{6}{(5x-4)^2}$ **h** $y=\dfrac{4}{3x-x^2}$ **i** $y=2\left(x^2-\dfrac{2}{x}\right)^3$

3 Find the slope of the tangent to:

a $y=\sqrt{1-x^2}$ at $x=\frac{1}{2}$ **b** $y=(3x+2)^6$ at $x=-1$

c $y=\dfrac{1}{(2x-1)^4}$ at $x=1$ **d** $y=6\times\sqrt[3]{1-2x}$ at $x=0$

e $y=\dfrac{4}{x+2\sqrt{x}}$ at $x=4$ **f** $y=\left(x+\dfrac{1}{x}\right)^3$ at $x=1$

4 If $y=x^3$ then $x=y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ and hence show that $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1$.

b Explain why $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1$ whenever these derivatives exist for any general function $y=f(x)$.

F PRODUCT AND QUOTIENT RULES

If $f(x)=u(x)+v(x)$ then $f'(x)=u'(x)+v'(x)$.

That is, the derivative of a sum of two functions is the sum of the derivatives.

But, what if $f(x)=u(x)v(x)$? Is $f'(x)=u'(x)v'(x)$?

That is, is the derivative of a product of two functions equal to the product of the derivatives of the two functions?

The following example shows that this cannot be true:

If $f(x)=x\sqrt{x}$ we could say $f(x)=u(x)v(x)$ where $u(x)=x$ and $v(x)=\sqrt{x}$.

Now $f(x)=x^{\frac{3}{2}}$ $\therefore$ $f'(x)=\frac{3}{2}x^{\frac{1}{2}}$.

But $u'(x)v'(x)=1\times\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2}x^{-\frac{1}{2}}\neq f'(x)$

THE PRODUCT RULE

If $u(x)$ and $v(x)$ are two functions of x and $y=uv$ then

$$\frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx} \quad \text{or} \quad y'=u'(v)\,v(x)+u(x)\,v'(x).$$

Consider the example $f(x) = x\sqrt{x}$ again.

This is a product $u(x)v(x)$ where $u(x) = x$ and $v(x) = x^{\frac{1}{2}}$

$\therefore \quad u'(x) = 1$ and $v'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

According to the product rule

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= 1 \times x^{\frac{1}{2}} + x \times \tfrac{1}{2}x^{-\frac{1}{2}}. \\ &= x^{\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}} \\ &= \tfrac{3}{2}x^{\frac{1}{2}} \quad \text{which is correct} \quad \checkmark \end{aligned}$$

For completeness we now prove the product rule.

Proof: Let $y = u(x)v(x)$ and consider the effect of a small change in x of Δx.

Corresponding changes of Δu in u, Δv in v and Δy in y occur and as $y = uv$,

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$\therefore \quad y + \Delta y = uv + (\Delta u)v + u(\Delta v) + \Delta u \Delta v$$

$$\Delta y = (\Delta u)v + u(\Delta v) + \Delta u \Delta v$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \left(\frac{\Delta u}{\Delta x}\right)v + u\left(\frac{\Delta v}{\Delta x}\right) + \left(\frac{\Delta u}{\Delta x}\right)\Delta v \qquad \{\text{dividing each term by } \Delta x\}$$

$$\therefore \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right)v + u\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) + 0 \qquad \{\text{as } \Delta x \to 0,\ \Delta v \to 0 \text{ also}\}$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

Example 15

Find $\dfrac{dy}{dx}$ if: **a** $y = \sqrt{x}(2x+1)^3$ **b** $y = x^2(x^2 - 2x)^4$

a $y = \sqrt{x}(2x+1)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (2x+1)^3$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(2x+1)^2 \times 2 = 6(2x+1)^2$

Now
$$\begin{aligned} \frac{dy}{dx} &= u'v + uv' \qquad \{\text{product rule}\} \\ &= \tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + x^{\frac{1}{2}} \times 6(2x+1)^2 \\ &= \tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2 \end{aligned}$$

b $y = x^2(x^2 - 2x)^4$ is the product of $u = x^2$ and $v = (x^2 - 2x)^4$

$\therefore \quad u' = 2x$ and $v' = 4(x^2 - 2x)^3(2x - 2)$

Now
$$\begin{aligned} \frac{dy}{dx} &= u'v + uv' \qquad \{\text{product rule}\} \\ &= 2x(x^2 - 2x)^4 + x^2 \times 4(x^2 - 2x)^3(2x - 2) \\ &= 2x(x^2 - 2x)^4 + 4x^2(x^2 - 2x)^3(2x - 2) \end{aligned}$$

EXERCISE 21F.1

1 Find $\dfrac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$ **b** $y = 4x(2x+1)^3$ **c** $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$ **e** $y = 5x^2(3x^2-1)^2$ **f** $y = \sqrt{x}(x-x^2)^3$

2 Find the slope of the tangent to:

a $y = x^4(1-2x)^2$ at $x=-1$ **b** $y = \sqrt{x}(x^2-x+1)^2$ at $x=4$

c $y = x\sqrt{1-2x}$ at $x=-4$ **d** $y = x^3\sqrt{5-x^2}$ at $x=1$

3 If $y = \sqrt{x}(3-x)^2$ show that $\dfrac{dy}{dx} = \dfrac{(3-x)(3-5x)}{2\sqrt{x}}$.

Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

THE QUOTIENT RULE

Expressions like $\dfrac{x^2+1}{2x-5}$, $\dfrac{\sqrt{x}}{1-3x}$ and $\dfrac{x^3}{(x-x^2)^4}$ are called **quotients**.

Quotient functions have form $Q(x) = \dfrac{u(x)}{v(x)}$.

Notice that $u(x) = Q(x)v(x)$ and by the product rule,

$$u'(x) = Q'(x)v(x) + Q(x)v'(x)$$

$$\therefore \quad u'(x) - Q(x)v'(x) = Q'(x)v(x)$$

$$\text{i.e.,} \quad Q'(x)v(x) = u'(x) - \frac{u(x)}{v(x)}v'(x)$$

$$\therefore \quad Q'(x)v(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)}$$

$$\therefore \quad Q'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

and this formula is called the **quotient rule**.

So, if $Q(x) = \dfrac{u(x)}{v(x)}$ then $Q'(x) = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$

or if $y = \dfrac{u}{v}$ where u and v are functions of x then $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

Example 16

Use the quotient rule to find $\dfrac{dy}{dx}$ if: **a** $y = \dfrac{1+3x}{x^2+1}$ **b** $y = \dfrac{\sqrt{x}}{(1-2x)^2}$

a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore \quad u' = 3 \quad$ and $\quad v' = 2x$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$

$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times -2$
$\qquad\qquad = -4(1-2x)$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times -4(1-2x)}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{\cancel{(1-2x)}\left[\dfrac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\dfrac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}\,3}} \qquad \{\text{look for common factors}\}$$

$$= \frac{1 - 2x + 8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Note: Most of the time, simplification of $\dfrac{dy}{dx}$ as in the above example is unnecessary, especially if you want to find the slope of a tangent at a given point, because you can substitute a value for x without simplifying.

EXERCISE 21F.2

1 Use the quotient rule to find $\dfrac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{2-x}$ **b** $y = \dfrac{x^2}{2x+1}$ **c** $y = \dfrac{x}{x^2-3}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ **e** $y = \dfrac{x^2-3}{3x-x^2}$ **f** $y = \dfrac{x}{\sqrt{1-3x}}$

2 Find the slope of the tangent to:

a $y = \dfrac{x}{1-2x}$ at $x = 1$

b $y = \dfrac{x^3}{x^2+1}$ at $x = -1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ at $x = 4$

d $y = \dfrac{x^2}{\sqrt{x^2+5}}$ at $x = -2$

3 **a** If $y = \dfrac{2\sqrt{x}}{1-x}$, show that $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$.

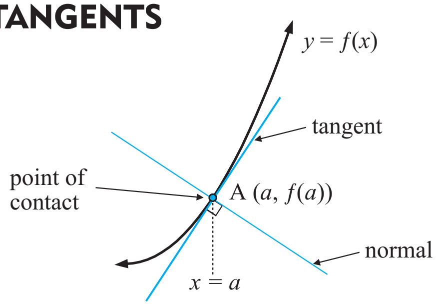
For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

b If $y = \dfrac{x^2-3x+1}{x+2}$, show that $\dfrac{dy}{dx} = \dfrac{x^2+4x-7}{(x+2)^2}$.

For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

G TANGENTS AND NORMALS

TANGENTS



Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the slope of the tangent at this point is $f'(a)$.

The equation of the tangent is

$$\frac{y - f(a)}{x - a} = f'(a) \qquad \{\text{equating slopes}\}$$

or $y - f(a) = f'(a)(x - a)$

NORMALS

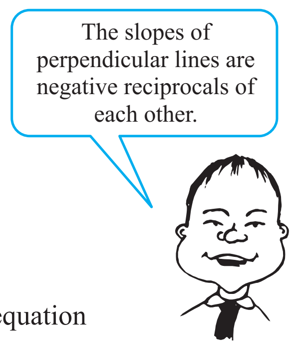
A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

Thus, the slope of a normal at $x = a$ is $-\dfrac{1}{f'(a)}$.

For example, if $f(x) = x^2$ then $f'(x) = 2x$.

At $x = 2$, $f'(2) = 4$ and $-\dfrac{1}{f'(2)} = -\frac{1}{4}$.

So, at $x = 2$ the tangent has slope 4 and the normal has slope $-\frac{1}{4}$.



Note:

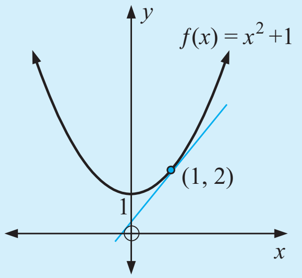
- If a tangent touches $y = f(x)$ at (a, b) then it has equation

$$\frac{y-b}{x-a} = f'(a) \quad \text{or} \quad y - b = f'(a)(x - a).$$

- Vertical and horizontal lines have equations $x = k$ and $y = c$ respectively.

Example 17

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1 + 1 = 2$, the point of contact is (1, 2).

Now $f'(x) = 2x$

$\therefore$ $f'(1) = 2$

$\therefore$ the tangent has equation

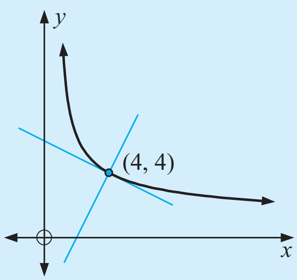
$$\frac{y-2}{x-1} = 2$$

i.e., $y - 2 = 2x - 2$

or $y = 2x$

Example 18

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.



When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$

$\therefore$ the point of contact is (4, 4).

Now as $y = 8x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = -4x^{-\frac{3}{2}}$$

and when $x = 4$, $\dfrac{dy}{dx} = -4 \times 4^{-\frac{3}{2}}$

$$= -\tfrac{1}{2}$$

$\therefore$ the normal at (4, 4) has slope $\frac{2}{1}$.

So, the equation of the normal is

$$\frac{y-4}{x-4} = 2$$

i.e., $y - 4 = 2x - 8$

i.e., $y = 2x - 4$.

EXERCISE 21G

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at (1, 4)

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at the point $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = 2\sqrt{x} + 3$ at $x = 1$

d $y = \dfrac{3}{\sqrt{x}}$ at the point where $x = 9$

e $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

f $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$

Example 19

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Let $f(x) = x^3 - 12x + 2$

$\therefore$ $f'(x) = 3x^2 - 12$

But $f'(x) = 0$ for horizontal tangents and so

$$3x^2 - 12 = 0$$
$$\therefore \quad 3(x^2 - 4) = 0$$
$$\therefore \quad 3(x+2)(x-2) = 0$$
$$\therefore \quad x = -2 \text{ or } 2$$

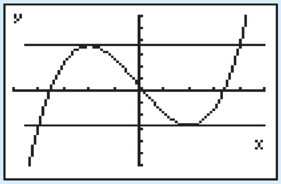
Now $f(2) = 8 - 24 + 2 = -14$ and

$f(-2) = -8 + 24 + 2 = 18$

i.e., points of contact are

$(2, -14)$ and $(-2, 18)$

$\therefore$ tangents are $y = -14$ and $y = 18$.



3 a Find the equations of the horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

b Find all points of contact of horizontal tangents to the curve $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

c Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has slope 4.

d Find the equation of the tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

4 a The tangent to the curve $y = x^2 + ax + b$, where a and b are constants, is $2x + y = 6$ at the point where $x = 1$. Find the values of a and b.

b The normal to the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$, where a and b are constants, has equation $4x + y = 22$ at the point where $x = 4$. Find the values of a and b.

Example 20

Find the equation of the tangent to $y = \sqrt{10-3x}$ at the point where $x = 3$.

Let $f(x) = (10-3x)^{\frac{1}{2}}$

$\therefore \quad f'(x) = \frac{1}{2}(10-3x)^{-\frac{1}{2}} \times (-3)$

$\therefore \quad f'(3) = \frac{1}{2}(1)^{-\frac{1}{2}} \times (-3)$

$= -\frac{3}{2}$

When $x = 3$, $y = \sqrt{10-9} = 1$

$\therefore$ point of contact is (3, 1).

So, the tangent has equation $\dfrac{y-1}{x-3} = -\dfrac{3}{2}$

i.e., $2y - 2 = -3x + 9$

or $3x + 2y = 11$

5 Find the equation of the:

a tangent to $y = \sqrt{2x+1}$ at the point where $x = 4$

b tangent to $y = \dfrac{1}{2-x}$ at the point where $x = -1$

c normal to $y = \dfrac{1}{(x^2+1)^2}$ at the point $(1, \frac{1}{4})$

d normal to $y = \dfrac{1}{\sqrt{3-2x}}$ at the point where $x = -3$

6 $y = a\sqrt{1-bx}$ where a and b are constants, has a tangent with equation $3x + y = 5$ at the point where $x = -1$. Find a and b.

Example 21

Find the equation of the normal to $y = \dfrac{x^2+1}{1-2x}$ at the point where $x = 1$

Let $f(x) = \dfrac{x^2+1}{1-2x}$

$f(1) = \frac{2}{-1} = -2$

$\therefore \quad f'(x) = \dfrac{2x(1-2x) - (x^2+1)(-2)}{(1-2x)^2}$ {quotient rule}

$\therefore \quad f'(1) = \dfrac{2(-1) - (2)(-2)}{(-1)^2} = \dfrac{-2+4}{1} = 2$

The point of contact is $(1, f(1))$, i.e., $(1, -2)$

$\therefore$ the equation of the normal is $\dfrac{y - -2}{x-1} = -\dfrac{1}{2}$

i.e., $2y + 4 = -x + 1$

i.e., $x + 2y = -3$

7 Find the equation of:

a the tangent to $f(x) = \dfrac{x}{1-3x}$ at the point $(-1, -\frac{1}{4})$

b the normal to $f(x) = \sqrt{x}(1-x)^2$ at the point where $x = 4$

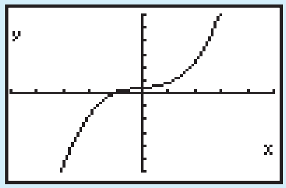
c the tangent to $f(x) = \dfrac{x^2}{1-x}$ at the point $(2, -4)$

d the normal to $f(x) = \dfrac{x^2-1}{2x+3}$ at the point where $x = -1$.

Example 22

Find the coordinates of the point(s) where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

$$f(x) = x^3 + x + 2$$
$$\therefore \quad f'(x) = 3x^2 + 1$$
$$\therefore \quad f'(1) = 3 + 1 = 4$$



$\therefore$ the tangent at $(1, 4)$ has slope 4

and therefore its equation is $\dfrac{y-4}{x-1} = 4$

i.e., $y - 4 = 4x - 4$

or $y = 4x$

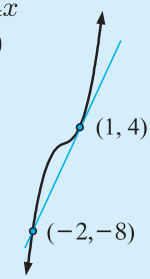
Now $y = 4x$ meets $y = x^3 + x + 2$ where $x^3 + x + 2 = 4x$

$\therefore \quad x^3 - 3x + 2 = 0$

and this cubic must have a repeated zero of $x = 1$ because of the tangent

$\therefore \quad (x-1)^2(x+2) = 0$

$x^2 \times x = x^3$ $\quad (-1)^2 \times 2 = 2$



$\therefore$ $x = 1$ or -2 and when $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ tangent meets the curve again at $(-2, -8)$.

8 **a** Find where the tangent to the curve $y = x^3$, at the point where $x = 2$, meets the curve again.

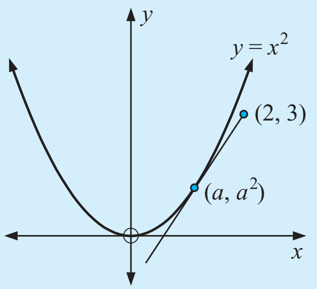
b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$, at the point where $x = -1$, meets the curve again.

c Find where the tangent to the curve $y = x^2 - \dfrac{3}{x}$, at $x = 3$, meets the curve again.

d Find where the tangent to the curve $y = x^3 + \dfrac{4}{x}$, at the point where $x = 1$, meets the curve again.

Example 23

Find the equations of the tangents to $y = x^2$ from the point (2, 3).



Let (a, a^2) lie on $f(x) = x^2$.

$$\text{Now} \quad f'(x) = 2x$$
$$\therefore \quad f'(a) = 2a$$

$\therefore$ at (a, a^2) the slope of the tangent is $\dfrac{2a}{1}$

$\therefore$ equation is $2ax - y = 2a(a) - (a^2)$

i.e., $2ax - y = a^2$.

But this tangent passes through (2, 3).

$$\therefore \quad 2a(2) - 3 = a^2$$
$$\therefore \quad 4a - 3 = a^2$$
$$\text{i.e.,} \quad a^2 - 4a + 3 = 0$$
$$(a-1)(a-3) = 0$$
$$\therefore \quad a = 1 \text{ or } 3$$

If $a = 1$, the tangent equation is $2x - y = 1$, with point of contact $(1, 1)$.

If $a = 3$, the tangent equation is $6x - y = 9$, with point of contact $(3, 9)$.

9 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$. Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

b Find the equations of the tangents to $y = x^3$ from the point $(-2, 0)$.

c Find the equation(s) of the normal(s) to $y = \sqrt{x}$ from the point $(4, 0)$.

H THE SECOND DERIVATIVE

The **second derivative** of a function $f(x)$ is the derivative of $f'(x)$, i.e., **the derivative of the first derivative**.

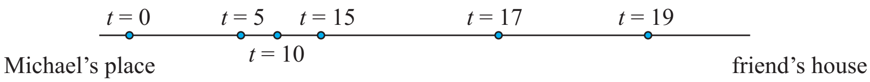
Notation: We use $f''(x)$, or y'' or $\dfrac{d^2y}{dx^2}$ to represent the second derivative.

Note that:

- $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$
- $\dfrac{d^2y}{dx^2}$ reads "*dee two y by dee x squared*".

THE SECOND DERIVATIVE IN CONTEXT

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.

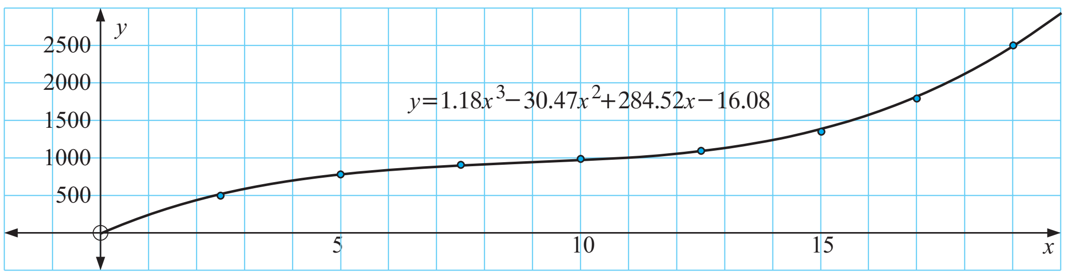


The distance travelled by Michael from his place is given at various times in the following table:

Time of ride (t min)	0	2.5	5	7.5	10	12.5	15	17	19
Distance travelled (s m)	0	498	782	908	989	1096	1350	1792	2500

A cubic model seems to fit this data well.
The model is $s \doteq 1.18t^3 - 30.47t^2 + 284.52t - 16.08$ metres.

Notice that the model gives $s(0) = -16.08$ m whereas the actual data gives $s(0) = 0$.
This sort of problem often occurs when modelling from data.



Now $\dfrac{ds}{dt} \doteq 3.54t^2 - 60.94t + 284.52$ metres/minute is the instantaneous rate of change in displacement per unit of time, i.e., instantaneous velocity.

The instantaneous rate of change in velocity at any point in time is the acceleration of the moving object and so,

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \quad \text{is the instantaneous acceleration,}$$

$$\text{i.e.,} \quad \frac{d^2s}{dt^2} = 7.08t - 60.94 \quad \text{metres/minute per minute.}$$

Notice that, when $t = 12$, $s \doteq 1050$ m

$$\frac{ds}{dt} \doteq 63 \quad \text{metres/minute}$$

$$\text{and} \quad \frac{d^2s}{dt^2} \doteq 24 \quad \text{metres/minute/minute}$$

We will examine displacement, velocity and acceleration in greater detail in the next chapter.

Example 24

Find $f''(x)$ given that $f(x) = x^3 - \dfrac{3}{x}$.

Now $f(x) = x^3 - 3x^{-1}$

$\therefore\ f'(x) = 3x^2 + 3x^{-2}$

$\therefore\ f''(x) = 6x - 6x^{-3}$

$= 6x - \dfrac{6}{x^3}$

EXERCISE 21H

1 Find $f''(x)$ given that:

a $f(x) = 3x^2 - 6x + 2$

b $f(x) = 2x^3 - 3x^2 - x + 5$

c $f(x) = \dfrac{2}{\sqrt{x}} - 1$

d $f(x) = \dfrac{2-3x}{x^2}$

e $f(x) = (1-2x)^3$

f $f(x) = \dfrac{x+2}{2x-1}$

2 Find $\dfrac{d^2y}{dx^2}$ given that:

a $y = x - x^3$

b $y = x^2 - \dfrac{5}{x^2}$

c $y = 2 - \dfrac{3}{\sqrt{x}}$

d $y = \dfrac{4-x}{x}$

e $y = (x^2 - 3x)^3$

f $y = x^2 - x + \dfrac{1}{1-x}$

3 Find x when $f''(x) = 0$ for:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$

b $f(x) = \dfrac{x}{x^2+2}$.

REVIEW SET 21A

1 Find the equation of the tangent to $y = -2x^2$ at the point where $x = -1$.

2 Find $\dfrac{dy}{dx}$ for:

a $y = 3x^2 - x^4$

b $y = \dfrac{x^3 - x}{x^2}$

3 Find, from first principles, the derivative of $f(x) = x^2 + 2x$.

4 Find the equation of the normal to $y = \dfrac{1-2x}{x^2}$ at the point where $x = 1$.

5 Find where the tangent to $y = 2x^3 + 4x - 1$ at (1, 5) cuts the curve again.

6 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at the point where $x = 1$ is $2x - y = 1$.
Find a and b.

7 Find a given that the tangent to $y = \dfrac{4}{(ax+1)^2}$ at $x = 0$ passes through the point (1, 0).

8 Find the equation of the normal to $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$.

9 Determine the derivative with respect to t of:

a $M = (t^2+3)^4$ **b** $A = \dfrac{\sqrt{t+5}}{t^2}$

10 Use the rules of differentiation to find $\dfrac{dy}{dx}$ for:

a $y = \dfrac{4}{\sqrt{x}} - 3x$ **b** $y = (x - \frac{1}{x})^4$ **c** $y = \sqrt{x^2 - 3x}$

REVIEW SET 21B

1 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2+x)^4$ **c** $(x^2+1)(1-x^2)^3$.

2 Determine the equation of any horizontal tangents to the curve with equation $y = x^3 - 3x^2 - 9x + 2$.

3 Find the equation of the normal to $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$.

4 Differentiate with respect to x:

a $f(x) = \dfrac{(x+3)^3}{\sqrt{x}}$ **b** $f(x) = x^4\sqrt{x^2+3}$

5 Differentiate with respect to x:

a $\sqrt{x}(1-x)^2$ **b** $\sqrt{3x - x^2}$ **c** $\dfrac{1}{2-x}$

6 Find $f''(x)$ for:

a $f(x) = 3x^2 - \dfrac{1}{x}$ **b** $f(x) = \sqrt{x}$

7 The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at A and B. Determine the area of triangle OAB.

8 $y = 2x$ is a tangent to the curve $y = x^3 + ax + b$ at $x = 1$. Find a and b.

9 The tangent to $y = x^3 + ax^2 - 4x + 3$ at $x = 1$ is parallel to the line $y = 3x$. Find the value of a and the equation of the tangent at $x = 1$. Where does the tangent cut the curve again?

10 The curve $f(x) = 2x^3 + Ax + B$ has a tangent with slope 10 at the point $(-2, 33)$. Find the values of A and B.

REVIEW SET 21C

1 Differentiate with respect to x:

a $y = x^3\sqrt{1-x^2}$ **b** $y = \dfrac{x^2-3x}{\sqrt{x+1}}$

2 Find the equation of the normal to $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$.

3 Determine the values of x for which $f''(x) = 0$ where $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

4 If the normal to $f(x) = \dfrac{3x}{1+x}$ at (2, 2) cuts the axes at B and C, determine the length of BC.

5 Find $\dfrac{dy}{dx}$ for: **a** $y = \dfrac{x^2}{3-2x}$ **b** $y = \sqrt{x}(x^2-x)^3$

6 Find $\dfrac{d^2y}{dx^2}$ for: **a** $y = 3x^4 - \dfrac{2}{x}$ **b** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$

7 $y = \dfrac{x}{\sqrt{1-x}}$ has a tangent with equation $5x + by = a$ at the point where $x = -3$. Find the values of a and b.

8 The curve $f(x) = 3x^3 + Ax^2 + B$ has tangent with slope 0 at the point $(-2, 14)$. Find A and B and hence $f''(-2)$.

9 The line joining A(2, 4) to B(0, 8) is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find a.

10 Show that the curves whose equations are $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have the same slope at their point of intersection. Find the equation of the common tangent at this point.

Chapter 22

Applications of differential calculus

Contents:

A Functions of time
B Time rate of change
C General rates of change
D Motion in a straight line
Investigation: Displacement, velocity and acceleration graphs
E Curve properties
F Rational functions
G Inflections and shape type
H Optimisation
I Economic models

Review set 22A
Review set 22B

One application of differential calculus is the finding of equations of tangents and normals to curves. There are many other uses, but in this chapter we consider only:

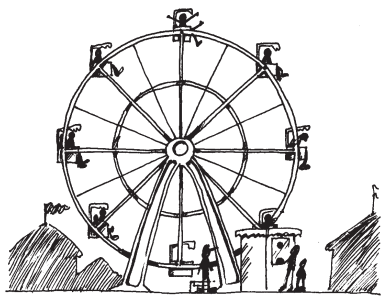
- **functions of time**
- **rates of change**
- **motion on a straight line** (displacement, velocity and acceleration)
- **curve properties** (monotonicity and concavity/convexity)
- **optimisation** (maxima and minima, local and global)
- **applications in economics**

A FUNCTIONS OF TIME

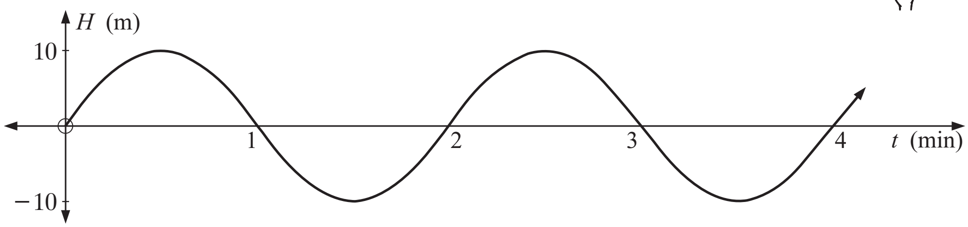
Earlier, we observed circular motion through a Ferris wheel.

We will revisit this demonstration and observe how far a particular point on the wheel is above the axle line and plot this height against the time of motion t.

Click on the icon to observe the motion.



The height function $h(t)$ metres is sinusoidal and its graph is:



The graph has time units on the horizontal axis.

DISCUSSION

From the graph, how do we determine the diameter of the wheel?

How long does it take for the wheel to complete one revolution?

Does the graph have any feature(s) which enable us to deduce that the wheel is rotating with constant angular velocity?

In this section we consider functions and quantities which vary with time. The height function for the Ferris wheel is one such function.

Functions of time may be determined in cases where a particular motion is regular or approximately so.

For example, the Ferris wheel's height above the axle line is

$$H(t) = 10\sin(\pi t) \text{ metres}, \quad t \geqslant 0 \quad \text{and in minutes.}$$

This function can then be used to solve problems. If we could differentiate this function then we could find the rate of increase or decrease in height at any instant. However, not all functions of time can be fitted to a functional equation.

Example 1

The population of brown bears on an island off the coast of Alaska over a twenty year period is shown in the table.

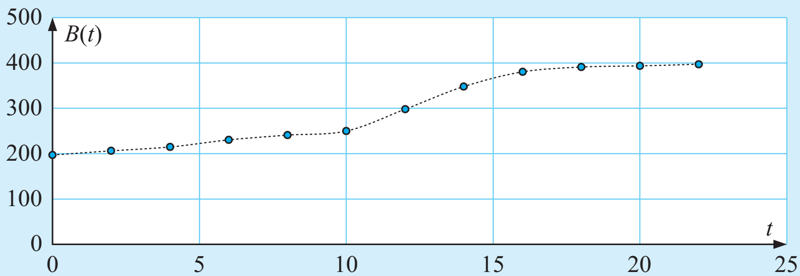
Year	t	$B(t)$	*Year*	t	$B(t)$
1980	0	197	1992	12	298
1982	2	206	1994	14	348
1984	4	215	1996	16	380
1986	6	230	1998	18	391
1988	8	241	2000	20	394
1990	10	250	2002	22	397

a Graph $B(t)$ against t.

b Find approximate values of $B'(t)$ and plot $B'(t)$ against t.

c What information can be deduced from $B'(t)$?

a



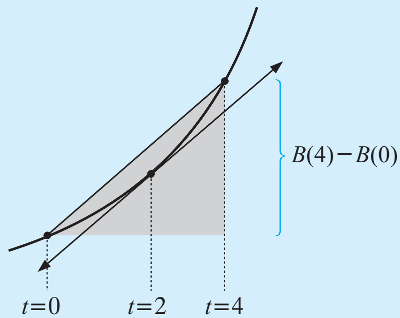
b

$$B'(2) \doteqdot \frac{B(4) - B(0)}{4} \doteqdot 4.5$$

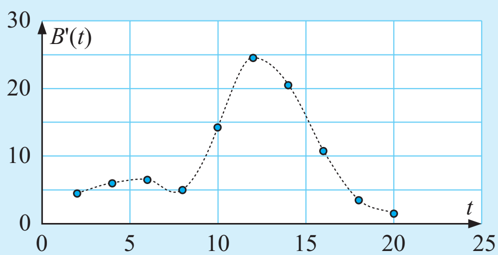
$$B'(4) \doteqdot \frac{B(6) - B(2)}{4} \doteqdot 6.0$$

$$B'(6) \doteqdot \frac{B(8) - B(4)}{4} \doteqdot 6.5$$

$\vdots$ etc.



So the graph of approximate $B'(t)$ values is:



c The population was steady initially, then increased rapidly over four years and since then has tailed off to a stable population size.

Note:

- $B'(2)$ was approximated by using the slope of the line segment between $t = 0$ and $t = 4$. This is approximately the slope of the tangent at $t = 2$.
- The dashed curves connecting the points give us an idea of the growth between them but could be highly inaccurate.

EXERCISE 22A

1 When the space shuttle Endeavour was launched in the early 1990's the velocity of the shuttle was measured at various times after the launch. The data is in the table provided.

Time (s)	*Velocity (m/s)*
0	0
4	24
8	50
12	78
16	106
20	136
24	166
28	196
32	226

a Plot the velocity function $v(t)$ against t.

b Use the technique of **Example 1** to find approximate values of $v'(t)$ for $t = 4, 8, ..., 28$. Graph the acceleration function $v'(t)$ over these values.

c Using modelling techniques applied to the velocity-time data, a power function best fits it and $v(t) = 5.242t^{1.0865}$. Find $v'(t)$ and check how accurately the estimated values of $v'(t)$ fit the $v'(t)$ values from the model.

2 The speed of an athlete is measured at half second intervals from the start of a sprint. The results were:

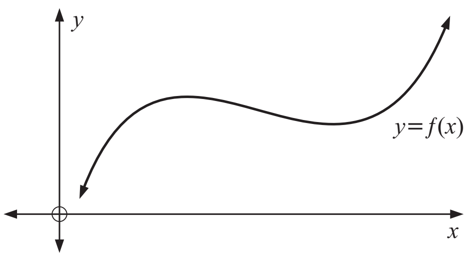
t (sec)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Speed (m/s)	0.0	2.0	3.6	5.0	5.8	6.3	6.6	6.8	6.9

a Use the data to estimate the athlete's acceleration at:

i $t = 1$ sec **ii** $t = 2.5$ sec

b Plot the athlete's acceleration graph.

B TIME RATE OF CHANGE



If we are given a function of x, $y = f(x)$ we know that $f'(x)$ or $\dfrac{dy}{dx}$ is the slope of the tangent at any value of x.

$\dfrac{dy}{dx}$ is also the rate of change in y with respect to x.

If we change the variable x to t which represents time and the variable y to s which represents displacement then $\dfrac{ds}{dt}$ would represent instantaneous velocity.

If y is changed to C where C represents the capacity of a person's lungs then as time changes $\dfrac{dC}{dt}$ represents the instantaneous rate of change in lung capacity per unit of time.

$\dfrac{ds}{dt}\begin{matrix}\leftarrow \text{metres}\\ \leftarrow \text{minute}\end{matrix}$ has units metres/minute

$\dfrac{dC}{dt}\begin{matrix}\leftarrow \text{litres}\\ \leftarrow \text{second}\end{matrix}$ has units litres/second

EXERCISE 22B

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

- **a** What is the current annual profit?
- **b** Find $\dfrac{dP}{dt}$ and state its units.
- **c** What is the significance of $\dfrac{dP}{dt}$?
- **d** When will the profit **i** decrease **ii** increase?
- **e** What is the minimum profit and when does it occur?
- **f** Find $\dfrac{dP}{dt}$ at $t = 4$, 10 and 25. What do these figures represent?

2 If A cm^2 is the area of a square with sides s cm, find:

- **a** the average rate of change in A as s changes from 4 cm to 4.5 cm
- **b** the instantaneous rate of change in A with respect to s when $s = 4$.

3 If water is draining from a swimming pool such that the volume of water after t minutes is $V = 200(50 - t)^2$ m^3 find:

- **a** the average rate at which the water leaves the pool in the first 5 minutes
- **b** the instantaneous rate of draining at $t = 5$ minutes.

4 Air is pumped into a balloon at a constant rate of 1.2 m^3 per minute.

- **a** If V represents the volume of air in the balloon write the above statement in terms of a derivative.
- **b** Given that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, find $\dfrac{dV}{dr}$.
- **c** Use the chain rule to show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.
- **d** At what rate is the radius of the balloon increasing when its radius is 3.6 m?

5 A ball is thrown vertically upwards and its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres.

- **a** At what distance above the ground was the throw released?
- **b** Find $s'(t)$ (i.e., $\dfrac{ds}{dt}$) and state what it represents.
- **c** Find t when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed:
  i when released **ii** at $t = 2$ sec **iii** at $t = 5$ sec
  State the significance of the sign of the derivative.
- **f** How long will it take for the ball to hit the ground?
- **g** What is the significance of $\dfrac{d^2s}{dt^2}$?

6 A shell is accidentally fired vertically upwards from ground level from a mortar and reaches the ground again after 14.2 seconds.

- **a** Given that its height above the ground at any time t is given by $s(t) = bt - 4.9t^2$ metres, show that the initial velocity of the shell is b m/s.
- **b** Find the initial velocity of the shell.

C GENERAL RATES OF CHANGE

Earlier we discovered that:

if $s(t)$ is a displacement function then $s'(t)$ or $\dfrac{ds}{dt}$ is the instantaneous rate of change in displacement with respect to time, which is of course the velocity function.

In general, $\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x**.

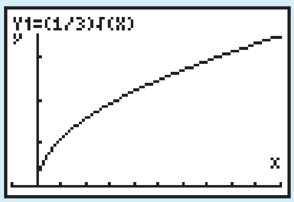
Note: If as x increases, y also increases, then $\dfrac{dy}{dx}$ will be positive, whereas if, as x increases, y decreases, then $\dfrac{dy}{dx}$ will be negative.

Example 2

According to a psychologist the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

- **a** Find the rate of improvement in ability to understand spatial concepts when the person is: **i** 9 years old **ii** 16 years old.
- **b** Explain why $\dfrac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$ and comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$ $\quad \therefore \quad \dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \dfrac{1}{6\sqrt{t}}$



i When $t = 9$, $\dfrac{dA}{dt} = \frac{1}{18}$

$\therefore$ rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\dfrac{dA}{dt} = \frac{1}{24}$

$\therefore$ rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b As $\sqrt{t}$ is never negative, $\dfrac{1}{6\sqrt{t}}$ is never negative,

i.e., $\dfrac{dA}{dt} > 0$ for all t in $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age. This is clearly shown by the graph.

Note that the rate of increase actually slows down as t increases.

EXERCISE 22C.1

You are encouraged to use technology to graph the function for each question.

1 The quantity of a chemical which is responsible for 'elasticity' in human skin is given by $Q = 100 - 10\sqrt{t}$ where t is the age of a person.

a Find Q at: **i** $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

b At what rate is the quantity of the chemical changing at the ages of:
i 25 years **ii** 50 years?

c Show that the rate at which the skin loses the chemical is decreasing for all values of t.

2 The height of *pinus radiata*, grown in ideal conditions, is given by $H = 20 - \dfrac{9}{t+5}$ metres, where t is the number of years after the tree was planted from an established juvenile tree.

a How high is the tree at planting?

b Find the height of the tree at $t = 4$, $t = 8$ and $t = 12$ years.

c Find the rate at which the tree is growing at $t = 0$, 5 and 10 years.

d Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$. What is the significance of this result?

3 The resistance to the flow of electricity in a certain metal is given by $R = 20 + \frac{1}{10}T + \frac{1}{200}T^2$ where T is the temperature (in ^{o}C) of the metal.

a Find the resistance R, at temperatures of 0^oC, 20^oC and 40^oC.

b Find the rate of change in the resistance at any temperature T.

c For what values of T does the resistance increase as the temperature increases?

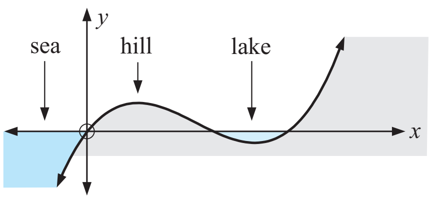
4 The total cost of running a train is given by $C(v) = 200v + \dfrac{10\,000}{v}$ dollars where v is the average speed of the train in kmph.

a Find the total cost of running the train at: **i** 20 kmph **ii** 40 kmph.

b Find the rate of change in the cost of running the train at speeds of:
i 10 kmph **ii** 30 kmph.

c At what speed will the cost be a minimum?

5



Alongside is a land and sea profile where the x-axis is sea level and y-values give the height of the land or sea bed above (or below) sea level and

$$y = \tfrac{1}{10}x(x-2)(x-3) \text{ km.}$$

a Find where the lake is located relative to the shore line of the sea.

b Find $\dfrac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake and the depth at this point.

6 A salt crystal has sides of length x mm and is growing slowly in a salt solution. V is the volume of the crystal (a cube).

a Find $\dfrac{dV}{dx}$ and explain what it represents.

b Find $\dfrac{dV}{dx}$ when $x = 2$. Interpret this result.

c Show that $\dfrac{dV}{dx} = \frac{1}{2}A$ where A is the surface area of the crystal. This means that $\dfrac{dV}{dx} \propto A$. Explain why this result seems reasonable.

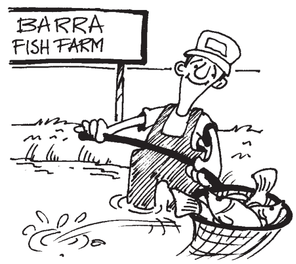
7 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \dfrac{t}{80}\right)^2$ where $0 \leqslant t \leqslant 80$.

a Find $\dfrac{dV}{dt}$ and draw the graph of $\dfrac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Find $\dfrac{d^2V}{dt^2}$ and interpret the fact that it is always constant and positive.

8 A fish farm grows and harvests barramundi in a large dam. The population P of fish at time t is of interest and the rate of change in the population $\dfrac{dP}{dt}$ is modelled by $\dfrac{dP}{dt} = aP\left(1 - \dfrac{P}{b}\right) - \left(\dfrac{c}{100}\right)P$ where a, b and c are known constants. a is the birth rate of the barramundi, b is the maximum carrying capacity of the dam and c is the percentage that is harvested.



a Explain why the fish population is stable when $\dfrac{dP}{dt} = 0$.

b If the birth rate is 6%, the maximum carrying capacity is 24 000 and 5% is harvested, find the stable population.

c If the harvest changes to 4%, what will the stable population increase to?

ECONOMIC MODELS

The cost of manufacturing x items has a **cost function** associated with it. Suppose this cost function is $C(x)$ dollars.

Now if the production increases from x to $x+h$ say, the additional cost is $C(x+h) - C(x)$ and the average rate of change in cost is $\dfrac{C(x+h) - C(x)}{h}$.

Letting $h \to 0$ we get the **instantaneous rate of change in cost** with respect to the number of items made.

This is $\dfrac{dC}{dx}$ and is known by economists as the **marginal cost** function.

Notice that in reality the smallest value of h is 1 and x is large and positive. Consequently, $\dfrac{dC}{dx} \doteqdot C(x+1) - C(x)$. Thus the marginal cost is approximately the additional cost of producing one more item, i.e., $x+1$ items instead of x of them.

COST MODELS

Most often cost functions are polynomial models.

For example, the cost of producing x items per day may be given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{cost of labour (including overtime) and other factors}} + \underbrace{5x}_{\text{raw material costs}} + \underbrace{2200}_{\text{fixed or overhead costs such as heating, cooling, maintenance, rent, etc.}}$$

Example 3

For the cost model $C(x) = 0.000\,13x^3 + 0.002x^2 + 5x + 2200$:

a find the marginal cost function

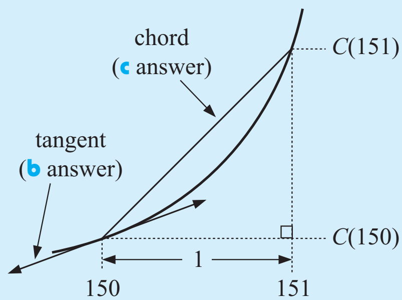
b find the marginal cost when 150 are produced. Interpret this result.

c Show that $C(151) - C(150)$ gives the approximate answer to **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$

b $C'(150) = \$14.38$ and is the rate at which the costs are increasing with respect to the production level x. It gives an estimated cost for making the 151st shirt.

c $C(151) - C(150) \doteqdot \$3448.19 - \$3433.75$
$\doteqdot \$14.44$



EXERCISE 22C.2

1 Seablue make jeans and the cost model for making x of them each day is
$$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250 \text{ dollars.}$$

a Find the marginal cost function $C'(x)$.

b Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. What does this represent?

d Find $C''(x)$ and the value of x when $C''(x) = 0$. What is the significance of this point?

2 The cost function for producing x items each day is
$$C(x) = 0.000\,072x^3 - 0.000\,61x^2 + 0.19x + 893 \text{ dollars.}$$

a Find $C'(x)$ and explain what it represents.

b Find $C'(300)$ and explain what it estimates.

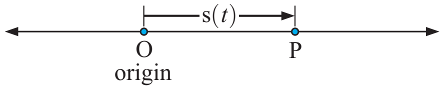
c Find the actual cost of producing the 301st item.

d Find $C''(x)$ and solve $C''(x) = 0$. What is the graphical significance of this result?

D MOTION IN A STRAIGHT LINE

Suppose an object P moves along a straight line so that its position from an origin s, is given as some function of time t,

i.e., $s = s(t)$ where $t \geqslant 0$.



$s(t)$ is a **displacement function** and for any value of t it gives the displacement from O.

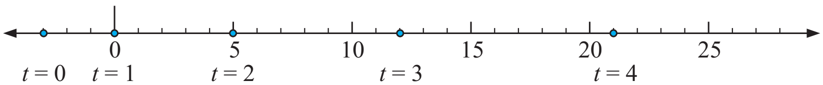
> It is clear that if $s(t) > 0$, P is located to the **right of O**
> if $s(t) = 0$, P is located **at O**
> if $s(t) < 0$, P is located to the **left of O**.

MOTION GRAPHS

Consider $s(t) = t^2 + 2t - 3$ cm, say, then

$s(0) = -3$ cm, $s(1) = 0$ cm, $s(2) = 5$ cm, $s(3) = 12$ cm, $s(4) = 21$ cm.

To appreciate the motion of P we draw a **motion graph**.



Click on the demo icon to get a better idea of the motion.

DEMO

Fully animated, we not only get a good idea of the position of P but also of what is happening with regard to velocity and acceleration.

VELOCITY AND ACCELERATION

AVERAGE VELOCITY

Recall that:

> The **average velocity** of an object moving in a straight line, in the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in displacement to the time taken,
>
> i.e., **average velocity** $= \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$, where $s(t)$ is the displacement function.

INSTANTANEOUS VELOCITY

In **Chapter 20** we established that $\dfrac{s(1+h) - s(1)}{h}$ approached a fixed value as h approached 0 and this value must be the instantaneous velocity at $t = 1$.

In general:

> If $s(t)$ is a displacement function of an object moving in a straight line, then
>
> $v(t) = s'(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$ is the **instantaneous velocity** of the object at time t.

Example 4

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity in the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity in the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2} \text{ ms}^{-1}$$
$$= \frac{(15 - 25) - (6 - 4)}{3} \text{ ms}^{-1}$$
$$= \frac{-10 - 2}{3} \text{ ms}^{-1}$$
$$= -4 \text{ ms}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ ms}^{-1} \text{ as } h \neq 0$$

c $$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} (-1 - h)$$
$$= -1 \text{ ms}^{-1}$$

and this is the instantaneous velocity at time $t = 2$ seconds.

EXERCISE 22D.1

1 A particle P moves in a straight line with a displacement function of $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.

d Find the average velocity from time t to time $t + h$ seconds and interpret

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}.$$

2 A particle P moves in a straight line with a displacement function of $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 2$ to $t = 5$ seconds.

b Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.

d Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

AVERAGE ACCELERATION

If an object moves in a straight line with velocity function $v(t)$ then its **average acceleration** on the time interval from $t = t_1$ to $t = t_2$ is the ratio of its *change in velocity* to the time taken,

i.e., **average acceleration** $= \dfrac{v(t_2) - v(t_1)}{t_2 - t_1}$.

INSTANTANEOUS ACCELERATION

If a particle moves in a straight line with velocity function $v(t)$, then

the **instantaneous acceleration** at time t is $a(t) = v'(t) = \lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$

3 A particle moves in a straight line with velocity function $v(t) = 2\sqrt{t} + 3$ cm s^{-1}, where $t \geqslant 0$.

- **a** Find the average acceleration from $t = 1$ to $t = 4$ seconds.
- **b** Find the average acceleration from $t = 1$ to $t = 1 + h$ seconds.
- **c** Find the value of $\lim\limits_{h \to 0} \dfrac{v(1+h) - v(1)}{h}$. Interpret this value.
- **d** Interpret $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

4 An object moves in a straight line with displacement function $s(t)$, and velocity function $v(t)$, $t \geqslant 0$. State the meaning of:

- **a** $\lim\limits_{h \to 0} \dfrac{s(3+h) - s(3)}{h}$
- **b** $\lim\limits_{h \to 0} \dfrac{v(5+h) - v(5)}{h}$
- **c** $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$
- **d** $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$

VELOCITY AND ACCELERATION FUNCTIONS

If a particle P, moves in a straight line and its position is given by the displacement function $s(t)$, $t \geqslant 0$, then:

- the **velocity** of P, at time t, is given by

  $v(t) = s'(t)$ {the derivative of the displacement function}

- the **acceleration** of P, at time t, is given by

  $a(t) = v'(t) = s''(t)$ {the derivative of the velocity function}

Note: $s(0)$, $v(0)$ and $a(0)$ give us the position, velocity and acceleration of the particle at time $t = 0$, and these are called the **initial conditions**.

SIGN INTERPRETATION

Suppose a particle P, moves in a straight line with displacement function $s(t)$ for locating the particle relative to an origin O, and has velocity function $v(t)$ and acceleration function $a(t)$.

We can use **sign diagrams** to interpret:

- where the particle is located relative to O
- the direction of motion and where a change of direction occurs
- when the particle's velocity is increasing/decreasing.

SIGNS OF $s(t)$:

$s(t)$	*Interpretation*
$= 0$	P is at O
> 0	P is located to the right of O
< 0	P is located to the left of O

SIGNS OF $v(t)$:

$v(t)$	*Interpretation*
$= 0$	P is instantaneously at rest
> 0	P is moving to the right
< 0	P is moving to the left

Note: $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$. If $h > 0$, so that $t + h > t$,

then $v(t) > 0$ implies that $s(t+h) - s(t) > 0$

$\therefore \quad s(t+h) > s(t)$

i.e.,

$\therefore$ P is moving to the right.

SIGNS OF $a(t)$:

$a(t)$	*Interpretation*
> 0	velocity is increasing
< 0	velocity is decreasing
$= 0$	velocity may be a maximum or minimum

A useful table:

Phrase used in a question	t	s	v	a
initial conditions	0			
at the origin		0		
stationary			0	
reverses			0	
maximum height			0	
constant velocity				0
max. / min. velocity				0

SPEED

As we have seen, velocities have size (magnitude) and sign (direction). The speed of a particle is a measure of how fast it is travelling regardless of the direction of travel.

Thus the speed at any instant is the modulus of the particle's velocity,

i.e., if S represents speed then $S = |v|$.

The question arises: "How can we determine when the speed of a particle is increasing or decreasing?"

We employ a **sign test**. This is:

- If the signs of $v(t)$ and $a(t)$ are the same, (i.e., both positive or both negative), then the **speed** of P is **increasing**.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the **speed** of P is decreasing.

We prove *the first* of these as follows:

Proof: Let $S = |v|$, be the speed of P at any instant

$$\therefore \quad S = \begin{cases} v & \text{if } v \geqslant 0 \\ -v & \text{if } v < 0 \end{cases} \qquad \{\text{definition of modulus}\}$$

Case 1: If $v > 0$, $S = v$ and $\therefore$ $\dfrac{dS}{dt} = \dfrac{dv}{dt} = a(t)$

and if $a(t) > 0$, $\dfrac{dS}{dt} > 0$ which implies that S is increasing.

Case 2: If $v < 0$, $S = -v$ and $\therefore$ $\dfrac{dS}{dt} = -\dfrac{dv}{dt} = -a(t)$

and if $a(t) < 0$, $\dfrac{dS}{dt} > 0$ which also implies that S is increasing.

Thus if $v(t)$ and $a(t)$ have the same sign, the speed of P is increasing.

INVESTIGATION — DISPLACEMENT, VELOCITY AND ACCELERATION GRAPHS

In this investigation we examine the motion of a projectile which is fired in a vertical direction under gravity. Other functions of a different kind will be examined.

What to do:

1. Click on the icon to examine vertical projectile motion in a straight line. Observe first the displacement along the line, then look at the velocity or rate of change in displacement.
2. Examine the three graphs
   - *displacement* v *time*
   - *velocity* v *time*
   - *acceleration* v *time*
3. Pick from the menu or construct functions of your own choosing to investigate their displacement, velocity and acceleration functions.

You are encouraged to use the motion demo in the questions of the following exercise.

Example 5

A particle moves in a straight line with position, relative to some origin O, given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds ($t \geqslant 0$).

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the position of the particle when changes in direction occur.

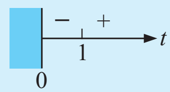
e Draw a motion diagram for the particle.

f For what time interval(s) is the particle's speed increasing?

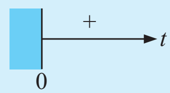
Note: $t \geqslant 0$
$\therefore$ critical value $t = -1$ is not required.

a Since $s(t) = t^3 - 3t + 1$ cm

$\therefore$ $v(t) = 3t^2 - 3$ cm s^{-1} $\{$as $v(t) = s'(t)\}$
$= 3(t^2 - 1)$
$= 3(t+1)(t-1)$ which has sign diagram



Also $a(t) = 6t$ cm s^{-2} $\{$as $a(t) = v'(t)\}$
which has sign diagram



b When $t = 0$, $s(0) = 1$ cm
$v(0) = -3$ cm s^{-1}
$a(0) = 0$ cm s^{-2}

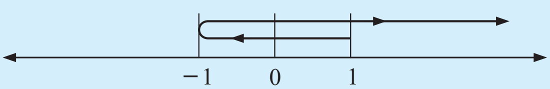
$\therefore$ particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.
$\{$speed $= |v|\}$

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm
$v(2) = 12 - 3 = 9$ cm s^{-1}
$a(2) = 12$ cm s^{-2}

$\therefore$ particle is 3 cm right of O, moving to the right at a speed of 9 cm s^{-1} and the speed is increasing. $\{$as a and v have the same sign$\}$

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant and $s(1) = 1 - 3 + 1 = -1$
$\therefore$ changes direction when $t = 1$ and is 1 cm left of O.

e



as $t \to \infty$, $s(t) \to \infty$ and $v(t) \to \infty$

Note: The motion is actually **on the line**, not above it as shown.

f Speed is increasing when $v(t)$ and $a(t)$ have the same sign i.e., $t \geqslant 1$.

EXERCISE 22D.2

1 An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from an origin O, $t \geqslant 0$, t in seconds.

a Find expressions for its velocity and acceleration at any instant and draw sign diagrams for each function.

b Find the initial conditions and explain what is happening to the object at that instant.

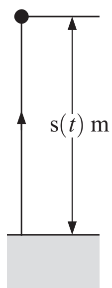
c Describe the motion of the object at time $t = 2$ seconds.

d At what time(s) does the object reverse direction? Find the position of the object at these instants.

e Draw a motion diagram of the object.

f For what time intervals is the speed of the object decreasing?

2 A stone is projected vertically upwards so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.



a Find the velocity and acceleration functions for the stone and draw sign diagrams for each function.

b Find the initial position and velocity of the stone.

c Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.

d Find the maximum height reached by the stone.

e Find the time taken for the stone to hit the ground.

3 A particle moves in a straight line with displacement function

$$s(t) = 12t - 2t^3 - 1 \text{ centimetres, } t \geqslant 0, \ t \text{ in seconds.}$$

a Find velocity and acceleration functions for the particle's motion.

b Find the initial conditions and interpret their meaning.

c Find the times and positions when the particle reverses direction.

d At what times is the particle's: **i** speed increasing **ii** velocity increasing?

4 The position of a particle moving along the x-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres, $t \geqslant 0$, t in seconds.

a Draw sign diagrams for the particle's velocity and acceleration functions.

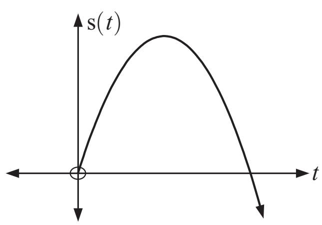
b Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.

c At what times is the particle's: **i** speed decreasing **ii** velocity decreasing?

d Find the total distance travelled by the particle in the first 5 seconds of motion.

5 An experiment to determine the position of an object fired vertically upwards from the earth's surface was performed. From the results, a two dimensional graph of position above the earth's surface $s(t)$ metres, against time t seconds, was plotted.

It was noted that the graph was *parabolic*.
Assuming a constant gravitational acceleration g, show that if the initial velocity is $v(0)$ then:



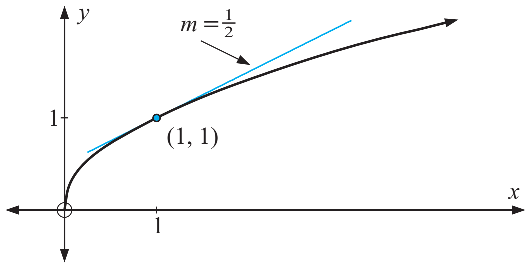
a $v(t) = v(0) + gt$, and **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

[**Hint:** Assume $s(t) = at^2 + bt + c$.]

E CURVE PROPERTIES

Recall that $f'(x)$ or $\dfrac{dy}{dx}$ is the **slope function** of a curve.

The derivative of a function is another function which enables us to find the slope of a tangent to the curve at any point on it.



For example, if $f(x) = \sqrt{x}$ then

$f(x) = x^{\frac{1}{2}}$ and

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Substituting $x = \frac{1}{4}, \frac{1}{2}$, 1 and 4 gives:

$$f'\left(\tfrac{1}{4}\right) = 1, \quad f'\left(\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{2}}, \quad f'(1) = \tfrac{1}{2}, \quad f'(4) = \tfrac{1}{4}$$

i.e., the slopes are 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively.

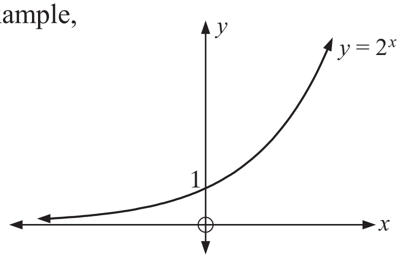
Notice also that a tangent to the graph at any point, provided that $x > 0$, has a positive slope.

This fact is also observed from $f'(x) = \dfrac{1}{2\sqrt{x}}$ as $\sqrt{x}$ is never negative and $x > 0$.

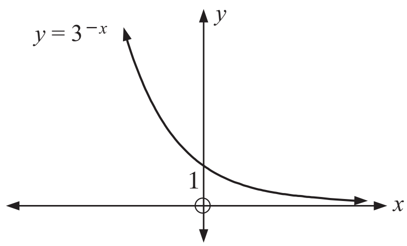
MONOTONICITY

Many functions are **increasing** *for all* x whereas others are **decreasing** *for all* x.

For example,



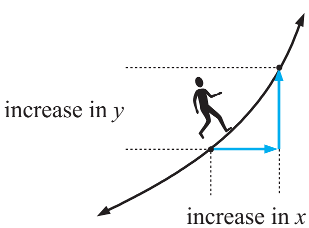
$y = 2^x$ is increasing for all x.

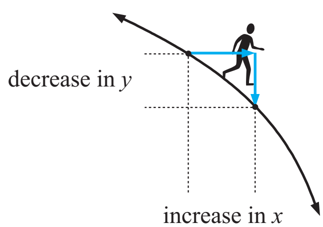


$y = 3^{-x}$ is decreasing for all x.

Notice that:

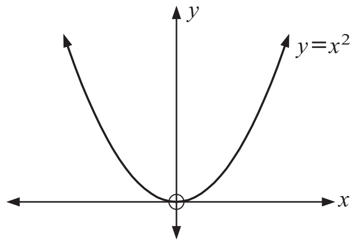
- for an increasing function an increase in x produces an increase in y
- for a decreasing function an increase in x produces a decrease in y.





The majority of other functions have intervals where the function is increasing and intervals where it is decreasing.

For example:



$y = x^2$ decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.

Note: $x \leqslant 0$ is an interval of x values. So is $x \geqslant 0$.

INTERVALS

Some examples of intervals and their graphical representations are:

Algebraic form	*Means*	*Alternative notation*
$x \geqslant 4$		$[4, \infty[$
$x > 4$		$]4, \infty[$
$x \leqslant 2$		$]-\infty, 2]$
$x < 2$		$]-\infty, 2[$
$2 \leqslant x \leqslant 4$		$[2, 4]$
$2 \leqslant x < 4$		$[2, 4[$

INCREASING / DECREASING INTERVALS

Definition

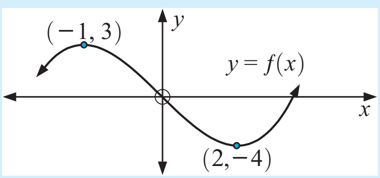
If S is an interval of real numbers and $f(x)$ is defined for all x in S, then:

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f'(x) \geqslant 0$ for all x in S, and
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f'(x) \leqslant 0$ for all x in S.

Note: $\Leftrightarrow$ is read as 'if and only if'

Example 6

Find intervals where $f(x)$ is:
a increasing
b decreasing.



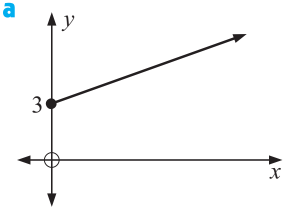
a $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 2$.
{since tangents have slopes $\geqslant 0$ on these intervals}

b $f(x)$ is decreasing for $-1 \leqslant x \leqslant 2$.

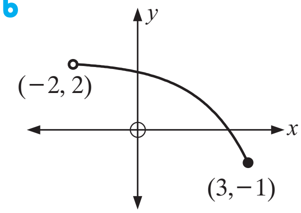
EXERCISE 22E.1

1 Find intervals where $f(x)$ is **i** increasing **ii** decreasing:

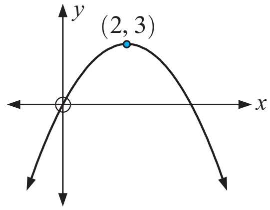
a



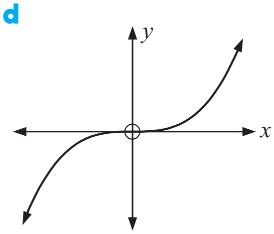
b



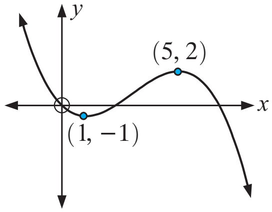
c



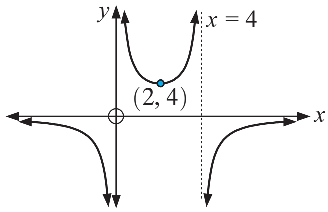
d



e

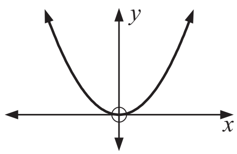


f



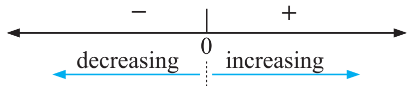
Sign diagrams for the derivative are extremely useful for determining intervals where a function is increasing/decreasing. Consider the following examples:

- $f(x) = x^2$



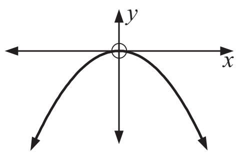
DEMO

$f'(x) = 2x$ which has sign diagram



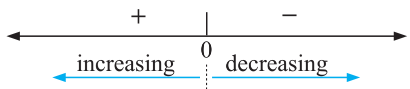
So $f(x) = x^2$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.

- $f(x) = -x^2$

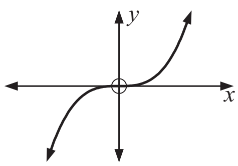


DEMO

$f'(x) = -2x$ which has sign diagram

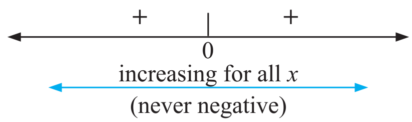


- $f(x) = x^3$

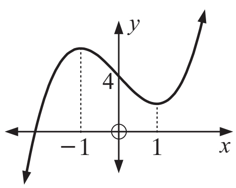


DEMO

$f'(x) = 3x^2$ which has sign diagram



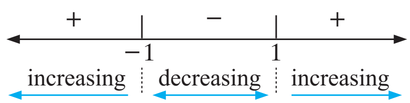
- $f(x) = x^3 - 3x + 4$



DEMO

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x+1)(x-1) \end{aligned}$$

which has sign diagram



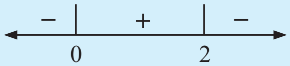
Example 7

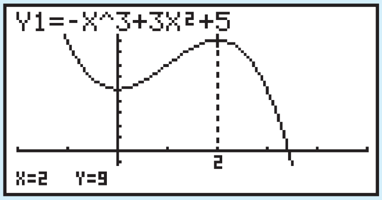
Find the intervals where the following functions are increasing/deceasing:

a $f(x) = -x^3 + 3x^2 + 5$ **b** $f(x) = 3x^4 - 8x^3 + 2$

a
$$\begin{aligned} f(x) &= -x^3 + 3x^2 + 5 \\ \therefore\ f'(x) &= -3x^2 + 6x \\ \therefore\ f'(x) &= -3x(x-2) \end{aligned}$$

which has sign diagram

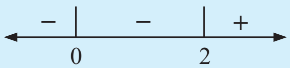


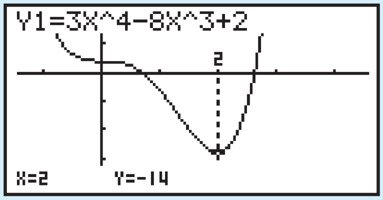


So, $f(x)$ is decreasing for $x \leqslant 0$ and for $x \geqslant 2$ and is increasing for $0 \leqslant x \leqslant 2$.

b
$$\begin{aligned} f(x) &= 3x^4 - 8x^3 + 2 \\ \therefore\ f'(x) &= 12x^3 - 24x^2 \\ &= 12x^2(x-2) \end{aligned}$$

which has sign diagram





So, $f(x)$ is decreasing for $x \leqslant 2$ and is increasing for $x \geqslant 2$.

EXERCISE 22E.2

1 Find intervals where $f(x)$ is increasing/decreasing:

a $f(x) = x^2$

b $f(x) = -x^3$

c $f(x) = 2x^2 + 3x - 4$

d $f(x) = \sqrt{x}$

e $f(x) = \dfrac{2}{\sqrt{x}}$

f $f(x) = x^3 - 6x^2$

g $f(x) = -2x^3 + 4x$

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$

k $f(x) = x^3 - 6x^2 + 3x - 1$

l $f(x) = x - 2\sqrt{x}$

m $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

n $f(x) = x^4 - 4x^3 + 2x^2 + 4x + 1$

Example 8

Consider $f(x) = \dfrac{3x-9}{x^2 - x - 2}$.

a Show that $f'(x) = \dfrac{-3(x-5)(x+1)}{(x-2)^2(x+1)^2}$ and draw its sign diagram.

b Hence, find intervals where $y = f(x)$ is increasing/decreasing.

a $f(x) = \dfrac{3x-9}{x^2-x-2}$

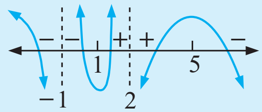
$$f'(x) = \frac{3(x^2-x-2)-(3x-9)(2x-1)}{(x-2)^2(x+1)^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{3x^2-3x-6-[6x^2-21x+9]}{(x-2)^2(x+1)^2}$$

$$= \frac{-3x^2+18x-15}{(x-2)^2(x+1)^2}$$

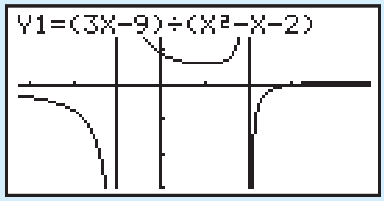
$$= \frac{-3(x^2-6x+5)}{(x-2)^2(x+1)^2}$$

$$= \frac{-3(x-5)(x-1)}{(x-2)^2(x+1)^2}$$ which has sign diagram



b $f(x)$ is increasing for $1 \leqslant x < 2$
and for $2 < x \leqslant 5$
$f(x)$ is decreasing for $x < -1$ and for $-1 < x \leqslant 1$ and for $x \geqslant 5$.

Note: A screen dump of $y = f(x)$ is:



2 **a** Consider $f(x) = \dfrac{4x}{x^2+1}$.

i Show that $f'(x) = \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$ and draw its sign diagram.

ii Hence, find intervals where $y = f(x)$ is increasing/decreasing.

b Consider $f(x) = \dfrac{4x}{(x-1)^2}$.

i Show that $f'(x) = \dfrac{-4(x+1)}{(x-1)^3}$ and draw its sign diagram.

ii Hence, find intervals where $y = f(x)$ is increasing/decreasing.

c Consider $f(x) = \dfrac{-x^2+4x-7}{x-1}$.

i Show that $f'(x) = \dfrac{-(x+1)(x-3)}{(x-1)^2}$ and draw its sign diagram.

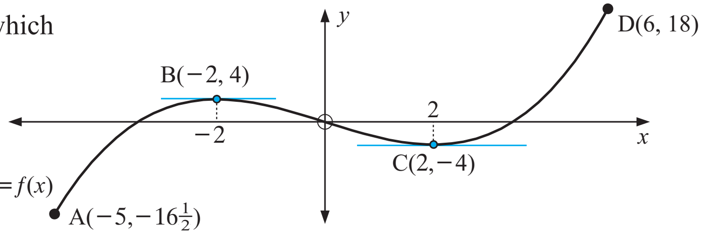
ii Hence, find intervals where $y = f(x)$ is increasing/decreasing.

3 Find intervals where $f(x)$ is increasing/decreasing if:

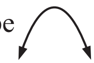
a $f(x) = \dfrac{x^3}{x^2-1}$ **b** $f(x) = x^2 + \dfrac{4}{x-1}$

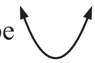
MAXIMA/MINIMA

Consider the following graph which has a restricted domain of $-5 \leqslant x \leqslant 6$.



A is a **global minimum** as it is the minimum value of y and occurs at an endpoint.

B is a **local maximum** as it is a turning point where the curve has shape  and $f'(x) = 0$ at that point.

C is a **local minimum** as it is a turning point where the curve has shape  and $f'(x) = 0$ at that point.

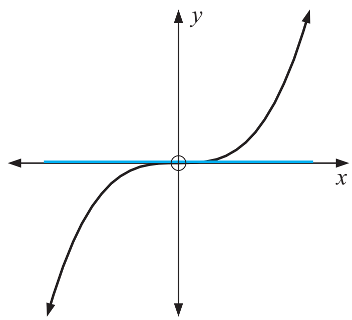
D is a **global maximum** as it is the maximum value of y and occurs at the endpoint of the domain.

Note: For local maxima/minima, tangents at these points are **horizontal** and thus have a slope of 0, i.e., $f'(x) = 0$.

HORIZONTAL INFLECTIONS

It is not true that whenever we find a value of x where $f'(x) = 0$ we have a local maximum or minimum.

For example, $f(x) = x^3$ has $f'(x) = 3x^2$
and $f'(x) = 0$ when $x = 0$.



Notice that the x-axis is a tangent to the curve which actually crosses over the curve at O(0, 0).

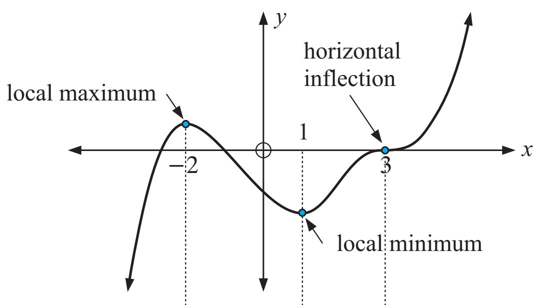
This tangent is horizontal and O(0, 0) is not a local maximum or minimum.

It is called a **horizontal inflection** (or **inflexion**).

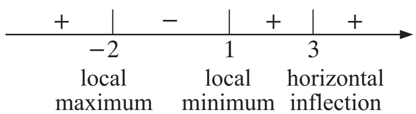
STATIONARY POINTS

A **stationary point** is a point where $f'(x) = 0$. It could be a local maximum, local minimum or a horizontal inflection.

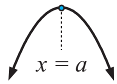
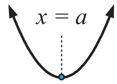
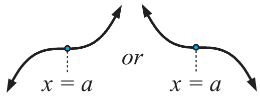
Consider the following graph:



Its **slope sign diagram** is:



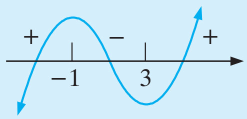
Summary:

Stationary point	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum	$+ \mid -$ at a	
local minimum	$- \mid +$ at a	
horizontal inflection	$+ \mid +$ at a *or* $- \mid -$ at a	

Example 9

Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

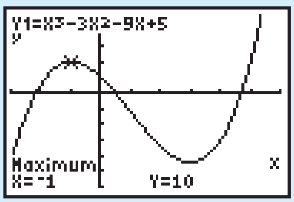
$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 5 \\ \therefore \quad f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1), \text{ which has sign diagram:} \end{aligned}$$

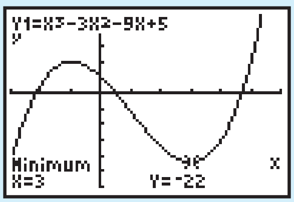


So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) + 5 \\ &= -1 - 3 + 9 + 5 \\ &= 10 \qquad \therefore \text{ local maximum at } (-1, 10) \end{aligned}$$

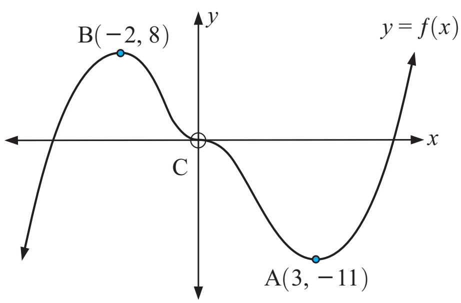
$$\begin{aligned} f(3) &= 3^3 - 3 \times 3^2 - 9 \times 3 + 5 \\ &= 27 - 27 - 27 + 5 \\ &= -22 \qquad \therefore \text{ local minimum at } (3, -22) \end{aligned}$$





EXERCISE 22E.3

1 A, B and C are points where tangents are horizontal.



a Classify points A, B and C.

b Draw a sign diagram for the slope of $f(x)$ for all x.

c State intervals where $y = f(x)$ is:

i increasing **ii** decreasing.

2 For each of the following functions, find and classify the stationary points and hence sketch the function showing all important features.

a $f(x) = x^2 - 2$ **b** $f(x) = x^3 + 1$

c $f(x) = x^3 - 3x + 2$ **d** $f(x) = x^4 - 2x^2$

e $f(x) = x^3 - 6x^2 + 12x + 1$ **f** $f(x) = \sqrt{x} + 2$

g $f(x) = x - \sqrt{x}$ **h** $f(x) = x^4 - 6x^2 + 8x - 3$

i $f(x) = 1 - x\sqrt{x}$ **j** $f(x) = x^4 - 2x^2 - 8$

3 At what value of x does the quadratic function, $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

4 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$. Find a.

5 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b.

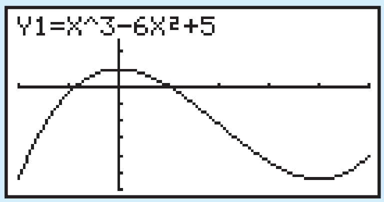
b Find the position and nature of all stationary points.

6 A cubic polynomial $P(x)$, touches the line with equation $y = 9x + 2$ at the point $(0, 2)$ and has a stationary point at $(-1, -7)$. Find $P(x)$.

Example 10

Find the greatest and least value of $x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

First we graph $y = x^3 - 6x^2 + 5$ on $[-2, 5]$.



The greatest value is clearly when $\dfrac{dy}{dx} = 0$.

Now $\dfrac{dy}{dx} = 3x^2 - 12x$
$= 3x(x-4)$
$= 0$ when $x = 0$ or 4.

So, the greatest value is $f(0) = 5$ when $x = 0$.

The least value is either $f(-2)$ or $f(4)$, whichever is smaller.

Now $f(-2) = -27$ and $f(4) = -27$

$\therefore$ least value is -27 when $x = -2$ and $x = 4$.

7 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

8 A manufacturing company makes door hinges. The cost function for making x hinges per hour is $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$ dollars where $50 \leqslant x \leqslant 150$. The condition $50 \leqslant x \leqslant 150$ applies as the company has a standing order filled by producing 50 each hour, but knows that production of more than 150 an hour is useless as they will not sell. Find the minimum and maximum hourly costs and the production levels when each occurs.

F RATIONAL FUNCTIONS

Rational functions are functions of the form $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials.

For example, $f(x) = \dfrac{2x-1}{x^2+2}$ and $f(x) = \dfrac{x^2-4}{x^2-3x+2}$ are rational functions.

One feature of a rational function is the presence of **asymptotes**.

These are lines (or curves) that a function's graph approaches when either x or y takes large values.

Vertical asymptotes are vertical lines which the graph of a function approaches.

These can be found by solving $h(x) = 0$ in the case where $f(x) = \dfrac{g(x)}{h(x)}$.

Horizontal asymptotes are horizontal lines which the graph of a function approaches.

These can be found by finding what value $f(x)$ approaches as $|x| \to \infty$.

FUNCTIONS OF THE FORM $y = \dfrac{\textbf{linear}}{\textbf{linear}}$.

Example 11

Consider $f(x) = \dfrac{2x-3}{x+2}$.

a Write $f(x)$ as $a + \dfrac{b}{x+2}$ and then find its asymptote equations.

b Find $f'(x)$ and its sign diagram.

c Find the axis intercepts.

d Sketch its graph.

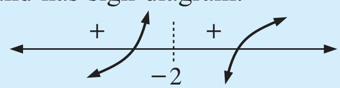
a
$$f(x) = \frac{2x-3}{x+2} = \frac{2(x+2)-7}{x+2} = 2 - \frac{7}{x+2}$$

Vertical asymptote is $x+2=0$
i.e., $x = -2$

Horizontal asymptote is $y = 2$
$\{$as $|x| \to \infty$, $f(x) \to 2\}$

b
$$f'(x) = \frac{2(x+2)-(2x-3)1}{(x+2)^2} = \frac{2x+4-2x+3}{(x+2)^2} = \frac{7}{(x+2)^2}$$

and has sign diagram:



As $f'(x)$ is never zero, $y = f(x)$ has no turning points.

c It cuts the x-axis when $y = 0$

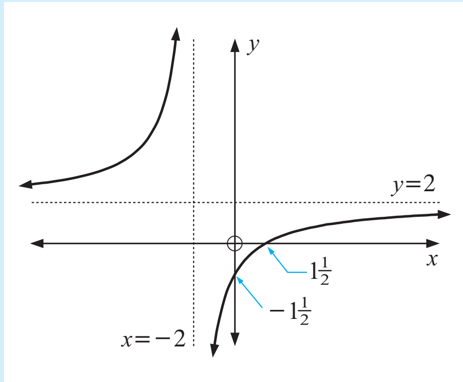
$\therefore \quad 2x - 3 = 0$

$\therefore \quad x = \frac{3}{2} = 1\frac{1}{2}$

It cuts the y-axis when $x = 0$

$\therefore \quad y = -\frac{3}{2} = -1\frac{1}{2}$

d



EXERCISE 22F.1

1 Write the following functions in the form $f(x) = a + \dfrac{b}{cx+d}$ and hence determine the equations of the asymptotes of:

a $f(x) = \dfrac{3x-2}{x+1}$ **b** $f(x) = \dfrac{x-4}{2x-1}$ **c** $f(x) = \dfrac{4-2x}{x-1}$

2 For each of the following functions:

i state the equations of the asymptotes (giving reasons)
ii find $f'(x)$ and draw a sign diagram of it
iii find the axis intercepts
iv sketch the graph of $y = f(x)$.

a $f(x) = -3 + \dfrac{1}{4-x}$ **b** $f(x) = \dfrac{x}{x+2}$

c $f(x) = \dfrac{4x+3}{x-2}$ **d** $f(x) = \dfrac{1-x}{x+2}$

FUNCTIONS OF THE FORM $f(x) = \dfrac{\textbf{linear}}{\textbf{quadratic}}$

Example 12

Consider $f(x) = \dfrac{3x-9}{x^2-x-2}$.

a Determine the equations of any asymptotes.
b Find $f'(x)$ and determine the position and nature of any stationary points.
c Find the axis intercepts.
d Sketch the graph of the function.

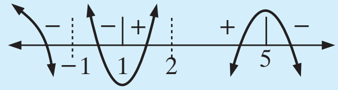
a $f(x) = \dfrac{3x-9}{x^2-x-2} = \dfrac{3x-9}{(x-2)(x+1)}$

Vertical asymptotes are $x = 2$ and $x = -1$ {when the denominator is 0}

Horizontal asymptote is $y = 0$ {as $|x| \to \infty$, $f(x) \to 0$}

b
$$\begin{aligned} f'(x) &= \frac{3(x^2-x-2)-(3x-9)(2x-1)}{(x-2)^2(x+1)^2} \\ &= \frac{3x^2-3x-6-[6x^2-21x+9]}{(x-2)^2(x+1)^2} \\ &= \frac{-3x^2+18x-15}{(x-2)^2(x+1)^2} \\ &= \frac{-3(x^2-6x+5)}{(x-2)^2(x+1)^2} \\ &= \frac{-3(x-5)(x-1)}{(x-2)^2(x+1)^2} \end{aligned}$$

So, $f'(x)$ has sign diagram:



$\therefore$ a local maximum when $x = 5$ and a local minimum when $x = 1$

local max. $(5, \frac{1}{3})$

local min. $(1, 3)$

c Cuts the x-axis when $y = 0$

$\therefore$ $3x - 9 = 0$, i.e., $x = 3$

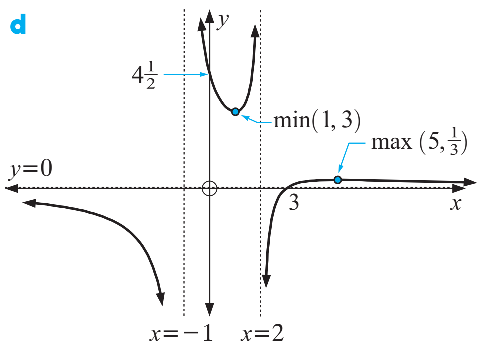
So, the x-intercept is 3.

Cuts the y-intercept when $x = 0$

$\therefore$ $y = \frac{-9}{-2} = 4\frac{1}{2}$

So, the y-intercept is $4\frac{1}{2}$.

d



EXERCISE 22F.2

1 Determine the equations of the asymptotes of:

a $y = \dfrac{2x}{x^2-4}$ **b** $y = \dfrac{1-x}{(x+2)^2}$ **c** $y = \dfrac{3x+2}{x^2+1}$

2 For each of the following functions:

i determine the equation(s) of the asymptotes

ii find $f'(x)$ and hence determine the position and nature of any stationary points

iii find the axis intercepts

iv sketch the graph of the function, showing all information in **a**, **b** and **c**.

a $f(x) = \dfrac{4x}{x^2+1}$ **b** $f(x) = \dfrac{4x}{x^2-4x-5}$

c $f(x) = \dfrac{4x}{(x-1)^2}$ **d** $f(x) = \dfrac{3x-3}{(x+2)^2}$

FUNCTIONS OF THE FORM $y = \dfrac{\textbf{quadratic}}{\textbf{quadratic}}$

Functions such as $y = \dfrac{2x^2 - x + 3}{x^2 + x - 2}$ have a **horizontal asymptote** which can be found by dividing every term by x^2.

Notice that $y = \dfrac{2 - \frac{1}{x} + \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$ and as $|x| \to \infty$, $y \to \dfrac{2}{1}$ i.e., $y \to 2$.

EXERCISE 22F.3

1 Determine the equations of the asymptotes of:

a $y = \dfrac{2x^2 - x + 2}{x^2 - 1}$ **b** $y = \dfrac{-x^2 + 2x - 1}{x^2 + x + 1}$ **c** $y = \dfrac{3x^2 - x + 2}{(x+2)^2}$

Example 13

Consider $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 3x + 2}$.

a Determine the equations of its asymptotes.
b Find $f'(x)$ and determine the position and nature of any turning points.
c Find the axis intercepts. **d** Sketch the graph of the function.

a $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 3x + 2} = \dfrac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 + \frac{3}{x} + \frac{2}{x^2}}$

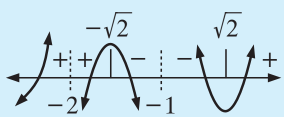
$\therefore$ horizontal asymptote is $y = \dfrac{1}{1}$ i.e., $y = 1$ $\{$as $|x| \to \infty$, $y \to 1\}$

$f(x) = \dfrac{x^2 - 3x + 2}{(x+1)(x+2)}$ $\therefore$ vertical asymptotes are $x = -1$ and $x = -2$

b $$f'(x) = \frac{(2x-3)(x^2+3x+2) - (x^2-3x+2)(2x+3)}{(x+1)^2(x+2)^2}$$

$$= \frac{6x^2 - 12}{(x+1)^2(x+2)^2} \quad \{\text{on simplifying}\}$$

$$= \frac{6(x+\sqrt{2})(x-\sqrt{2})}{(x+1)^2(x+2)^2}$$ and has sign diagram



So, we have a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$.
The local max. is $(-\sqrt{2}, -33.971)$. The local min. is $(\sqrt{2}, -0.029)$.

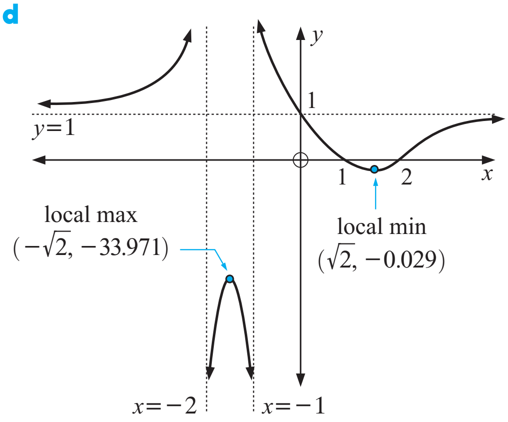
c Cuts the x-axis when $y = 0$

$$\therefore \quad x^2 - 3x + 2 = 0$$
$$\therefore \quad (x-1)(x-2) = 0$$
$$\text{i.e., at } x = 1 \text{ or } 2$$

Cuts the y-axis when $x = 0$

$$\therefore \quad y = \tfrac{2}{2} = 1$$

d



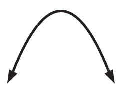
2 For each of the following functions:

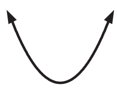
i determine the equation(s) of the asymptotes
ii find $f'(x)$ and hence determine the position and nature of any turning points
iii find the axis intercepts
iv sketch the function, showing all information obtained in **i**, **ii** and **iii**.

a $y = \dfrac{x^2 - x}{x^2 - x - 6}$ **b** $y = \dfrac{x^2 - 1}{x^2 + 1}$

c $y = \dfrac{x^2 - 5x + 4}{x^2 + 5x + 4}$ **d** $y = \dfrac{x^2 - 6x + 5}{(x+1)^2}$

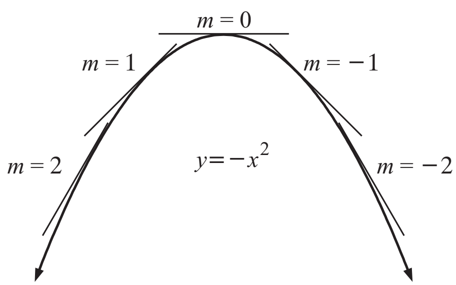
G INFLECTIONS AND SHAPE TYPE

When a curve, or part of a curve, has shape  we say that the shape is **concave downwards**.

If a curve, or part of a curve, has shape  we say that the shape is **concave upwards**.

TEST FOR SHAPE

Consider the **concave downwards** curve:



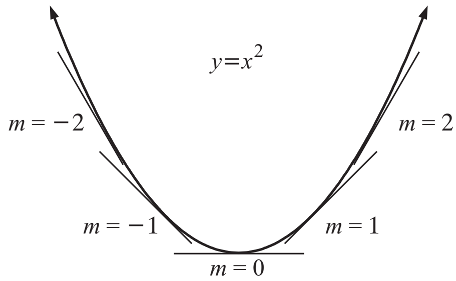
Notice that as x increases for all points on the curve the slope is decreasing,

i.e., $f'(x)$ is decreasing,

$\therefore$ its derivative is negative,

i.e., $f''(x) < 0$.

Likewise, if the curve is **concave upwards**:



As the values of x increase for all points on the curve the slope is increasing,

i.e., $f'(x)$ is increasing,

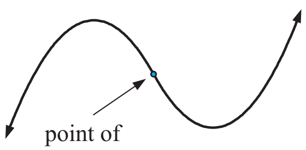
$\therefore$ its derivative is positive,

i.e., $f''(x) > 0$.

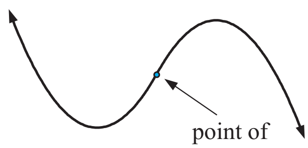
POINTS OF INFLECTION (INFLEXION)

A **point of inflection** is a point on a curve at which a change of shape occurs.

i.e.,



or

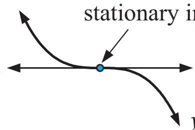


DEMO

Notes:

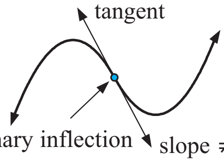
- If the tangent at a point of inflection is horizontal we say that we have a **horizontal** or **stationary inflection**.

  For example,



- If the tangent at a point of inflection is not horizontal we say that we have a **non-horizontal** or **non-stationary inflection**.

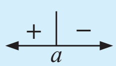
  For example,

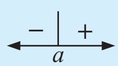


- Notice that the tangent at the point of inflection (the **inflecting tangent**) crosses the curve at that point.

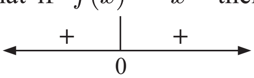
Consequently,

we have a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes on either side of $x = a$,

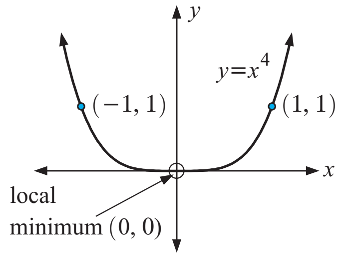
i.e., $f''(x)$ has **sign diagram**, in the vicinity of a, of either 

or 

Observe that if $f(x) = x^4$ then $f'(x) = 4x^3$ and $f''(x) = 12x^2$ and $f''(x)$ has sign diagram



Although $f''(0) = 0$ we do not have a point of inflection at $(0, 0)$ since the sign of $f''(x)$ does not change on either side of $x = 0$. In fact the graph of $f(x) = x^4$ is:



Summary:

<table>
<tr><td>
concave downwards</td><td>For a curve (or part curve) which is concave downwards in an interval S, $f''(x) < 0$ for all x in S.</td></tr>
<tr><td>
concave upwards</td><td>For a curve (or part curve) which is concave upwards in an interval S, $f''(x) > 0$ for all x in S.</td></tr>
<tr><td colspan="2">If $f''(x)$ has a sign change on either side of $x = a$, and $f''(a) = 0$, then
• we have a horizontal inflection if $f'(a) = 0$ also,
• we have a non-horizontal inflection if $f'(a) \neq 0$.</td></tr>
</table>

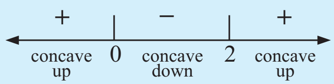
Click on the demo icon to examine some standard functions for turning points, points of inflection and intervals where the function is increasing, decreasing, concave up/down.

DEMO

Example 14

Find and classify all points of inflection of $f(x) = x^4 - 4x^3 + 5$.

$$f(x) = x^4 - 4x^3 + 5$$
$$\therefore \quad f'(x) = 4x^3 - 12x^2$$
$$\therefore \quad f''(x) = 12x^2 - 24x$$
$$= 12x(x-2) \quad \text{which has sign diagram}$$



$$f''(x) = 0 \quad \text{when} \quad x = 0 \text{ or } 2$$

and since the signs of $f''(x)$ change about $x = 0$ and $x = 2$, we have points of inflection at these two points.

$$\text{Also} \quad f'(0) = 0 \quad \text{and} \quad f'(2) = 32 - 48 \neq 0$$
$$\text{and} \quad f(0) = 5, \quad f(2) = 16 - 32 + 5 = -11$$

Thus $(0, 5)$ is a horizontal inflection and $(2, -11)$ is a non-horizontal inflection.

EXERCISE 22G

1 Find and classify, if they exist, all points of inflection of:

a $f(x) = x^2 + 3$

b $f(x) = 2 - x^3$

c $f(x) = x^3 - 6x^2 + 9x + 1$

d $f(x) = x^3 + 6x^2 + 12x + 5$

e $f(x) = -3x^4 - 8x^3 + 2$

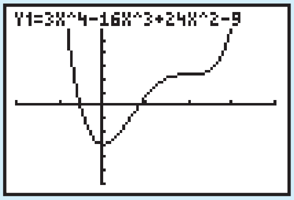
f $f(x) = 3 - \dfrac{1}{\sqrt{x}}$

Example 15

For $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$:

a find and classify all points where $f'(x) = 0$
b find and classify all points of inflection
c find intervals where the function is increasing/decreasing
d find intervals where the function is concave up/down.
e Hence, sketch the graph showing *all* important features.

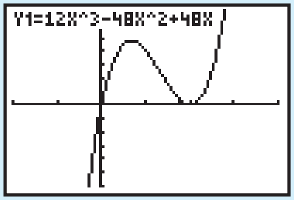
$f(x) = 3x^4 - 16x^3 + 24x^2 - 9$



a $\therefore \quad f'(x) = 12x^3 - 48x^2 + 48x$
$= 12x(x^2 - 4x + 4)$
$= 12x(x-2)^2$

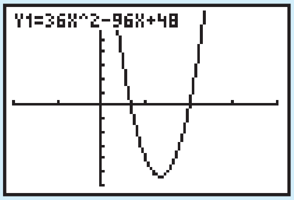
which has sign diagram $\quad - \;|_{0}\; + \;|_{2}\; +$

$\therefore$ $(0, f(0))$ is a local minimum
and $(2, f(2))$ is a horizontal inflection
i.e., $(0, -9)$ is a local minimum and
$(2, 7)$ is a horizontal inflection



b $f''(x) = 36x^2 - 96x + 48$
$= 12(3x^2 - 8x + 4)$
$= 12(x-2)(3x-2)$

which has sign diagram $\quad + \;|_{\frac{2}{3}}\; - \;|_{2}\; +$

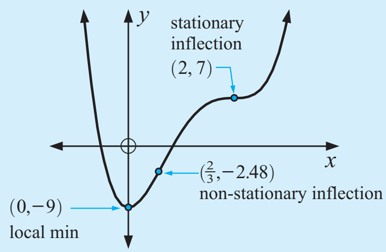


$(2, 7)$ is a horizontal inflection
$\left(\frac{2}{3}, f(\frac{2}{3})\right)$ i.e., $\left(\frac{2}{3}, -2.48\right)$ is a non-horizontal inflection.

c $f(x)$ is decreasing for $x \leqslant 0$
$f(x)$ is increasing for $x \geqslant 0$.

d $f(x)$ is concave up for $x \leqslant \frac{2}{3}$ and for $x \geqslant 2$
$f(x)$ is concave down for $\frac{2}{3} \leqslant x \leqslant 2$.

e



2 For each of the following functions:

i find and classify all points where $f'(x) = 0$
ii find and classify all points of inflection
iii find intervals where the function is increasing/decreasing
iv find intervals where the function is concave up/down.
v Sketch the graph showing *all* important features.

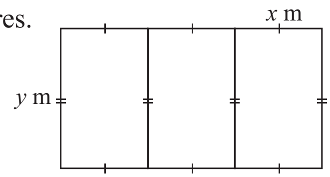
a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \sqrt{x}$
d $f(x) = x^3 - 3x^2 - 24x + 1$ **e** $f(x) = 3x^4 + 4x^3 - 2$ **f** $f(x) = (x-1)^4$
g $f(x) = x^4 - 4x^2 + 3$ **h** $f(x) = 3 - \dfrac{4}{\sqrt{x}}$

H OPTIMISATION

Many problems where we try to find the **maximum** or **minimum** value of a variable can be solved using differential calculus techniques. The solution is often referred to as the **optimum** solution. Consider the following problem.

An industrial shed is to have a total floor space of 600 m^2 and is to be divided into 3 rectangular rooms of equal size. The walls, internal and external, will cost \$60 per metre to build. What dimensions should the shed have to minimise the cost of the walls?

We let one room be x m by y m as shown.



Note: $x > 0$ and $y > 0$

The total length of wall material is L where $L = 6x + 4y$ metres.

However we do know that the total area is 600 m^2,

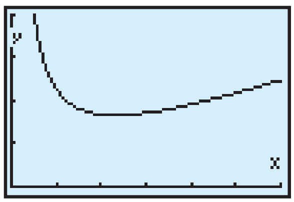
$\therefore \quad 3x \times y = 600 \quad$ and so $\quad y = \dfrac{200}{x}$

Knowing this relationship enables us to write L in terms of one variable (x in this case),

$$\text{i.e.,} \quad L = 6x + 4\left(\frac{200}{x}\right) \text{ m}, \qquad \text{i.e.,} \quad L = \left(6x + \frac{800}{x}\right) \text{ m}$$

and at \$60/metre, the total cost is $\quad C(x) = 60\left(6x + \dfrac{800}{x}\right)$ dollars.

When graphed we have:



Clearly, $C(x)$ is a minimum when $C'(x) = 0$.

$$\begin{aligned} \text{Now} \quad C(x) &= 360x + 48\,000x^{-1} \\ \therefore \quad C'(x) &= 360 - 48\,000x^{-2} \\ \therefore \quad C'(x) &= 0 \quad \text{when} \quad 360 = \frac{48\,000}{x^2} \end{aligned}$$

$$\text{i.e.,} \quad x^2 = \frac{48\,000}{360} \doteqdot 133.333 \quad \text{and so} \quad x \doteqdot 11.547$$

Now when $\quad x \doteqdot 11.547, \quad y \doteqdot \dfrac{200}{11.547} \doteqdot 17.321 \quad$ and $\quad C(11.547) \doteqdot 8313.84$ dollars.

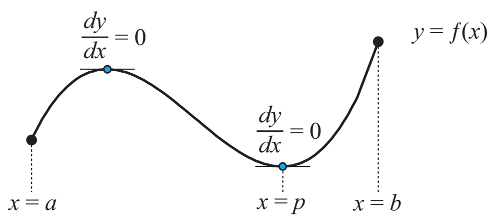
So, the minimum cost is about \$8310 when the shed is 34.6 m by 17.3 m.

WARNING

The maximum/minimum value does not always occur when the first derivative is zero.

It is essential to also examine the values of the function at the end point(s) of the domain for global maxima/minima, i.e., given $a \leqslant x \leqslant b$, you should also consider $f(a)$ and $f(b)$.

Example:



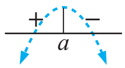
In the illustrated example, the maximum value of y occurs at $x = b$ and the minimum value of y occurs at $x = p$.

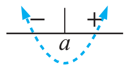
TESTING OPTIMAL SOLUTIONS

If one is trying to optimise a function $f(x)$ and we find values of x such that $f'(x) = 0$, how do we know whether we have a maximum or a minimum solution? The following are acceptable evidence.

SIGN DIAGRAM TEST

If near to $x = a$ where $f'(a) = 0$ the sign diagram is:

- 

we have a **local maximum**
- 

we have a **local minimum**.

SECOND DERIVATIVE TEST

If near $x = a$ where $f'(a) = 0$ and:

- $\dfrac{d^2y}{dx^2} < 0$ we have  shape, i.e., a **local maximum**
- $\dfrac{d^2y}{dx^2} > 0$ we have  shape, i.e., a **local minimum**.

GRAPHICAL TEST

If we have a graph of $y = f(x)$ showing  we have a **local maximum** and  we have a **local minimum**.

OPTIMISATION PROBLEM SOLVING METHOD

The following steps should be followed:

Step 1: Draw a large, clear diagram of the situation. Sometimes more than one diagram is necessary.

Step 2: Construct an equation with the variable to be **optimised** (**maximised** or **minimised**) as the subject of the formula in terms of **one** convenient **variable**, x say. Also find what restrictions there may be on x.

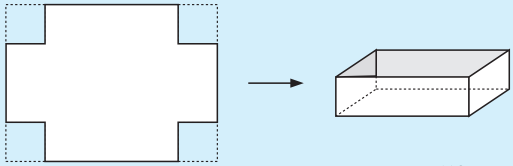
Step 3: Find the **first derivative** and find the value(s) of x when it is **zero**.

Step 4: If there is a restricted domain such as $a \leqslant x \leqslant b$, the maximum/minimum value of the function may occur either when the derivative is zero or at $x = a$ or at $x = b$. Show by the **sign diagram test**, the **second derivative test** or the **graphical test** that you have a maximum or a minimum situation.

To illustrate the method we consider the following example.

Example 16

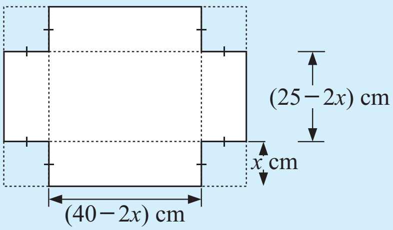
A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate and folding the metal to form the container.



DEMO

What size squares must be cut out in order to produce the cake dish of maximum volume?

Step 1:



Let x cm be the lengths of the sides of the squares cut out.

Step 2: Now volume = length × width × depth

$$= (40-2x)(25-2x)x \text{ cm}^3$$

i.e., $$V = (40-2x)(25x-2x^2) \text{ cm}^3$$

Notice that $x > 0$ and $25 - 2x > 0$

$$\therefore \quad x < 12.5$$

$$\therefore \quad 0 < x < 12.5$$

Step 3: Now $$\frac{dV}{dx} = -2(25x-2x^2) + (40-2x)(25-4x) \qquad \{\text{product rule}\}$$

$$= -50x + 4x^2 + 1000 - 50x - 160x + 8x^2$$

$$= 12x^2 - 260x + 1000$$

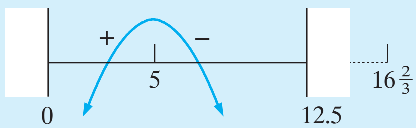
$$= 4(3x^2 - 65x + 250)$$

$$= 4(3x-50)(x-5)$$

$$\therefore \quad \frac{dV}{dx} = 0 \quad \text{when} \quad x = \tfrac{50}{3} = 16\tfrac{2}{3} \quad \text{or} \quad x = 5$$

Step 4: **Sign diagram test**

$\dfrac{dV}{dx}$ has sign diagram:



or **Second derivative test**

$$\frac{d^2V}{dx^2} = 24x - 260 \quad \text{and at} \quad x = 5, \quad \frac{d^2V}{dx^2} = -140 \quad \text{which is } < 0$$

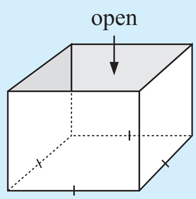
$\therefore$ shape is and we have a local maximum.

So, the maximum volume is obtained when $x = 5$,

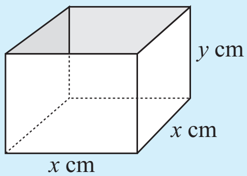
i.e., 5 cm squares are cut from the corners.

Example 17

Find the most economical shape (minimum surface area) for a box with a square base, vertical sides and an open top, given that it must contain 4 litres.



Step 1:



Let the base lengths be x cm and the depth be y cm. Now the volume

$$V = \text{length} \times \text{width} \times \text{depth}$$

$$\therefore \quad V = x^2y$$

$$\therefore \quad 4000 = x^2y \quad .... \ (1) \quad \{\text{as } 1 \text{ litre} \equiv 1000 \text{ cm}^3\}$$

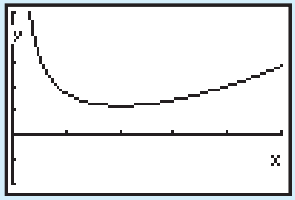
Step 2: Now total surface area,

$$A = \text{area of base} + 4 \text{ (area of one side)}$$

$$\therefore \quad A(x) = x^2 + 4xy$$

$$\therefore \quad A(x) = x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using (1)}\}$$

$$\therefore \quad A(x) = x^2 + 16\,000x^{-1}$$



Notice:
$x > 0$ as x is a length.

Step 3: Thus $A'(x) = 2x - 16\,000x^{-2}$

and $A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

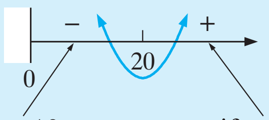
i.e., $2x^3 = 16\,000$

$$x^3 = 8000$$

$$x = \sqrt[3]{8000}$$

$$x = 20$$

Step 4: **Sign diagram test** *or* **Second derivative test**



if $x = 10$

$$A'(10) = 20 - \frac{16\,000}{100}$$
$$= 20 - 160$$
$$= -140$$

if $x = 30$

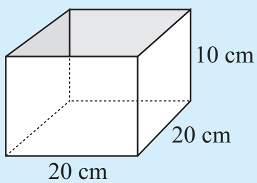
$$A'(30) = 60 - \frac{16\,000}{900}$$
$$\doteqdot 60 - 17.8$$
$$\doteqdot 42.2$$

$$A''(x) = 2 + 32\,000x^{-3}$$
$$= 2 + \frac{32\,000}{x^3}$$

which is always positive as $x^3 > 0$ for all $x > 0$.

Each of these tests establishes that minimum material is used to make the container when $x = 20$, and $y = \dfrac{4000}{20^2} = 10$,

i.e.,



is the shape.

EXERCISE 22H

Use **calculus techniques** in the following problems.

1 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides as the fourth side is an existing fence of another duck yard. Naturally the farmer wishes to minimise the length (and therefore the cost) of the fencing required to complete the job.

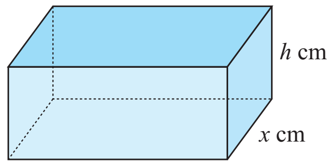
a If the shorter sides are of length x m, show that the required length of wire netting to be purchased is $L = 2x + \dfrac{100}{x}$.

b Use **technology** to help you sketch the graph of $y = 2x + \dfrac{100}{x}$.

c Find the minimum value of L and the corresponding value of x when this occurs.

d Sketch the optimum situation with its dimensions.

2 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of the box has dimensions in the ratio 2 : 1.



a What is the inner length of the box?

b Explain why $x^2h = 100$.

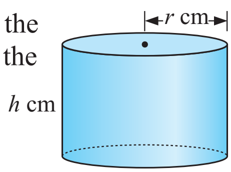
c Explain why the inner surface area of the box is given by $A(x) = 4x^2 + \dfrac{600}{x}$ cm^2.

d Use technology to help sketch the graph of $y = 4x^2 + \dfrac{600}{x}$.

e Find the minimum inner surface area of the box and the corresponding value of x.

f Draw a sketch of the optimum box shape with dimensions shown.

3 Consider the manufacture of 1 L capacity tin cans where the cost of the metal used to manufacture them is to be minimised. This means that the surface area is to be as small as possible but still must hold a litre.



a Explain why the height h, is given by $h = \dfrac{1000}{\pi r^2}$ cm.

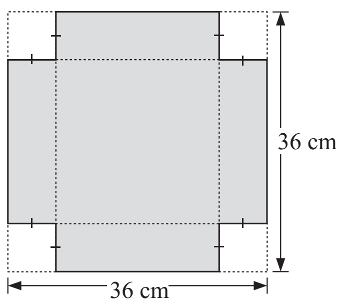
b Show that the total surface area A, is given by $A = 2\pi r^2 + \dfrac{2000}{r}$ cm^2.

c Use technology to help you sketch the graph of A against r.

d Find the value of r which makes A as small as possible.

e Draw a sketch of the dimensions of the can of smallest surface area.

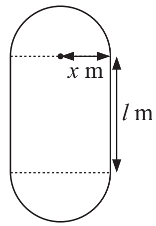
4 Sam has sheets of metal which are 36 cm by 36 cm square and wishes to use them. He cuts out identical squares which are x cm by x cm from the corners of each sheet. The remaining shape is then bent along the dashed lines to form an open container.



a Show that the capacity of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

5 An athletics track has two 'straights' of length l m and two semi-circular ends of radius x m. The perimeter of the track is 400 m.

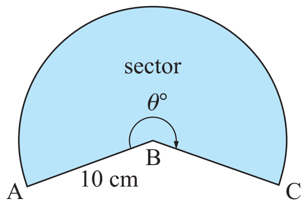


a Show that $l = 200 - \pi x$, and hence write down the possible values that x may have.

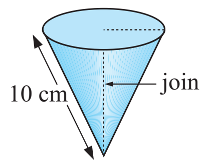
b Show that the area inside the track is $A = 400x - \pi x^2$.

c What values of l and x produce the largest area inside the track?

6 A sector of radius 10 cm is bent to form a conical cup as shown.



becomes



when edges AB and CB are joined with tape

Suppose the resulting cone has base radius r cm and height h cm.

a Show that in the sector, arc AC $= \dfrac{\theta\pi}{18}$.

b If r is the radius of the cone, explain why $r = \dfrac{\theta}{36}$.

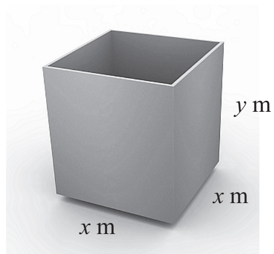
c If h is the height of the cone show that $h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$.

d If $V(\theta)$ is the cone's capacity, find $V(\theta)$ in terms of θ only.

e Use technology to sketch the graph of $V(\theta)$.

f Find θ when $V(\theta)$ is a maximum.

7 Special boxes are constructed from lead. Each box is to have an internal capacity of one cubic metre and the base is to be square. The cost per square metre of lining each of the 4 sides is twice the cost of lining the base.

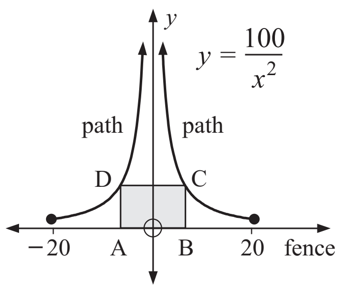


a Show that $y = \dfrac{1}{x^2}$.

b If the base costs \$25 per m^2 to line, show that the total cost of lining the box is $C(x) = 25(x^2 + 8x^{-1})$ dollars.

c What are the dimensions of the box costing least to line?

8 A retired mathematics teacher has a garden in which the paths are modelled by $y = \dfrac{100}{x^2}$ (as shown), $-20 \leqslant x \leqslant 20$.



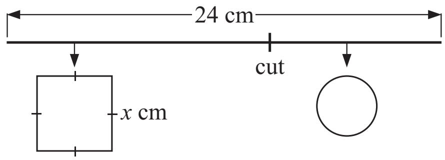
He plans a rose garden as shown by the shaded region.

a If OB $= x$ units, find the dimensions of rectangle ABCD.

b Show that as x increases, the area of the rectangle decreases for all $x > 0$.

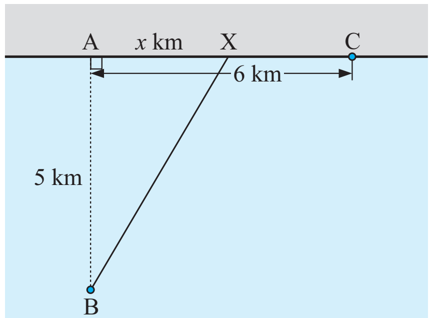
c Show that the perimeter P, of the rectangle is given by $P = 4x + \dfrac{200}{x^2}$ for $x > 0$ and hence find the dimensions of the rectangle of least perimeter.

9 Colin bends sheet steel into square section piping and circular section piping. A client supplies him with 24 cm wide sheets of steel which must be cut into two pieces, where one piece is bent into square-section tubing and the other into circular tubing.



However, the client insists that the sum of the cross-sectional areas is to be as small as possible. Where could Colin cut the sheet so that the client's wishes are fulfilled?

10 B is a row boat 5 km out at sea from A. AC is a straight sandy beach, 6 km long. Peter can row the boat at 8 kmph and run along the beach at 17 kmph. Suppose Peter rows directly from B to X, where X is some point on AC and AX $= x$ km.



a Explain why $0 \leqslant x \leqslant 6$.

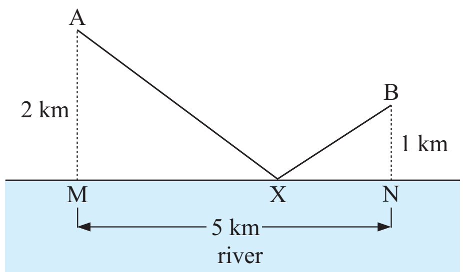
b If $T(x)$ is the *total time* Peter takes to row to X and then run along the beach to C, show that $T(x) = \dfrac{\sqrt{x^2+25}}{8} + \dfrac{6-x}{17}$ hrs.

c Find x when $\dfrac{dT}{dx} = 0$. What is the significance of this value of x? Prove your statement.

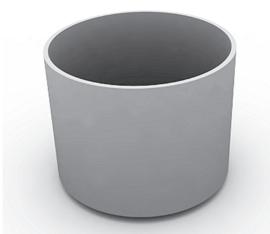
11 A pipeline is to be placed so that it connects point A to the river to point B.

A and B are two homesteads and X is the pumphouse.

How far from M should point X be so that the pipeline is as short as possible?

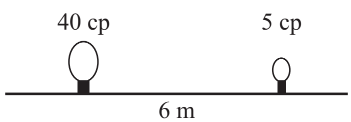


12



Open cylindrical bins are to contain 100 litres. Find the radius and the height of the bin made from the least amount of material (i.e., minimise the surface area).

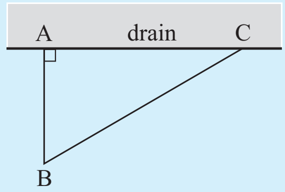
13 Two lamps are of intensities 40 and 5 candle-power respectively and are 6 m apart. If the intensity of illumination I, at any point is directly proportional to the power of the source and inversely proportional to the square of the distance from the source, find the darkest point on the line joining the two lamps.



Sometimes the variable to be optimised is in the form of a single square root function. In these situations it is convenient to square the function and use the fact that "if $A > 0$, the optimum value of $A(x)$ occurs at the same value of x as the optimum value of $[A(x)]^2$."

Example 18

An animal enclosure is a right angled triangle with one leg being a drain. The farmer has 300 m of fencing available for the other two sides, AB and BC.



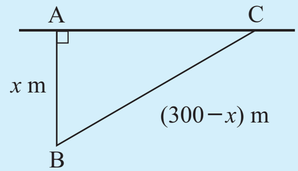
a Show that $\text{AC} = \sqrt{90\,000 - 600x}$ if $\text{AB} = x$ m.

b Find the maximum area of the triangular enclosure.
(**Hint:** If the area is A m^2, find A^2 in terms of x.
Notice that A is a maximum when A^2 takes its maximum value.)

a $(\text{AC})^2 + x^2 = (300 - x)^2$ {Pythagoras}

$\therefore\ (\text{AC})^2 = 90\,000 - 600x + x^2 - x^2$
$= 90\,000 - 600x$

$\therefore\ \text{AC} = \sqrt{90\,000 - 600x}$



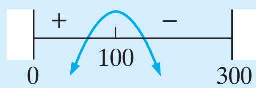
b The area of triangle ABC is

$$\begin{aligned} A(x) &= \tfrac{1}{2}(\text{base} \times \text{altitude}) \\ &= \tfrac{1}{2}(\text{AC} \times x) \\ &= \tfrac{1}{2}x\sqrt{90\,000 - 600x} \end{aligned}$$

So $[A(x)]^2 = \dfrac{x^2}{4}(90\,000 - 600x) = 22\,500x^2 - 150x^3$

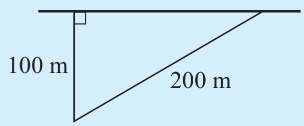
$$\begin{aligned} \therefore\ \frac{d}{dx}[A(x)]^2 &= 45\,000x - 450x^2 \\ &= 450x(100 - x) \end{aligned}$$

with sign diagram:

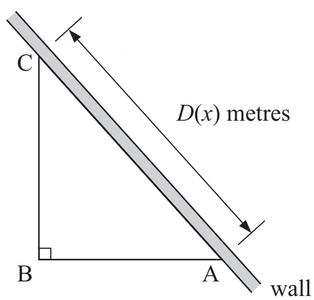


So $A(x)$ is maximised when $x = 100$

$A_{\max} = \tfrac{1}{2}(100)\sqrt{90\,000 - 60\,000}$
$\doteqdot 8660$ m^2



14 A right angled triangular pen is made from 24 m of fencing used for sides AB and BC. Side AC is an existing brick wall.



a If $\text{AB} = x$ m, find $D(x)$, the distance AC, in terms of x.

b Find $\dfrac{d[D(x)]^2}{dx}$ and hence draw a sign diagram for it.

c Find the smallest and the greatest value of $D(x)$ and the design of the pen in each case.

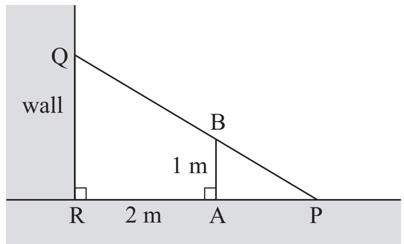
15 At 1.00 pm a ship A leaves port P, and sails in the direction 30^oT at 12 kmph. Also, at 1.00 pm ship B is 100 km due East of P and is sailing at 8 kmph towards P. Suppose t is the number of hours after 1.00 pm.

a Show that the distance $D(t)$ km, between the two ships is given by
$$D(t) = \sqrt{304t^2 - 2800t + 10\,000} \text{ km}$$

b Find the minimum value of $[D(t)]^2$ for all $t \geqslant 0$.

c At what time, to the nearest minute, are the ships closest?

16 AB is a 1 m high fence which is 2 m from a vertical wall, RQ. An extension ladder PQ is placed on the fence so that it touches the ground at P and the wall at Q.



a If $\text{AP} = x$ m, find QR in terms of x.

b If the ladder has length $L(x)$ m show that $[L(x)]^2 = (x+2)^2\left(1 + \dfrac{1}{x^2}\right)$.

c Show that $\dfrac{d[L(x)]^2}{dx} = 0$ only when $x = \sqrt[3]{2}$.

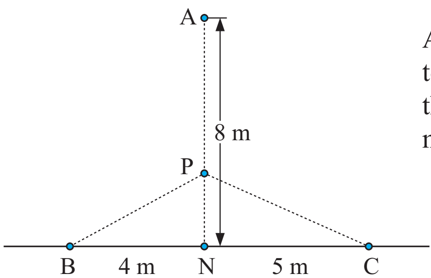
d Find, correct to the nearest centimetre, the shortest length of the extension ladder. You must prove that this length is the shortest.

Sometimes the derivative finding is difficult and technology use is recommended.

Use the **graphing package** or your **graphics calculator** to help solve the following problems.

GRAPHING PACKAGE

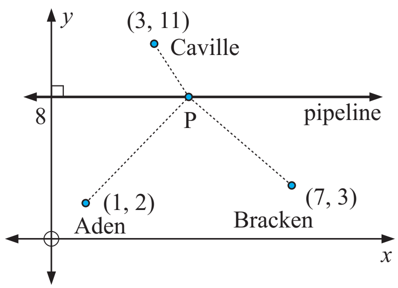
17



A, B and C are computers. A printer P is networked to each computer. Where should P be located so that the total cable length $\text{AP} + \text{BP} + \text{CP}$ is a minimum?

18 Three towns and their grid references are marked on the diagram alongside. A pumping station is to be located at P on the pipeline, to pump water to the three towns. Grid units are kilometres.

Exactly where should P be located so that pipelines to Aden, Bracken and Caville in total are as short as possible?

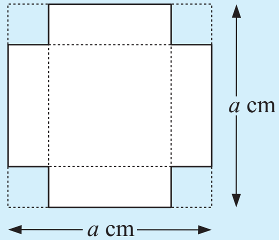


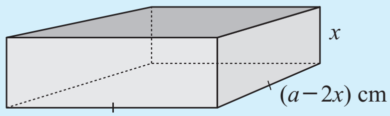
Sometimes technology does not provide easy solutions to problems where optimisation is required. This occurs when at least one quantity is unknown.

Example 19

A square sheet of metal has squares cut from its corners as shown.

What sized square should be cut out so that when bent into an open box shape the container holds maximum liquid?





Let x cm by x cm squares be cut out.

$$\begin{aligned}\text{Volume} &= \text{length} \times \text{width} \times \text{depth}\\ &= (a-2x) \times (a-2x) \times x\end{aligned}$$

i.e., $V(x) = x(a-2x)^2$

$$\begin{aligned}\text{Now} \quad V'(x) &= 1(a-2x)^2 + x \times 2(a-2x)^1 \times (-2) \quad \{\text{product rule}\}\\ &= (a-2x)[a-2x-4x]\\ &= (a-2x)(a-6x)\end{aligned}$$

and $V'(x) = 0$ when $x = \dfrac{a}{2}$ or $\dfrac{a}{6}$

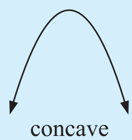
We notice that $a - 2x > 0$ i.e., $a > 2x$ or $x < \dfrac{a}{2}$, So, $0 < x < \dfrac{a}{2}$

Thus $x = \dfrac{a}{6}$ is the only value in $0 < x < \dfrac{a}{2}$ with $V'(x) = 0$.

Second derivative test:

$$\begin{aligned}\text{Now} \quad V''(x) &= -2(a-6x) + (a-2x)(-6) \quad \{\text{product rule}\}\\ &= -2a + 12x - 6a + 12x\\ &= 24x - 8a\end{aligned}$$

$\therefore \; V''(\frac{a}{6}) = 4a - 8a = -4a$ which is < 0

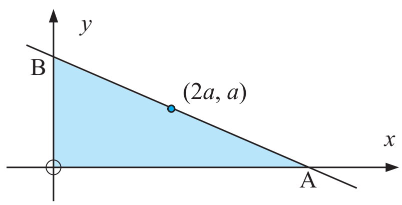


So, volume is maximised when $x = \dfrac{a}{6}$.

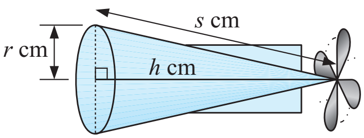
Conclusion: When $x = \dfrac{a}{6}$, the resulting container has maximum capacity.

19 Infinitely many lines can be drawn through the fixed point $(2a, a)$ where $a > 0$.

Find the position of point A on the x-axis so that triangle AOB has minimum area.

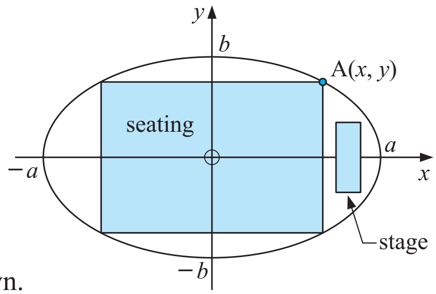


20 The trailing cone of a guided long range torpedo is to be conical with slant edge s cm where s is fixed, but unknown. The cone is hollow and must contain maximum volume of fuel.



Find the ratio of $s : r$ when maximum fuel carrying capacity occurs.

21 A company constructs rectangular seating plans and arranges seats for pop concerts on AFL grounds. All AFL grounds used are elliptical in shape and the equation of the ellipse illustrated is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where a and b are the lengths of its semi-major and semi-minor axes as shown.



a Show that $y=\dfrac{b}{a}\sqrt{a^2-x^2}$ for A as shown.

b Show that the seating area is given by $A(x)=\dfrac{4bx}{a}\sqrt{a^2-x^2}$.

c Show that $A'(x)=0$ when $x=\dfrac{a}{\sqrt{2}}$.

d Prove that the seating area is a maximum when $x=\dfrac{a}{\sqrt{2}}$.

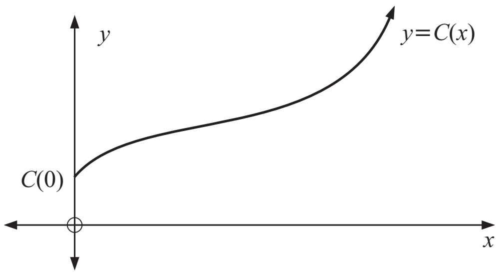
e Given that the area of the ellipse is πab, what percentage of the ground is occupied by the seats in the optimum case?

I ECONOMIC MODELS

In a previous section, we discussed the idea of a **cost function** $C(x)$ where $C(x)$ is the cost of producing x items of a product.

The **marginal cost** is the instantaneous rate of change in cost with respect to the number of items made, x, i.e., $C'(x)$ and is the slope function of the cost function.

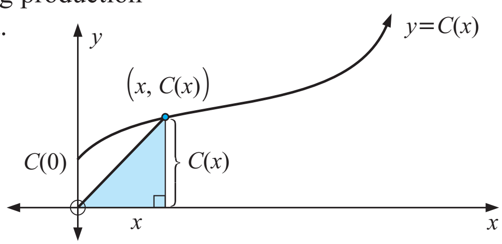
We saw that a typical cost function was modelled by a polynomial.



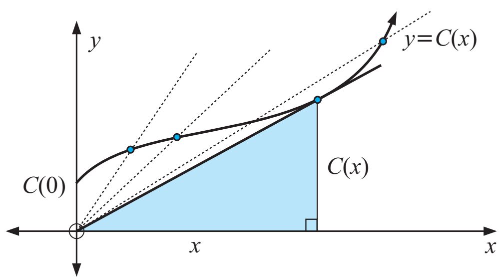
AVERAGE COST FUNCTIONS

Many manufacturers will be interested in gearing production so that the **average cost** of an item is minimised.

$A(x)=\dfrac{C(x)}{x}$ is the cost per item when x of them are made.



Notice that $\dfrac{C(x)}{x}$ is the slope of the line from O(0, 0) to $y=C(x)$.



From the graph it appears that the average cost is a minimum when

average cost $=$ marginal cost.

This is easily checked using calculus.

$$A(x) = \frac{C(x)}{x} \quad \therefore \quad A'(x) = \frac{C'(x)x - C(x) \times 1}{x^2} \quad \text{and}$$

$$A'(x) = 0 \quad \text{when} \quad xC'(x) = C(x)$$

$$\text{i.e.,} \quad C'(x) = \frac{C(x)}{x}$$

$$\text{i.e.,} \quad C'(x) = A(x)$$

So, average cost is a minimum when marginal cost = average cost

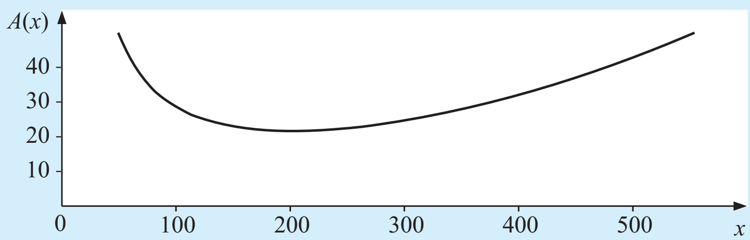
Example 20

For the cost model $C(x) = 0.000\,13x^3 + 0.002x^2 + 5x + 2200$:

a find the average cost function and draw a sketch graph of it.

b At what production level is average cost minimised?

a $A(x) = \frac{C(x)}{x} = 0.000\,13x^2 + 0.002x + 5 + \frac{2200}{x}$ dollars/item



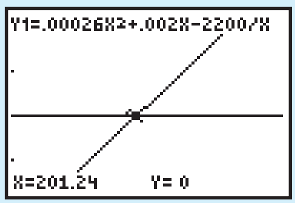
b $C'(x) = 0.000\,39x^2 + 0.004x + 5$

Now $C'(x) = A(x)$ when

$$0.000\,39x^2 + 0.004x + 5 = 0.000\,13x^2 + 0.002x + 5 + \frac{2200}{x}$$

$$\text{i.e.,} \quad 0.000\,26x^2 + 0.002x - \frac{2200}{x} = 0$$

$$\text{i.e.,} \quad x \doteqdot 201$$



DEMAND, REVENUE AND PROFIT FUNCTIONS

Suppose $p(x)$ is the price per item that the business charges when selling x items. $p(x)$ is called the **demand function** (or **price function**) and is a decreasing function as cost of production reduces as volume increases.

If we sell x items the total revenue raised would be $R(x) = xp(x)$ where $R(x)$ is called the **revenue function** (or **sales function**).

Finally, if x items are sold, the total profit is $P(x) = R(x) - C(x)$ and $P(x)$ is called the **profit function**.

Note:
- The rate of change in revenue $R'(x)$ is the **marginal revenue function**.
- The rate of change in profit $P'(x)$ is the **marginal profit function**.

MAXIMISING PROFIT

Profit is maximised when $P'(x) = 0$.

Since $P(x) = R(x) - C(x)$ then $P'(x) = R'(x) - C'(x)$.

$\therefore$ $P'(x) = 0$ when $R'(x) = C'(x)$ i.e., marginal revenue = marginal cost

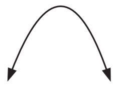
By the **second derivative test**,

as $P''(x) = R''(x) - C''(x)$

we require $P''(x) < 0$

i.e., $R''(x) - C''(x) < 0$

i.e., $R''(x) < C''(x)$



Example 21

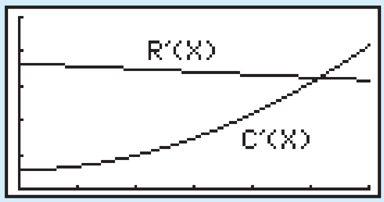
For cost function $C(x) = 0.000\,13x^3 + 0.002x^2 + 5x + 2200$ dollars and demand function $p(x) = 36.5 - 0.008x$ dollars, $x \geqslant 0$:

a state the revenue function and find the marginal revenue function.

b What level of production will maximise profits?

a $R(x) = xp(x) = 36.5x - 0.008x^2$

and $R'(x) = 36.5 - 0.016x$



b If $C'(x) = R'(x)$ then

$0.000\,39x^2 + 0.004x + 5 = 36.5 - 0.016x$

$\therefore$ $0.000\,39x^2 + 0.02x - 31.5 = 0$

which has solutions $x \doteqdot -315.07$ or 259.71

But $x \geqslant 0$, so $C'(x) = R'(x)$ only when $x \doteqdot 260$

And as $R''(x) = -0.016$ and $C''(x) = 0.000\,78x + 0.004$

$R''(260) = -0.016$ and $C''(260) \doteqdot 0.207$

i.e., $R''(260) < C''(260)$

So maximum profit is made when 260 items are produced.

EXERCISE 22I

1 $C(x) = 38\,000 + 250x + x^2$ dollars is the cost function for producing x items. Find:

a the cost, average cost per item and marginal cost for producing 800 items

b the production level needed to minimise average cost and the corresponding average cost.

2 A cost function for producing x items is $C(x) = 295 + 24x - 0.08x^2 + 0.0008x^3$ dollars. Find:

- **a** the average cost and marginal cost functions
- **b** the minimum average cost
- **c** the minimum marginal cost.

3 A small manufacturer can produce x fittings per day where $0 \leqslant x \leqslant 10\,000$. The costs are:

- \$1000 per day for the workers
- \$2 per day per fitting
- $\$\dfrac{5000}{x}$ per day for running costs and maintenance.

How many fittings should be produced daily to minimise costs?

4 For the cost function $C(x) = 720 + 4x + 0.02x^2$ dollars and price function $p(x) = 15 - 0.002x$ dollars, find the production level that will maximise profits.

5 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars and each blanket may be sold at $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

6 A manufacturer of DVD players has been selling 800 each week at \$150 each. From a market survey it is discovered that for each \$5 reduction in price, they will sell an extra 40 DVD players each week.

- **a** What is the demand function?
- **b** How large a reduction in price (rebate) should the manufacturer give a buyer so that revenue will be maximised?
- **c** If the weekly cost function is $C(x) = 20\,000 + 30x$ dollars, what rebate would maximise the profit?

7 The cost of running a boat is $\dfrac{v^2}{10}$ dollars per hour where v is the speed of the boat.
All other costs amount to \$62.50 per hour. Find the speed which will minimise the total cost per kilometre.

REVIEW SET 22A

1 A particle P, moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds $(t \geqslant 0)$.

- **a** Find expressions for the particle's velocity and acceleration and draw sign diagrams for each of them.
- **b** Find the initial conditions.
- **c** Describe the motion of the particle at time $t = 2$ seconds.
- **d** Find the times and positions where the particle changes direction.
- **e** Draw a diagram to illustrate the motion of P.
- **f** Determine the time intervals when the particle's speed is increasing.

2 The cost per hour of running a freight train is given by $C(v) = 10v + \dfrac{90}{v}$ dollars where v is the average speed of the train.

a Find the cost of running the train for:

i two hours at 15 kmph **ii** 5 hours at 24 kmph.

b Find the rate of change in the cost of running the train at speeds of:

i 10 kmph **ii** 6 kmph.

c At what speed will the cost be a minimum?

3 For the function $f(x) = 2x^3 - 3x^2 - 36x + 7$:

a find and classify all stationary points and points of inflection

b find intervals where the function is increasing and decreasing

c find intervals where the function is concave up/down

d sketch the graph of $y = f(x)$, showing all important features.

4 For $f(x) = \dfrac{3x-2}{x+3}$:

a state the equation of the vertical asymptote

b find the axis intercepts

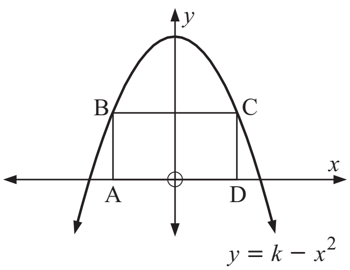
c find $f'(x)$ and draw a sign diagram for it

d find the position and nature of any stationary points.

5 Rectangle ABCD is inscribed within the parabola $y = k - x^2$ and the x-axis, as shown.

a If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 2kx - 2x^3$.

b If the area of ABCD is a maximum when $\text{AD} = 2\sqrt{3}$, find k.



6 For the function $f(x) = x^3 + x^2 + 2x - 4$:

a state the y-intercept

b find the x-intercept(s), given that $x = 1$ is one of them

c find and classify any stationary points and points of inflection

d on a sketch of the cubic, show the features found in **a**, **b** and **c**.

7 A rectangle has a fixed area of 500 m^2, but its length y m, and width x m may vary.

a Find y in terms of x.

b Find $\dfrac{dy}{dx}$ and explain why $\dfrac{dy}{dx} < 0$ for all values of x.

c Interpret your results of **b**.

8 The height of a tree at time t years after the tree was planted is given by:

$$H(t) = 6\left(1 - \frac{2}{t+3}\right) \text{ metres}, \quad t \geqslant 0.$$

a How high was the tree when it was planted?

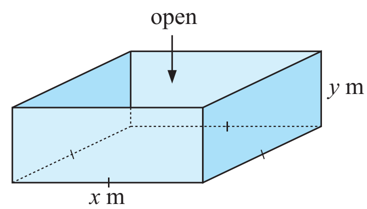
b Determine the height of the tree after $t = 3$, 6 and 9 years.

c Find the rate at which the tree is growing at $t = 0$, 3, 6 and 9 years.

d Show that $H'(t) > 0$. What is the significance of this result?

e Sketch the graph of $H(t)$ against t.

9 A manufacturer of open steel boxes has to make one with a square base and a volume of 1 m^3. The steel costs \$2 per square metre.

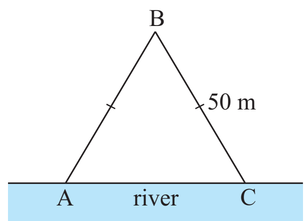


a If the base measures x m by x m and the height is y m, find y in terms of x.

b Hence, show that the total cost of the steel is $C(x) = 2x^2 + \dfrac{8}{x}$ dollars.

c Find the dimensions of the box costing the manufacturer least to make.

10 A triangular pen is enclosed by two fences AB and BC each of length 50 m, and the river is the third side.



a If $\text{AC} = 2x$ m, show that the area of triangle ABC is $A(x) = x\sqrt{2500 - x^2}$.

b Find $\dfrac{d[A(x)]^2}{dx}$ and hence find x when the area is a maximum.

11 A particle P, moves in a straight line with position from O given by

$s(t) = 15t - \dfrac{60}{(t-1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

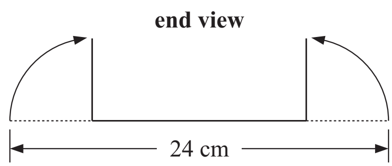
a Find velocity and acceleration functions for P's motion.

b Describe the motion of P at $t = 3$ secs.

c For what values of t is the particle's speed increasing?

12 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown in the illustration.

Where must the bends be made in order to maximise the water carried by the gutter?



REVIEW SET 22B

1 The weight of a fish t weeks after it is released from a breeding pond is given by

$$W(t) = 5000 - \frac{4900}{t^2 + 1} \text{ grams}, \quad t \geqslant 0.$$

a How heavy is the fish at the time of release?

b How heavy is the fish after:

i 1 week **ii** 4 weeks **iii** 26 weeks?

c Find the rate at which the fish is growing at time:
i 0 days ii 10 days iii 20 days.

d Sketch the graph of $W(t)$ against t.

2 A particle moves along the x-axis with position relative to origin O, given by $x(t) = 3t - \sqrt{t}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration at any time t and draw sign diagrams for each function.

b Find the initial conditions and hence describe the motion at that instant.

c Describe the motion of the particle at $t = 9$ seconds.

d Find the time(s) and position(s) when the particle reverses direction.

e Determine the time intervals when the particle's speed is decreasing.

3 Given that $G = \dfrac{(t-2)^2}{3} + 5$ units where t is the time in seconds:

a find the values of t for which G is increasing

b for what values of t is $\dfrac{dG}{dt} > 10$ units per second?

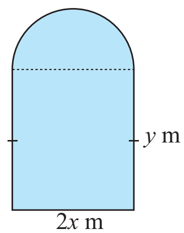
4 For the function $f(x) = x^3 - 4x^2 + 4x$:

a find all axis intercepts

b find and classify all stationary points and points of inflection

c sketch the graph of $y = f(x)$ showing features from **a** and **b**.

5 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.



a Using the dimensions shown on the figure show that $y = 100 - x - \frac{\pi}{2}x$.

b Hence, find the area of the lawn $A(x)$, in terms of x only.

c Find the dimensions of the lawn if it has maximum area.

6 Consider the function, $f(x) = x\sqrt{x} - x$.

a For what values of x does $f(x)$ have meaning?

b Find the axis intercepts for $y = f(x)$.

c Find the positon and nature of any stationary points and points of inflection.

d Sketch the graph of $y = f(x)$ showing all special features found in **a**, **b** and **c**.

7 For the function $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$:

a find the axis intercepts.

b Why does $f(x)$ have no vertical asymptotes?

c Find the position and nature of any stationary points.

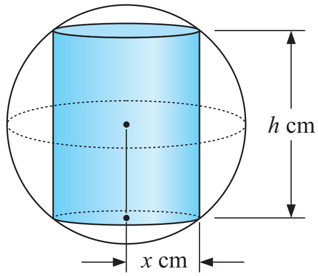
d Show that $y = f(x)$ has non-stationary inflections at $x = \pm\sqrt{\frac{1}{3}}$.

e Sketch the graph of $y = f(x)$ showing all features found in **a**, **b**, **c** and **d** above.

8 Consider $f(x) = \dfrac{x-2}{x^2+x-2}$.

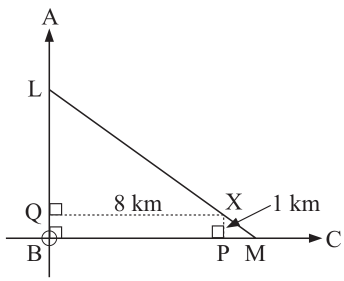
- **a** Determine the equations of any vertical asymptotes.
- **b** Find the position and nature of its turning points.
- **c** Find its axis intercepts.
- **d** Sketch the graph of the function showing all important features found in **a**, **b** and **c** above.
- **e** For what values of p does $\dfrac{x-2}{x^2+x-2} = p$ have two real distinct roots?

9 A machinist has a spherical ball of brass with diameter 10 cm. The ball is placed in a lathe and machined into a cylinder.



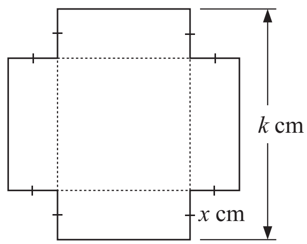
- **a** If the cylinder has radius x cm, show that the cylinder's volume is given by
$$V(x) = \pi x^2\sqrt{100-4x^2}\ \text{cm}^3.$$
- **b** Hence, find the dimensions of the cylinder of largest volume which can be made.

10 Two roads AB and BC meet at right angles. A straight pipeline LM is to be laid between the two roads with the requirement that it must pass through point X.



- **a** If $\text{PM} = x$ km, find LQ in terms of x.
- **b** Hence show that the length of the pipeline is given by $L(x) = \sqrt{x^2+1}\left(1+\dfrac{8}{x}\right)$ km
- **c** Find $\dfrac{d[L(x)]^2}{dx}$ and hence find the shortest possible length for the pipeline.

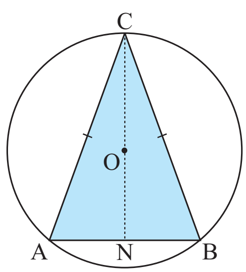
11 A square sheet of tin-plate is k cm by k cm and four squares each with sides x cm are cut from its corners.



The remainder is bent into the shape of an open square-based container.

Show that the capacity of the container is maximised when $x = \dfrac{k}{6}$.

12 Infinitely many isosceles triangles can be inscribed in a circle, one of which has maximum area.



- **a** If $\text{ON} = x$ cm and the circle's radius is fixed at r cm, show that the area of ΔABC is given by
$$A(x) = \sqrt{r^2-x^2}\,(r+x)$$
- **b** Hence, prove that the triangle of maximum area is equilateral.

Chapter 23

Derivatives of exponential and logarithmic functions

Contents:

A Derivatives of exponential functions

Investigation 1: The derivative of $y = a^x$

Investigation 2: Finding a when $y = a^x$ and $\frac{dy}{dx} = a^x$

B Using natural logarithms

C Derivatives of logarithmic functions

Investigation 3: The derivative of $\ln x$

D Applications

Review set 23A

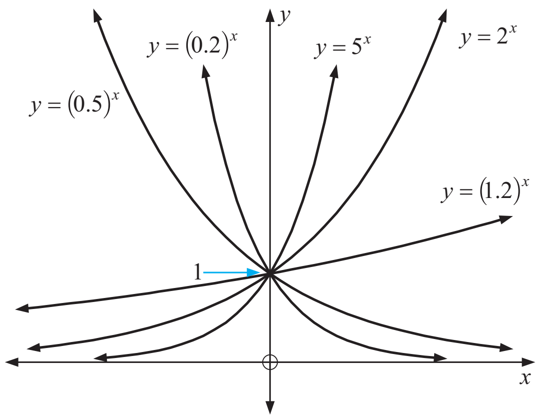
Review set 23B

The simplest **exponential functions** are of the form $f(x) = a^x$ where a is any positive constant, $a \neq 1$.

All members of the exponential family $f(x) = a^x$ have the properties that:

- their graphs pass through the point (0, 1)
- their graphs are asymptotic to the x-axis at one end
- their graphs are above the x-axis for all values of x
- their graphs are concave up for all x
- their graphs are increasing for $a > 1$ and decreasing for $0 < a < 1$.

For example,



A DERIVATIVES OF EXPONENTIAL FUNCTIONS

INVESTIGATION 1 THE DERIVATIVE OF $y = a^x$

This investigation could be done by using a **graphics calculator** or by clicking on the icon for a **computer** method.

The purpose of this investigation is to observe the nature of the derivative of $f(x) = a^x$ for $a = 2, 3, 5$ and $\frac{1}{2}$.

CALCULUS DEMO

TI

C

What to do:

MODELLING SOFTWARE

1 For $y = 2^x$ find the gradient of the tangent at $x = 0, 0.5, 1, 1.5, 2$ and 2.5. Use modelling techniques from your graphics calculator or by clicking on the icon to show that

$$\frac{dy}{dx} \doteqdot 0.693 \times 2^x.$$

2 Repeat **1** for $y = 3^x$.

3 Repeat **1** for $y = 5^x$.

4 Repeat **1** for $y = (0.5)^x$.

5 Use **1**, **2**, **3** and **4** to help write a statement about the derivative of the general exponential $y = a^x$ for $a > 0, a \neq 1$.

From the previous investigation you should have discovered that:

- if $f(x) = a^x$ then $f'(x) = ka^x$ where k is a constant
- k is the derivative of $y = a^x$ at $x = 0$, i.e., $k = f'(0)$.

This result is easily proved algebraically.

If $f(x)=a^x$, then $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ {first principles definition}

$$=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

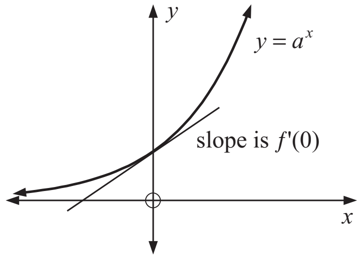
$$=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$=a^x\times\left(\lim_{h\to 0}\frac{a^h-1}{h}\right)$$ {as a^x is independent of h}

But $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{a^h-1}{h}$$

$\therefore\quad f'(x)=a^x f'(0)$



At this stage we realise that if we can find a value of a such that $f'(0)=1$, then we have found *a function which is its own derivative*.

INVESTIGATION 2 FINDING a WHEN $y=a^x$ AND $\dfrac{dy}{dx}=a^x$

Click on the icon to graph $f(x)=a^x$ and its derivative function $y=f'(x)$.

DEMO

Experiment with different values of a until the graphs of $f(x)=a^x$ and $y=f'(x)$ are the same.

Because of the nature of this investigation only an approximate value of a (to 2 decimal places) can be found.

From **Investigation 2** you should have discovered that if $a \doteqdot 2.72$ and $f(x)=a^x$ then $f'(x)=a^x$ also.

To find this value of a more accurately we could return to the algebraic approach.

We showed that if $f(x)=a^x$ then $f'(x)=a^x\left(\lim\limits_{h\to 0}\dfrac{a^h-1}{h}\right)$.

So if $f'(x)=a^x$ also we require $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}=1$.

Now if $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}=1$ then roughly speaking,

$\dfrac{a^h-1}{h}\doteqdot 1$ for values of h which are close to 0

$\therefore\quad a^h \doteqdot 1+h$ for h close to 0.

Thus $a^{\frac{1}{n}} \doteqdot 1 + \dfrac{1}{n}$ for large values of n $\quad \{h = \dfrac{1}{n} \to 0 \quad \text{as} \quad n \to \infty\}$

$$\therefore \quad a \doteqdot \left(1 + \frac{1}{n}\right)^n \quad \text{for large values of } n.$$

We now examine $\left(1 + \dfrac{1}{n}\right)^n$ as $n \to \infty$.

Notice:

n	$\left(1+\frac{1}{n}\right)^n$	n	$\left(1+\frac{1}{n}\right)^n$
10	2.593 742 460	10^7	2.718 281 693
10^2	2.704 813 829	10^8	2.718 281 815
10^3	2.716 923 932	10^9	2.718 281 827
10^4	2.718 145 927	10^{10}	2.718 281 828
10^5	2.718 268 237	10^{11}	2.718 281 828
10^6	2.718 280 469	10^{12}	2.718 281 828

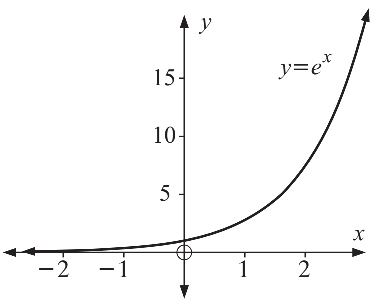
In fact as $n \to \infty$, $\left(1 + \dfrac{1}{n}\right)^n \to 2.718\,281\,828\,459\,045\,235\ldots.$

and this irrational number is given the symbol e to represent it,

i.e., $e = 2.718\,281\,828\,459\,045\,235\ldots.$ and is called **exponential** e.

Thus, if $f(x) = e^x$ then $f'(x) = e^x$.

We have discussed $y = e^x$ near to the origin. But, what happens to the graph for large positive and negative x-values?



Notice that $\dfrac{dy}{dx} = e^x = y$.

For x large and positive, we write $x \to \infty$.

As $y = e^x$, $y \to \infty$ very rapidly.

So $\dfrac{dy}{dx} \to \infty$.

This means that the slope of the curve is very large for large x values.

For x large and negative, we write $x \to -\infty$.

As $y = e^x$, $y \to 0$ and so $\dfrac{dy}{dx} \to 0$.

This means for large negative x, the graph becomes flatter.

Observe that $e^x > 0$ for all x.

THE DERIVATIVE OF $e^{f(x)}$

Functions such as e^{-x}, e^{2x+3}, e^{-x^2}, x^2e^{2x}, etc need to be differentiated in problem solving. How do we differentiate such functions?

Consider $y = e^{f(x)}$.

Now $y = e^u$ where $u = f(x)$.

$$\text{But } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = e^u \frac{du}{dx}$$

$$= e^{f(x)} \times f'(x)$$

Summary:

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$

Alternative notation: e^x is sometimes written as $\exp(x)$.
For example, $\exp(1-x) = e^{1-x}$.

Example 1

Find the slope function for y equal to:

a $2e^x + e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a if $y = 2e^x + e^{-3x}$, then
$$\frac{dy}{dx} = 2e^x + e^{-3x}(-3) = 2e^x - 3e^{-3x}$$

b if $y = x^2e^{-x}$, then
$$\frac{dy}{dx} = 2xe^{-x} + x^2e^{-x}(-1) \qquad \{\text{product rule}\}$$
$$= 2xe^{-x} - x^2e^{-x}$$

c if $y = \dfrac{e^{2x}}{x}$, then
$$\frac{dy}{dx} = \frac{e^{2x}(2)x - e^{2x}(1)}{x^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{e^{2x}(2x-1)}{x^2}$$

EXERCISE 23A

1 Find the slope function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** $\exp(-2x)$ **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1 + e^{2x})$ **l** $20(1 - e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Use the product or quotient rules to find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $\sqrt{x}e^{-x}$ **h** $\dfrac{e^x+2}{e^{-x}+1}$

Example 2

Find the slope function of $f(x)$ equal to:

a $(e^x-1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a $y=(e^x-1)^3=u^3$ where $u=e^x-1$

$$\text{Now } \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$=3u^2\frac{du}{dx}$$
$$=3(e^x-1)^2\times e^x$$
$$=3e^x(e^x-1)^2$$

b $y=(2e^{-x}+1)^{-\frac{1}{2}}=u^{-\frac{1}{2}}$ where $u=2e^{-x}+1$

$$\text{Now } \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$
$$=-\tfrac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx}$$
$$=-\tfrac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}}\times 2e^{-x}(-1)$$
$$=e^{-x}(2e^{-x}+1)^{-\frac{3}{2}}$$
$$=\frac{1}{e^x(2e^{-x}+1)^{\frac{3}{2}}}$$

3 Find the slope function of $f(x)$ equal to:

a $(e^x+2)^4$ **b** $\dfrac{1}{1-e^{-x}}$ **c** $\sqrt{e^{2x}+10}$

d $\dfrac{1}{(1-e^{3x})^2}$ **e** $\dfrac{1}{\sqrt{1-e^{-x}}}$ **f** $x\sqrt{1-2e^{-x}}$

4 If $y=Ae^{kx}$, where A and k are constants:

a show that **i** $\dfrac{dy}{dx}=ky$ **ii** $\dfrac{d^2y}{dx^2}=k^2y$.

b Predict the connection between $\dfrac{d^ny}{dx^n}$ and y. No proof is required.

5 If $y=2e^{3x}+5e^{4x}$, show that $\dfrac{d^2y}{dx^2}-7\dfrac{dy}{dx}+12y=0$.

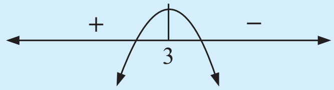
Example 3

Find the position and nature of any turning points of $y = (x-2)e^{-x}$.

$$\begin{aligned}\frac{dy}{dx} &= (1)e^{-x} + (x-2)e^{-x}(-1) \qquad \{\text{product rule}\}\\ &= e^{-x}(1-(x-2))\\ &= \frac{3-x}{e^x} \quad \text{where } e^x \text{ is positive for all } x.\end{aligned}$$

So, $\dfrac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore$ at $x = 3$ we have a maximum turning point.

But when $x = 3$, $y = (1)e^{-3} = \dfrac{1}{e^3}$

$\therefore$ the maximum turning point is $(3, \frac{1}{e^3})$.

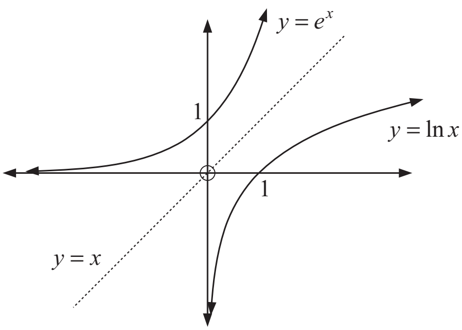
6 Find the position and nature of the turning point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \dfrac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

B USING NATURAL LOGARITHMS

Recall that:

if $e^x = a$ then $x = \ln a$ and vice versa
i.e., $e^x = a \Leftrightarrow x = \ln a$.



The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ in the mirror line $y = x$.

Functions which are reflections in the mirror line $y = x$ are called **inverse functions**. $y = e^x$ and $y = \ln x$ are inverse functions.

From the definition $e^x = a \Leftrightarrow x = \ln a$ we observe that $e^{\ln a} = a$

This means that: any positive real number can be written as a power of e,

or alternatively, the **natural logarithm** of any positive number is its power of e, i.e., $\ln e^n = n$.

Recall that the **laws of logarithms** in base e are identical to those for base 10.

These are: For $a > 0$, $b > 0$

- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$
- $\ln(a^n) = n \ln a$ **Note**: $\ln e^n = n$

Notice also that:

- $\ln 1 = 0$ and $\ln e = 1$
- $\ln\left(\dfrac{1}{a}\right) = -\ln a$

EXERCISE 23B (Mainly review)

1 Write as a natural logarithmic equation:

a $N = 50e^{2t}$ **b** $P = 8.69e^{-0.0541t}$ **c** $S = a^2e^{-kt}$

2 Write in exponential form:

a $\ln D \doteqdot 2.1 + 0.69t$ **b** $\ln G \doteqdot -31.64 + 0.0173t$

c $\ln P = \ln g - 2t$ **d** $\ln F = 2\ln x - 0.03t$

[**Hint:** In **c** find $\ln P - \ln g$.]

Example 4

Find, without a calculator, the exact values of: **a** $\ln e^3$ **b** $e^{3\ln 2}$

a $\ln e^3$
$= 3$

b $e^{3\ln 2}$
$= e^{\ln 2^3}$ {third log law}
$= e^{\ln 8}$ {as $2^3 = 8$}
$= 8$ {as $e^{\ln a} = a$}

3 Without using a calculator, evaluate:

a $\ln e^2$ **b** $\ln\sqrt{e}$ **c** $\ln\left(\dfrac{1}{e}\right)$ **d** $\ln\left(\dfrac{1}{\sqrt{e}}\right)$

e $e^{\ln 3}$ **f** $e^{2\ln 3}$ **g** $e^{-\ln 5}$ **h** $e^{-2\ln 2}$

Use your calculator to check the answers above.

4 Simplify to $\ln k$, using the laws of logarithms:

a $\ln 5 + \ln 6$ **b** $3\ln 4 - 2\ln 2$ **c** $2\ln 5$ **d** $2 + 3\ln 2$

5 Write as a power of e: **a** 2 **b** 10 **c** a **d** a^x

Example 5

Solve for x: **a** $e^x = 7$ **b** $e^x + 2 = 15e^{-x}$

a $e^x = 7$
$\therefore \ln e^x = \ln 7$ {find ln of both sides}
$\therefore x = \ln 7$

b
$$\begin{aligned}
e^x + 2 &= 15e^{-x} \\
\therefore\ e^x(e^x+2) &= 15e^{-x} \times e^x \quad \{\text{multiply both sides by } e^x\} \\
\therefore\ e^{2x} + 2e^x &= 15 \quad \{e^o = 1\} \\
\therefore\ e^{2x} + 2e^x - 15 &= 0 \\
\therefore\ (e^x+5)(e^x-3) &= 0 \\
\therefore\ e^x &= -5 \text{ or } 3 \\
\therefore\ x &= \ln 3 \quad \{\text{as } e^x = -5 \text{ is impossible}\}
\end{aligned}$$

6 Solve for x:

a $e^x = 2$ **b** $e^x = -2$ **c** $e^x = 0$

d $e^{2x} = 2e^x$ **e** $e^x = e^{-x}$ **f** $e^{2x} - 5e^x + 6 = 0$

g $e^x + 2 = 3e^{-x}$ **h** $1 + 12e^{-x} = e^x$ **i** $e^x + e^{-x} = 3$

7 Solve for x:

a $\ln x + \ln(x+2) = \ln 8$ **b** $\ln(x-2) - \ln(x+3) = \ln 2$

Example 6

Find algebraically, the points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.
Check your solution using technology.

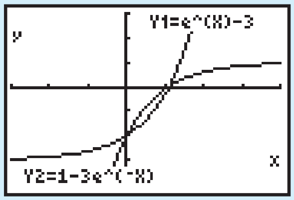
GRAPHING PACKAGE

TI

C

The functions meet where $e^x - 3 = 1 - 3e^{-x}$

$$\begin{aligned}
\therefore\ e^x - 4 + 3e^{-x} &= 0 \\
\therefore\ e^{2x} - 4e^x + 3 &= 0 \quad \{\text{multiplying each term by } e^x\} \\
\therefore\ (e^x-1)(e^x-3) &= 0 \\
\therefore\ e^x &= 1 \text{ or } 3 \\
\therefore\ x &= \ln 1 \text{ or } \ln 3 \\
\therefore\ x &= 0 \text{ or } \ln 3
\end{aligned}$$



when $x = 0$, $y = e^0 - 3 = -2$

when $x = \ln 3$, $e^x = 3$ $\therefore$ $y = 3 - 3 = 0$

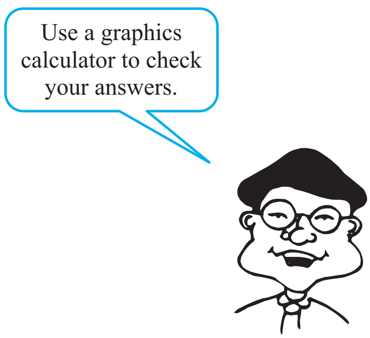
$\therefore$ functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

8 Find algebraically, the point(s) of intersection of:

a $y = e^x$ and $y = e^{2x} - 6$

b $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$



Example 7

Consider the function $y = 2 - e^{-x}$.

a Find the x-intercept. **b** Find the y-intercept.

c Show algebraically that the function is increasing for all x.

d Show algebraically that the function is concave down for all x.

e Use technology to help graph $y = 2 - e^{-x}$.

f Explain why $y = 2$ is a horizontal asymptote.

a Any graph cuts the x-axis when $y = 0$, i.e., $0 = 2 - e^{-x}$

$$\therefore \quad e^{-x} = 2$$
$$\therefore \quad -x = \ln 2$$
$$\therefore \quad x = -\ln 2$$

The graph cuts the x-axis at $-\ln 2$. (or $\doteqdot -0.69$)

b The y-intercept occurs when $x = 0$
i.e., $y = 2 - e^0 = 2 - 1 = 1$.

c $\dfrac{dy}{dx} = 0 - e^{-x}(-1) = e^{-x} = \dfrac{1}{e^x}$

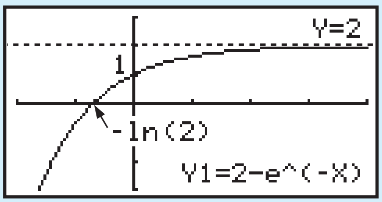
Now $e^x > 0$ for all x, $\therefore \dfrac{dy}{dx} > 0$ for all x.

$\therefore$ the function is increasing for all x.

d $\dfrac{d^2y}{dx^2} = e^{-x}(-1) = \dfrac{-1}{e^x}$ which is < 0 for all x.

$\therefore$ the function is concave down for all x.

e



f As $x \to \infty$, $e^x \to \infty$
and $e^{-x} \to 0$
$\therefore \quad 2 - e^{-x} \to 2$
i.e., $y \to 2$ (below)

9 The function $f(x) = 3 - e^x$ cuts the x-axis at A and the y-axis at B.

a Find the coordinates of A and B.

b Show algebraically that the function is decreasing for all x.

c Find $f''(x)$ and hence explain why $f(x)$ is concave down for all x.

d Use technology to help graph $y = 3 - e^x$.

e Explain why $y = 3$ is a horizontal asymptote.

10 The function $y = e^x - 3e^{-x}$ cuts the x-axis at P and the y-axis at Q.

a Determine the coordinates of P and Q.

b Prove that the function is increasing for all x.

c Show that $\dfrac{d^2y}{dx^2} = y$.

What can be deduced about the concavity of the function above and below the x-axis?

d Use technology to help graph $y = e^x - 3e^{-x}$.

Show the features of **a**, **b** and **c** on the graph.

11 $f(x) = e^x - 3$ and $g(x) = 3 - 5e^{-x}$.

a Find the x and y-intercepts of both functions.

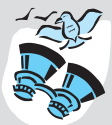
b Discuss $f(x)$ and $g(x)$ as $x \to \infty$ and as $x \to -\infty$.

c Find algebraically the point(s) of intersection of the function.

d Sketch the graph of both functions on the same set of axes. Show all important features on your graph.

C DERIVATIVES OF LOGARITHMIC FUNCTIONS

INVESTIGATION 3 THE DERIVATIVE OF $\ln x$



If $y = \ln x$, what is the slope function?

CALCULUS GRAPHING PACKAGE

What to do:

1 Click on the icon to see the graph of $y = \ln x$

A tangent is drawn to a point on the graph and the slope of this tangent is given.

The graph of the slope of the tangent is displayed as the point on the graph of $y = \ln x$ is dragged.

2 What do you suspect the equation of the slope is?

3 Find the slope at $x = 0.25$, $x = 0.5$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$. Do your results confirm your suspicion from **2**?

From the investigation you should have observed that if $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

An algebraic proof of this fact is as follows.

Proof: As $y = \ln x$ then $x = e^y$ and we differentiate with respect to x

$$\therefore\ 1 = e^y\frac{dy}{dx} \quad \{\text{by the chain rule}\}$$

$$\therefore\ 1 = x\frac{dy}{dx} \quad \{\text{as } e^y = x\}$$

$$\therefore\ \frac{1}{x} = \frac{dy}{dx}$$

Also, if $y = \ln f(x)$ then $y = \ln u$ where $u = f(x)$.

$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= \frac{1}{u} \times f'(x)$$
$$= \frac{f'(x)}{f(x)}$$

Summary

Function	*Derivative*
$\ln x$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

Example 8

Find the slope function of:
a $y = \ln (kx)$ where k is a constant
b $y = \ln (1 - 3x)$
c $y = x^3 \ln x$

a If $y = \ln (kx)$, then
$$\frac{dy}{dx} = \frac{k}{kx} \begin{array}{l} \leftarrow f'(x) \\ \leftarrow f(x) \end{array}$$
$$= \frac{1}{x}$$

b If $y = \ln (1 - 3x)$, then
$$\frac{dy}{dx} = \frac{-3}{1-3x} \begin{array}{l} \leftarrow f'(x) \\ \leftarrow f(x) \end{array}$$
$$= \frac{3}{3x-1}$$

c If $y = x^3 \ln x$, then
$$\frac{dy}{dx} = 3x^2 \ln x + x^3 \left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$
$$= 3x^2 \ln x + x^2$$
$$= x^2 (3 \ln x + 1)$$

EXERCISE 23C

1 Find the slope function of:

a $y = \ln(7x)$
b $y = \ln(2x + 1)$
c $y = \ln(x - x^2)$
d $y = 3 - 2\ln x$
e $y = x^2 \ln x$
f $y = \dfrac{\ln x}{2x}$
g $y = e^x \ln x$
h $y = (\ln x)^2$
i $y = \sqrt{\ln x}$
j $y = e^{-x} \ln x$
k $y = \sqrt{x}\ln(2x)$
l $y = \dfrac{2\sqrt{x}}{\ln x}$
m $y = 3 - 4\ln(1 - x)$
n $y = \sqrt{x}\ln(4x)$
o $y = x\ln(x^2 + 1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = x\ln 5$ **b** $y = \ln(x^3)$ **c** $y = \ln(x^4 + x)$

d $y = \ln(10 - 5x)$ **e** $y = [\ln(2x+1)]^3$ **f** $y = \dfrac{\ln(4x)}{x}$

g $y = \ln\left(\dfrac{1}{x}\right)$ **h** $y = \ln(\ln x)$ **i** $y = \dfrac{1}{\ln x}$

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

Example 9

Differentiate with respect to x:

a $y = \ln(xe^{-x})$ **b** $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a As $y = \ln(xe^{-x})$, then $y = \ln x + \ln e^{-x}$ {log of a product law}

$\therefore\ y = \ln x - x$ {$\ln e^a = a$}

Differentiating with respect to x, we get $\dfrac{dy}{dx} = \dfrac{1}{x} - 1$

b As $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$ then

$$\begin{aligned} y &= \ln x^2 - \ln[(x+2)(x-3)] \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \end{aligned}$$

$$\therefore\ \frac{dy}{dx} = 2\left(\frac{1}{x}\right) - \frac{1}{x+2} - \frac{1}{x-3}$$

Note: Differentiating **b** using the rule $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$ is extremely tedious and difficult.

3 After using logarithmic laws to write in an appropriate form, differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$ **b** $y = \ln\left(\dfrac{1}{2x+3}\right)$ **c** $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$ **e** $y = \ln\left(\dfrac{x+3}{x-1}\right)$ **f** $y = \ln\left(\dfrac{x^2}{3-x}\right)$

g $f(x) = \ln\left((3x-4)^3\right)$ **h** $f(x) = \ln\left(x(x^2+1)\right)$

i $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$ **j** $f(x) = \ln\left(\dfrac{x^3}{(2-3x)^2}\right)$

k $f(x) = \ln\left(\dfrac{\sqrt{x+1}}{x(x+2)}\right)$ **l** $f(x) = \ln\left(\dfrac{(x+1)(2x-1)}{3x^2}\right)$

4 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$ find $\dfrac{dy}{dx}$.

b Show that if $y = a^x$, then $\dfrac{dy}{dx} = a^x \times \ln a$

5 Consider $f(x) = \ln(2x - 1) - 3$.

a Find the x-intercept.

b Can $f(0)$ be found? What is the significance of this result?

c Find the slope of the tangent to the curve at $x = 1$.

d For what values of x does $f(x)$ have meaning?

e Find $f''(x)$ and hence explain why $f(x)$ is concave down whenever $f(x)$ has meaning.

f Graph the function.

6 Consider $f(x) = x \ln x$.

a For what values of x is $f(x)$ defined?

b Show that the smallest value of $x \ln x$ is $-\dfrac{1}{e}$.

7 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$. (**Hint:** Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.)

8 Consider the function $f(x) = x - \ln x$.
Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point. Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

D APPLICATIONS

The applications we consider here are:

- **tangents and normals**
- **rates of change**
- **curve properties**
- **displacement, velocity** and **acceleration**
- **optimisation (maxima/minima)**

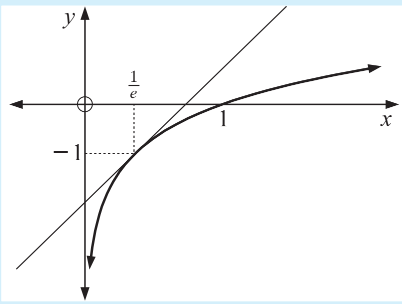
Example 10

Find the equation of the tangent to $y = \ln x$ at the point where $y = -1$.

When $y = -1$, $\ln x = -1$

$\therefore\ x = e^{-1} = \dfrac{1}{e}$

$\therefore$ point of contact is $(\frac{1}{e}, -1)$



Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ tangent has slope $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ tangent has equation $\dfrac{y - -1}{x - \frac{1}{e}} = e$

i.e., $y + 1 = e\left(x - \frac{1}{e}\right)$

i.e., $y = ex - 2$

EXERCISE 23D

1 Find the equation of the tangent to $y = e^{-x}$ at the point where $x = 1$.

2 Find the equation of the tangent to $y = \ln(2 - x)$ at the point where $x = -1$.

3 The tangent at $x = 1$ to $y = x^2 e^x$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

4 Find the equation of the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

5 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ from the origin.

6 Consider $f(x) = \ln x$.

- **a** For what values of x is $f(x)$ defined?
- **b** Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.
- **c** Sketch the graph of $f(x) = \ln x$ and find the equation of the normal at the point where $y = 1$.

7 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

8 The weight of bacteria in a culture is given by $W = 200e^{\frac{t}{2}}$ grams where t is the time in hours.

- **a** What is the weight of the culture at:
  i $t = 0$ **ii** $t = 30$ minutes **iii** $t = 1\frac{1}{2}$ hours?
- **b** How long would it take for the weight to reach 1 kg?
- **c** Find the rate of increase in the weight at time: **i** $t = 0$ **ii** $t = 2$ hours.
- **d** Sketch the graph of W against t.

9 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

- **a** Find k given that the weight is 10 grams after 50 hours.
- **b** Find the weight of radioactive substance present at:
  i $t = 0$ hours **ii** $t = 24$ hours **iii** $t = 1$ week.
- **c** How long will it take for the weight to reach 1 gram?
- **d** Find the rate of radioactive decay at:
  i $t = 100$ hours **ii** $t = 1000$ hours.
- **e** Show that $\dfrac{dW}{dt}$ is proportional to the weight of substance remaining.

10 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ degrees Celsius where k is a positive constant and t is the time in minutes.

- **a** Find k if the temperature of the liquid is 20°C after 15 minutes.
- **b** What was the temperature of the liquid when it was first placed in the refrigerator?

c Show that $\dfrac{dT}{dt}$ is directly proportional to $T-5$.

d At what rate is the temperature changing at:

i $t=0$ mins **ii** $t=10$ mins **iii** $t=20$ mins?

11 The height of a certain species of shrub t years after it is planted is given by

$$H(t)=20\ln(3t+2)+30 \text{ cm}, \quad t \geqslant 0.$$

a How high was the shrub when it was planted?

b How long would it take for the shrub to reach a height of 1 m?

c At what rate is the shrub's height changing:

i 3 years after being planted **ii** 10 years after being planted?

12 In the conversion of sugar solution to alcohol, the chemical reaction obeys the law $A=s(1-e^{-kt})$, $t \geqslant 0$ where t is the number of hours after the reaction commenced, s is the original sugar concentration (%) and A is the quantity of alcohol produced (in litres).

a Find A when $t=0$.

b If after 3 hours, $A=5$ when $s=10$, find k.

c Find the speed of the reaction at time 5 hours, when $s=10$.

d Show that the speed of the reaction is proportional to $A-s$.

13 Consider the function $f(x)=\dfrac{e^x}{x}$.

a Does the graph of $y=f(x)$ have any x or y-intercepts?

b Discuss $f(x)$ as $x\to\infty$ and as $x\to-\infty$.

c Find and classify any stationary points of $y=f(x)$.

d Sketch the graph of $y=f(x)$ showing all important features.

e Find the equation of the tangent to $f(x)=\dfrac{e^x}{x}$ at the point where $x=-1$.

14 Consider the function $f(x)=\dfrac{x}{e^x}$.

a Find x and y-intercepts.

b Discuss $f(x)$ as $x\to\infty$ and as $x\to-\infty$.

c Does $f(x)$ have any turning points?

d Sketch the graph of $f(x)=\dfrac{x}{e^x}$ from the information found in **a**, **b** and **c** above.

e Show that the normal to the curve at the point where $x=2$ cuts the x-axis at $2-\dfrac{2}{e^4}$.

15 The *distribution function* for experiments in radioactivity is $f(t)=1-e^{-at}$ where t is the time, $t>0$ and a is a positive constant.

a Does $f(t)$ have any stationary points?

b Graph $f(t)$ in the case where $a=1$.

c Find the signs of $f'(t)$ and $f''(t)$ in the case where $a=1$, and comment on the significance of each of them.

16 A body moves along the x-axis with displacement function $x(t) = 100(2 - e^{-\frac{t}{10}})$ cm where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions for the motion of the body.
- **b** Find the initial position, velocity and acceleration of the body.
- **c** Find the position, velocity and acceleration when $t = 5$ seconds.
- **d** Find t when $x(t) = 150$ cm.
- **e** Explain why the speed of the body is decreasing for all t and its velocity is always decreasing.

17 A particle P, moves in a straight line so that its displacement from the origin O, is given by $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions.
- **b** Find the initial position, velocity and acceleration of P.
- **c** Discuss the velocity of P as $t \to \infty$.
- **d** Sketch the graph of the velocity function.
- **e** Find when the velocity of P is 80 cm per second.

18 A psychologist claims that the ability $A(t)$ to memorise simple facts during infancy years can be calculated using the formula $A(t) = t\ln t + 1$ where $0 < t \leqslant 5$, t being the age of the child in years.

- **a** At what age is the child's memorising ability a minimum?
- **b** Sketch the graph of $A(t)$ against t.

19 The most common function used in statistics is the *normal distribution function* given by $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

- **a** Find the stationary points of the function and find intervals where the function is increasing and decreasing.
- **b** Find all points of inflection.
- **c** Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **d** Sketch the graph of $y = f(x)$ showing all important features.

20 In making electric kettles, the manufacturer performs a cost control study and discovers that to produce x kettles the cost per kettle $C(x)$ is given by

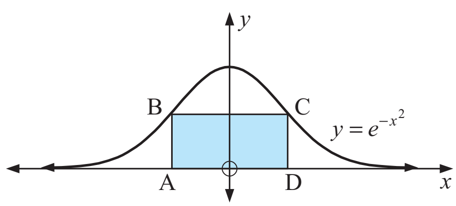
$$C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2 \text{ hundred dollars}$$

with a minimum production capacity per day of 10 kettles.

How many kettles should be manufactured to keep the cost per kettle a minimum?

21 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.

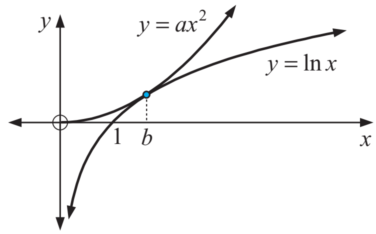
Determine the coordinates of C when rectangle ABCD has maximum area.



22 The revenue generated when a manufacturer sells x torches per day is given by
$R(x) \doteqdot 1000\ln\left(1+\dfrac{x}{400}\right)+600$ dollars.

Each torch costs the manufacturer \$1.50 to produce plus fixed costs of \$300 per day. How many torches should be produced daily to maximise the profits made?

23 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$.



- **a** If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.
- **b** Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.
- **c** What is the value of a?
- **d** What is the equation of the common tangent?

24 A small population of wasps is observed. After t weeks the population is modelled by
$P(t) = \dfrac{50\,000}{1+1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.

Find when the wasp population was growing fastest.

25 $f(t) = ate^{bt^2}$ has a maximum value of 1 when $t = 2$.

Find a and b given that they are constants.

REVIEW SET 23A

1 Differentiate with respect to x:

- **a** $y = e^{x^3+2}$
- **b** $y = \dfrac{e^x}{x^2}$

2 Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = 1$.

3 Sketch the graphs of $y = e^x + 3$ and $y = 9 - e^{-x}$ on the same set of axes. Determine the coordinates of the points of intersection.

4 Consider the function $f(x) = \dfrac{e^x}{x-1}$.

- **a** Find the x and y-intercepts.
- **b** For what values of x is $f(x)$ defined?
- **c** Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.
- **d** Sketch the graph of $y = f(x)$ and find the equation of the tangent at the point where $x = 2$.

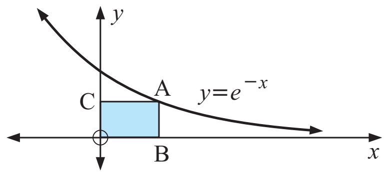
5 The height of a tree t years after it was planted is given by
$H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$.

- **a** How high was the tree when it was planted?
- **b** How long would it take for the tree to reach: **i** 150 cm **ii** 300 cm?
- **c** At what rate is the tree's height increasing after: **i** 2 years **ii** 20 years?

6 A particle, P, moves in a straight line such that its position is given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions.
- **b** Find the initial position, velocity and acceleration of P.
- **c** Discuss the velocity of P as $t \to \infty$.
- **d** Sketch the graph of the velocity function.
- **e** Find when the velocity is -44 metres per second.

7 Infinitely many rectangles can be inscribed under the curve $y = e^{-x}$ as shown. Determine the coordinates of A when the rectangle OBAC has maximum area.



8 A shirt maker sells x shirts per day with revenue function

$$R(x) = 200\ln\left(1 + \frac{x}{100}\right) + 1000 \text{ dollars.}$$

The manufacturing costs are determined by the cost function $C(x) = (x - 100)^2 + 200$ dollars. How many shirts should be sold daily to maximise profits? What is the maximum daily profit?

REVIEW SET 23B

1 Differentiate with respect to x:

- **a** $y = \ln(x^3 - 3x)$
- **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$

2 Find where the tangent to $y = \ln(x^2 + 3)$ at $x = 0$ cuts the x-axis.

3 Solve for x:

- **a** $e^{2x} = 3e^x$
- **b** $e^{2x} - 7e^x + 12 = 0$

4 Consider the function $f(x) = e^x - x$.

- **a** Find and classify any stationary points of $y = f(x)$.
- **b** Discuss $f(x)$ as $x \to -\infty$ and as $x \to \infty$.
- **c** Find $f''(x)$ and draw a sign diagram of it. State geometrical interpretations of the signs of $f''(x)$.
- **d** Sketch the graph of $y = f(x)$.
- **e** Deduce that $e^x \geqslant x + 1$ for all x.

5 Differentiate with respect to x:

- **a** $f(x) = \ln(e^x + 3)$
- **b** $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$

6 Solve for x:

- **a** $3e^x - 5 = -2e^{-x}$
- **b** $2\ln x - 3\ln\left(\dfrac{1}{x}\right) = 10$

7 A particle P, moves in a straight line such that its position is given by $s(t) = 25t - 10\ln t$ cm, $t \geqslant 1$, where t is the time in minutes.

- **a** Find the velocity and acceleration functions.
- **b** Find the position, velocity and acceleration when $t = e$ minutes.
- **c** Discuss the velocity as $t \to \infty$.
- **d** Sketch the graph of the velocity function.
- **e** Find when the velocity of P is 12 cm per minute.

8 When producing x clocks per day a manufacturer determines that the total weekly cost $C(x)$ thousand dollars is given by $C(x) = 10\ln x + \left(20 - \dfrac{x}{10}\right)^2$.

How many clocks per day should be produced to minimise the costs given that at least 50 clocks per day must be made to fill fixed daily orders?

Chapter 24

Derivatives of trigonometric functions

Contents:

A The derivative of $\sin x$, $\cos x$, $\tan x$

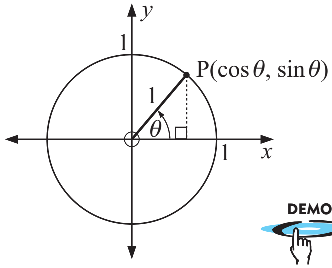
B Maxima/minima with trigonometry

Review set 24

INTRODUCTION

Recall from **Chapter 13** that sine and cosine curves arise naturally from a consideration of motion in a circle.

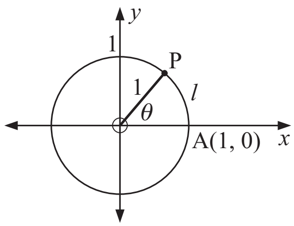
Click on the icon to observe the motion of point P around the unit circle and observe the graphs of P's height relative to the x-axis and then P's displacement from the y-axis. The resulting graphs are those of $y = \cos t$ and $y = \sin t$.



Suppose P moves anticlockwise around the unit circle with constant linear speed of 1 unit/second.

Then, after 2π seconds, P has covered 2π units which is one full revolution.

We therefore know that in t seconds P travels through t radians. Consequently at time t, P is at $(\cos t, \sin t)$.



Note:
- The **angular velocity** of P is the time rate of change in $\measuredangle$AOP,
  i.e., angular velocity of P is $\dfrac{d\theta}{dt}$ and
  $\dfrac{d\theta}{dt} = 1$ radian/sec in the above circular motion.
- If l is arc length AP, the **linear speed** of P is the time rate of change in l,
  i.e., linear speed is $\dfrac{dl}{dt}$.
  Notice that $l = r\theta = 1\theta = \theta$ and $\dfrac{dl}{dt} = \dfrac{d\theta}{dt} = 1$ radian/sec.

Angular velocity is only meaningful in motion along a circular or elliptical arc.

DERIVATIVES OF $\sin t$ AND $\cos t$

Click on the icon to observe the graph of $y = \sin t$ as a tangent of unit t-step moves across the curve. The y-step is translated onto the slope graph. Repeat for the graph of $y = \cos t$.

You should be able to guess what $(\sin t)'$ and $(\cos t)'$ are equal to.

From our observations from the previous computer demonstration we suspect that

$$(\sin t)' = \cos t \quad \text{and} \quad (\cos t)' = -\sin t.$$

$$\text{i.e.,} \quad \frac{d}{dt}(\sin t) = \cos t \quad \text{and} \quad \frac{d}{dt}(\cos t) = -\sin t$$

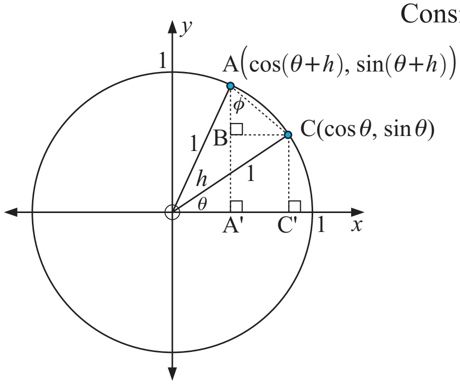
For completeness we will now look at a first principles argument to prove that

$$\frac{d}{dt}(\sin t) = \cos t.$$

Now consider the following algebraic argument as to why $\dfrac{d}{dx}(\sin x) = \cos x$.

A THE DERIVATIVE OF $\sin x$, $\cos x$, $\tan x$

THE DERIVATIVE OF $\sin x$



Consider $f(\theta) = \sin\theta$. Now

$$\frac{f(\theta+h)-f(\theta)}{h}$$
$$= \frac{\sin(\theta+h)-\sin\theta}{h}$$
$$= \frac{AA' - CC'}{h}$$
$$= \frac{AA' - BA'}{h}$$
$$= \frac{AB}{h}$$
$$= \frac{AB}{\text{arc } AC} \quad \text{...... (1)}$$

Now consider angle ϕ, $\phi = \measuredangle OAC - \measuredangle OAA'$

$$= \frac{\pi - h}{2} - (\tfrac{\pi}{2} - h - \theta)$$
$$= \theta + \frac{h}{2} \quad \text{on simplifying}$$

Now as $h \to 0$, $\phi \to \theta$ and $\cos\phi \to \cos\theta$ ......(2)

$$\frac{\sin(\theta+h)-\sin\theta}{h} \to \frac{AB}{AC} \quad \{\text{arc } AC \to AC\}$$

i.e., $$\frac{\sin(\theta+h)-\sin\theta}{h} \to \cos\phi \quad \{\text{using (2)}\}$$

i.e., $$\frac{\sin(\theta+h)-\sin\theta}{h} \to \cos\theta \quad \{\text{using (2)}\}$$

$$\therefore \quad f'(\theta) = \cos\theta$$

That is, for x in radians, if $f(x) = \sin x$ then $f'(x) = \cos x$.

THE DERIVATIVE OF $\cos x$

Consider $y = \cos x$

$\therefore$ $y = \sin\left(\frac{\pi}{2} - x\right)$ $\quad \{\sin(\frac{\pi}{2} - x) = \cos x\}$

$\therefore$ $y = \sin u$ where $u = \frac{\pi}{2} - x$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$= \cos u \times (-1)$$
$$= -\cos u$$
$$= -\cos(\tfrac{\pi}{2} - x)$$
$$= -\sin x$$

So, if $f(x) = \cos x$, then $f'(x) = -\sin x$. (x in radians)

THE DERIVATIVE OF $\tan x$

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$ $\qquad u = \sin x, \quad v = \cos x$

$$\frac{du}{dx} = \cos x, \quad \frac{dv}{dx} = -\sin x$$

$$\begin{aligned}
\therefore \quad \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \{\text{chain rule}\} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad (x \text{ in radians})
\end{aligned}$$

DERIVATIVE DEMO

Summary:

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$

Note: $\dfrac{1}{\cos x}$ is often called secant x or $\sec x$, so we observe that $\dfrac{d}{dx}(\tan x)$ is sometimes written as $\sec^2 x$.

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$ AND $\tan[f(x)]$

Consider $y = \sin[f(x)]$ where $f(x) = u$, say

i.e., $y = \sin u$

$$\begin{aligned}
\text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \qquad \{\text{chain rule}\} \\
&= \cos u \times f'(x) \\
&= \cos[f(x)] \times f'(x)
\end{aligned}$$

Thus

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]f'(x)$
$\tan[f(x)]$	$\dfrac{f'(x)}{\cos^2[f(x)]}$

Example 1

Differentiate with respect to x: **a** $x\sin x$ **b** $4\tan^2(3x)$

a If $y = x\sin x$, then

$$\frac{dy}{dx} = (1)\sin x + (x)\cos x \qquad \{\text{product rule}\}$$
$$= \sin x + x\cos x$$

b If $y = 4\tan^2(3x) = 4[\tan(3x)]^2$

then
$$\frac{dy}{dx} = 8[\tan(3x)]^1 \times \frac{d}{dx}[\tan(3x)] \qquad \{\text{chain rule}\}$$
$$= 8\tan(3x)\frac{3}{\cos^2(3x)}$$
$$= \frac{24\tan(3x)}{\cos^2(3x)}$$

EXERCISE 24A

1 Find $\dfrac{dy}{dx}$ for:

a $y = \sin(2x)$ **b** $y = \sin x + \cos x$ **c** $y = \cos(3x) - \sin x$
d $y = \sin(x+1)$ **e** $y = \cos(3-2x)$ **f** $y = \tan(5x)$
g $y = \sin\left(\frac{x}{2}\right) - 3\cos x$ **h** $y = 3\tan(\pi x)$ **i** $y = 4\sin x - \cos(2x)$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x\cos x$ **d** $e^{-x}\sin x$
e $\ln(\sin x)$ **f** $e^{2x}\tan x$ **g** $\sin(3x)$ **h** $\cos\left(\dfrac{x}{2}\right)$
i $3\tan(2x)$ **j** $x\cos x$ **k** $\dfrac{\sin x}{x}$ **l** $x\tan x$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos(\sqrt{x})$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$
e $\cos^3 x$ **f** $\cos x\sin(2x)$ **g** $\cos(\cos x)$ **h** $\cos^3(4x)$
i $\dfrac{1}{\sin x}$ **j** $\dfrac{1}{\cos(2x)}$ **k** $\dfrac{2}{\sin^2(2x)}$ **l** $\dfrac{8}{\tan^3\left(\frac{x}{2}\right)}$

4 If $y = x^4$ then $\dfrac{dy}{dx} = 4x^3$, $\dfrac{d^2y}{dx^2} = 12x^2$, $\dfrac{d^3y}{dx^3} = 24x$, $\dfrac{d^4y}{dx^4} = 24$ and higher derivatives are all zero. Consider $y = \sin x$.

a Find $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$, $\dfrac{d^4y}{dx^4}$.

b Explain why $\dfrac{d^ny}{dx^n}$ can have four different values.

5 **a** If $y = \sin(2x+3)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

b If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$. $\left(y'' \text{ is } \dfrac{d^2y}{dx^2}\right)$

c Show that the curve with equation $y = \dfrac{\cos x}{1+\sin x}$ cannot have horizontal tangents.

Example 2

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

Let $f(x) = \tan x$ $\quad f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$$\therefore \quad f'(x) = \frac{1}{\cos^2 x}$$

$$\therefore \quad f'(\tfrac{\pi}{4}) = \frac{1}{[\cos(\frac{\pi}{4})]^2} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

So at $(\frac{\pi}{4}, 1)$, the tangent has slope 2

$\therefore$ the equation is $\dfrac{y-1}{x-\frac{\pi}{4}} = 2$ i.e., $y - 1 = 2x - \frac{\pi}{2}$

i.e., $y = 2x + \left(1 - \frac{\pi}{2}\right)$

6 Find the equation of:

a the tangent to $y = \sin x$ at the origin

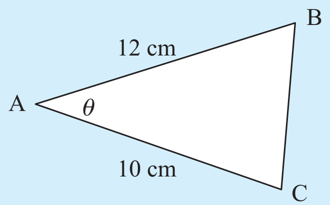
b the tangent to $y = \tan x$ at the origin

c the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$

d the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

Example 3

Find the rate of change in the area of triangle ABC when $\theta = 60^o$ given that θ is a variable angle.



Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\quad \{\text{Area} = \frac{1}{2}ab\sin C\}$

$$\therefore \quad A = 60\sin\theta$$

$$\therefore \quad \frac{dA}{d\theta} = 60\cos\theta$$

and when $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$ **Note**: θ is in radians

$$\therefore \quad \frac{dA}{d\theta} = 30 \text{ cm}^2 \text{ / radian}$$

7 In Indonesia large tides occur. The depth of water d metres at time t hours after midnight is given by $d = 9.3 + 6.8\cos(0.507t)$ metres.

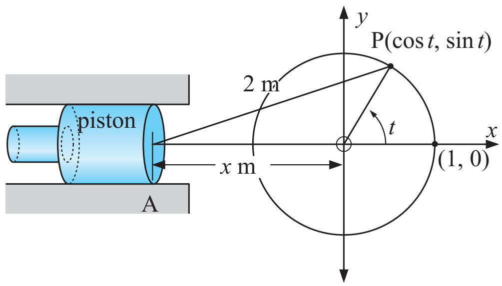
- **a** Is the tide rising or falling at 8.00 am?
- **b** What is the rate of change in the depth of water at 8.00 am?

8 The voltage in a circuit is given by $V(t) = 340\sin(100\pi t)$ where t is the time in seconds. At what rate is the voltage changing:

- **a** when $t = 0.01$
- **b** when $V(t)$ is a maximum?

($\sin\theta$ is a maximum when $\theta = \frac{\pi}{2}$)

9 A piston moves as a result of rod AP attached to a flywheel of radius 1 m. AP = 2 m.
P has coordinates $(\cos t, \sin t)$.



- **a** Point A is $(-x, 0)$.
  Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.
- **b** Find the rate at which x is changing at the instant when:
  - **i** $t = 0$
  - **ii** $t = \frac{\pi}{2}$
  - **iii** $t = \frac{2\pi}{3}$

10 Determine the position and nature of the stationary points of $y = f(x)$ on the interval $0 \leqslant x \leqslant 2\pi$ and show them on a sketch graph of the function:

- **a** $y = \sin x$
- **b** $y = \cos(2x)$
- **c** $y = \sin^2 x$

11 Consider the function $f(x) = \dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant 2\pi$.

- **a** For what values of x is $f(x)$ undefined in this interval?
- **b** Find the position and nature of any stationary points in this interval.
- **c** Prove that $f(x + 2\pi) = f(x)$ for all x. What is the geometrical significance of this result?
- **d** Sketch the graph of $y = \dfrac{1}{\cos x}$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{2}$ and show the stationary points on it.

12 Determine the position and nature of the stationary points of $y = \sin(2x) + 2\cos x$ on $0 \leqslant x \leqslant 2\pi$. Sketch the graph of the function on this interval, and show the positions of the stationary points you found on it.

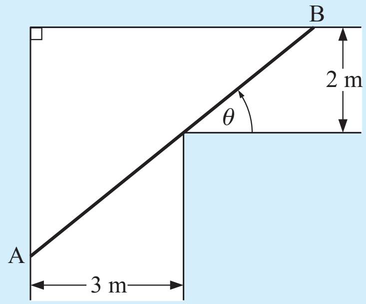
13 A particle P, moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm where t is the time in seconds.

- **a** State the initial position, velocity and acceleration of P.
- **b** Describe the motion when $t = \frac{\pi}{4}$ seconds.
- **c** Find the times when the particle reverses direction on $0 \leqslant t \leqslant 2\pi$ and find the position of the particle at these instances.
- **d** When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

B MAXIMA/MINIMA WITH TRIGONOMETRY

Example 4

Two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure and AB is a thin metal tube which must be kept horizontal as it moves around the corner from one corridor to the other without bending it.



a Show that the length AB, is given by

$$L = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}.$$

b Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \doteqdot 41.14^o$.

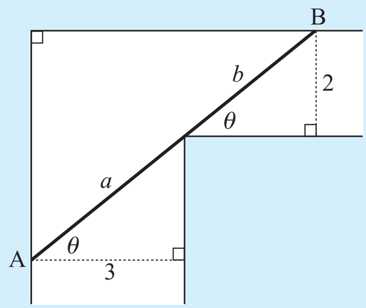
DEMO

c Find L when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ and comment on the significance of this value.

a $\cos\theta = \dfrac{3}{a}$ and $\sin\theta = \dfrac{2}{b}$ $\therefore$ $a = \dfrac{3}{\cos\theta}$ and $b = \dfrac{2}{\sin\theta}$

Now $L = a + b = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$

b
$$L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$$
$$\therefore \frac{dL}{d\theta} = -3[\cos\theta]^{-2} \times (-\sin\theta) - 2[\sin\theta]^{-2} \times \cos\theta$$
$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$
$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$



Thus $\dfrac{dL}{d\theta} = 0 \Leftrightarrow 3\sin^3\theta = 2\cos^3\theta$

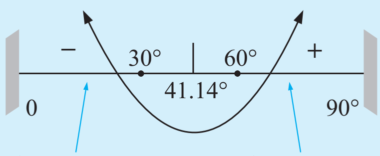
i.e., $\dfrac{\sin^3\theta}{\cos^3\theta} = \dfrac{2}{3}$

i.e., $\tan^3\theta = \frac{2}{3}$

$\therefore$ $\tan\theta = \sqrt[3]{\frac{2}{3}}$

$\therefore$ $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \doteqdot 41.14^o$

c *Sign diagram* of $\dfrac{dL}{d\theta}$:



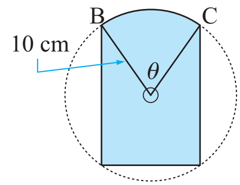
$\dfrac{dL}{d\theta} \doteqdot -4.93 < 0$, $\dfrac{dL}{d\theta} \doteqdot 9.06 > 0$

Thus, AB is minimised when $\theta \doteqdot 41.14^o$ and $L \doteqdot 7.023$ metres, and if we ignore the width of the rod, then the greatest length of rod able to be horizontally carried around the corner is of length 7.023 m.

EXERCISE 24B

1 A circular piece of tinplate of radius 10 cm has 3 segments removed (as illustrated). If θ is the measure of angle COB, show that the remaining area is given by $A = 50(\theta + 3\sin\theta)$.

Hence, find θ to the nearest $\frac{1}{10}$ of a degree when the area A is a maximum.



2 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice (as shown).

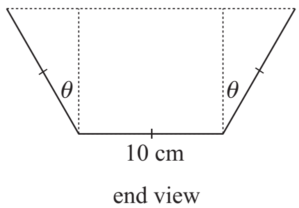
For θ as indicated:

a deduce that the cross-sectional area is given by

$$A = 100\cos\theta(1 + \sin\theta).$$

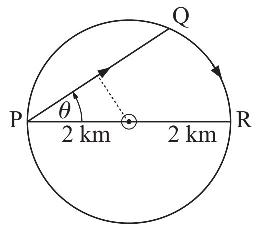
b Hence, show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

c What value is θ if the gutter has maximum carrying capacity?



3 Hieu can row a boat across a circular lake of radius 4 km at 3 kmph. He can walk around the edge of the lake at 5 kmph.

What is Hieu's longest possible time to get from P to R by rowing from P to Q and walking from Q to R?



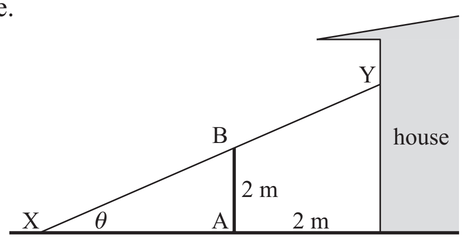
4 Fence AB is 2 m high and is 2 m from a house. XY is a ladder which touches the ground at X, the house at Y, and the fence at B.

a If L is the length of XY, show that

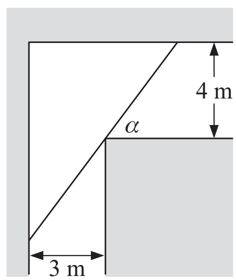
$$L = \frac{2}{\cos\theta} + \frac{2}{\sin\theta}.$$

b Show that $\dfrac{dL}{d\theta} = \dfrac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}$

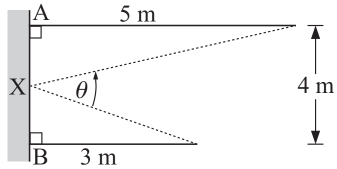
c What is the length of the shortest ladder XY which touches at X, B and Y?



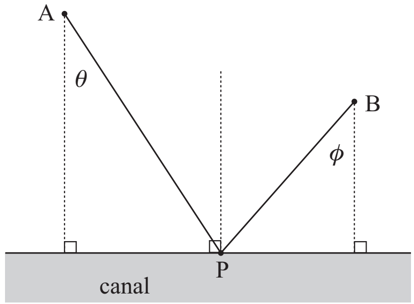
5 In **Example 4**, suppose the corridors are those in a hospital and are 4 m wide and 3 m wide respectively. What is the maximum length of thin metal tube that can be moved around the corner? Remember it must be kept horizontal and must not be bent.



6 How far should X be from A if angle θ is to be a maximum?



7 A and B are two homesteads. A pump house is to be located at P on the canal to pump water to both A and B.



a If A and B are a km and b km from the canal respectively, show that:

$$\text{AP} + \text{PB} = \frac{a}{\cos\theta} + \frac{b}{\cos\phi} = L, \quad \text{say.}$$

b Show that $\dfrac{dL}{d\theta} = \dfrac{a\sin\theta}{\cos^2\theta} + \dfrac{b\sin\phi}{\cos^2\phi}\dfrac{d\phi}{d\theta}$.

c Explain why $a\tan\theta + b\tan\phi$ is a constant and hence show that $\dfrac{d\phi}{d\theta} = \dfrac{-a\cos^2\phi}{b\cos^2\theta}$.

d Hence, show that $\dfrac{dL}{d\theta} = 0 \quad \Leftrightarrow \quad \sin\theta = \sin\phi$.

e What can be deduced from **d**? (All reasoning must be given with an appropriate 'test'.)

REVIEW SET 24

1 Differentiate with respect to x:

a $10x - \sin(10x)$ **b** $\sin(3x)\cos(2x)$ **c** $e^{-2x}\sin x$

d $\ln\left(\dfrac{1}{\cos x}\right)$ **e** $\sin(5x)\ln(2x)$

2 Show that the equation of the tangent to $y = x\tan x$ at $x = \frac{\pi}{4}$ is $(2+\pi)x - 2y = \dfrac{\pi^2}{4}$.

3 Find $f'(x)$ and $f''(x)$ for:

a $f(x) = 3\sin x - 4\cos(2x)$ **b** $f(x) = \sqrt{x}\cos(4x)$

4 A particle moves in a straight line along the x-axis in such a way that its position is given by $x(t) = 3 + \sin(2t)$ cm after t seconds.

a Find the initial position, velocity and acceleration of the particle.

b Find the times when the particle changes direction during $0 \leqslant t \leqslant \pi$ secs.

c Find the total distance travelled by the particle in the first π seconds.

5 Consider $f(x) = \sqrt{\cos x}$ for $0 \leqslant x \leqslant 2\pi$.

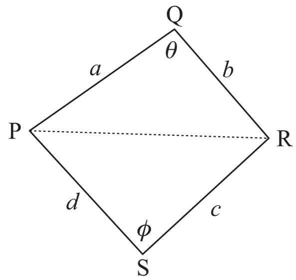
a For what values of x is $f(x)$ meaningful?

b Find $f'(x)$ and hence find intervals where $f(x)$ is increasing and decreasing.

c Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$.

6 Find $\dfrac{dy}{dx}$ given that: **a** $y = x\ln(\sin x)$ **b** $y = \sqrt{e^{\tan x}}$ **c** $y = \dfrac{\cos(3x)}{\sqrt{x}}$

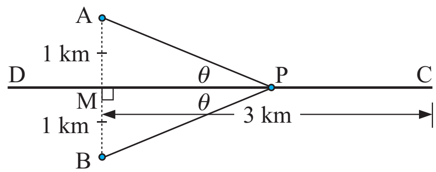
7 Four straight sticks of fixed length a, b, c and d are hinged together at P, Q, R and S.



a Use the cosine rule to find an equation which connects a, b, c, d, $\cos\theta$ and $\cos\phi$ and hence show that $\dfrac{d\theta}{d\phi} = \dfrac{cd\sin\phi}{ab\sin\theta}$

b Hence, show that the area of quadrilateral PQRS is a maximum when it is a cyclic quadrilateral.

8 A and B are two houses directly opposite one another and 1 km from a straight road CD. MC is 3 km and C is a house at the roadside.



A power unit is to be located on DC at P such that PA + PB + PC is to be a minimum so that the cost of trenching and cable will be as small as possible.

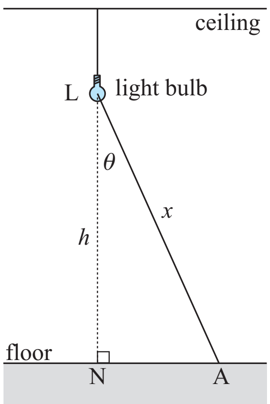
a What cable length would be required if P is at **i** M **ii** C?

b Show that, if $\theta = \measuredangle APM = \measuredangle BPM$, then the length of cable will be

$$L = \frac{2}{\sin\theta} + 3 - \frac{\cos\theta}{\sin\theta} \text{ metres.}$$

c Show that $\dfrac{dL}{d\theta} = \dfrac{1 - 2\cos\theta}{\sin^2\theta}$ and hence show that the minimum length of cable required is $(3 + \sqrt{3})$ km.

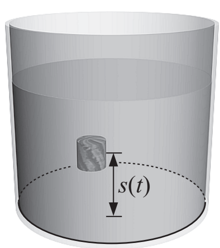
9 A light bulb hangs from the ceiling at a distance h metres above the floor directly above point N. At any point A on the floor, which is x metres from the light bulb, the illumination I is given by $I = \dfrac{\sqrt{8}\cos\theta}{x^2}$ units.



a If NA = 1 metre, show that at A, $I = \sqrt{8}\cos\theta\sin^2\theta$.

b The light bulb may be lifted or lowered to change the intensity at A. Find the height the bulb has to be above the floor for greatest illumination at A.

10 A cork moves up and down in a bucket of water such that the distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos(\pi t)$ cm where t is the time in seconds, $t \geqslant 0$.



a Find the cork's velocity at time, $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$ sec.

b Find the time intervals when the cork is falling.

Chapter 25

Areas within curved boundaries

Contents:

A Areas where boundaries are curved

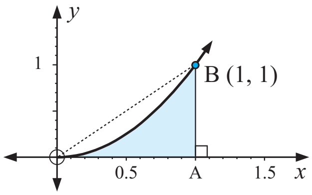
Investigation 1: Finding areas using rectangles

B Definite integrals

Investigation 2: $\int_0^1 (1-x^2)\,dx$, $\int_0^1 (x^2-1)\,dx$

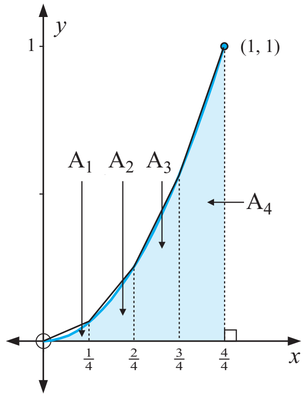
Review set 25

A AREAS WHERE BOUNDARIES ARE CURVED



We will begin with trying to find the area between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$. The function $y = x^2$ provides us with a curved boundary.

It is clear that the required area is less than the area of triangle OAB, i.e., area $< \frac{1}{2}$ unit^2.



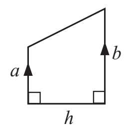
We cannot exactly subdivide the region into triangles, rectangles, trapezia, etc. so existing area formulae cannot be used.

However, we could find better approximations to the exact area by using them.

For example, if the region is subdivided into strips of width $\frac{1}{4}$, an over-estimate of the required area could be made by joining the points on the curve with straight lines as shown.

Now area $< A_1 + A_2 + A_3 + A_4$

i.e., area $< \left(\dfrac{\frac{0}{16}+\frac{1}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{1}{16}+\frac{4}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{4}{16}+\frac{9}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{9}{16}+\frac{16}{16}}{2}\right)\frac{1}{4}$

{using area of trapezium $= \left(\frac{a+b}{2}\right)h$  }

$\therefore$ area $< \frac{1}{4}\left[\frac{1}{32} + \frac{5}{32} + \frac{13}{32} + \frac{25}{32}\right]$

i.e., area $< \frac{11}{32}$ where $\frac{11}{32} = 0.343\,75$

An estimate of the area is $\doteqdot 0.34$ units^2, which is a better estimate than that which would have been obtained if we had used two strips.

DISCUSSION

How could the estimate of the area under $y = x^2$ be improved?

What factors must be considered when trying to improve area estimates?

EXERCISE 25A.1

1 Use the method outlined above to find an estimate of the area:

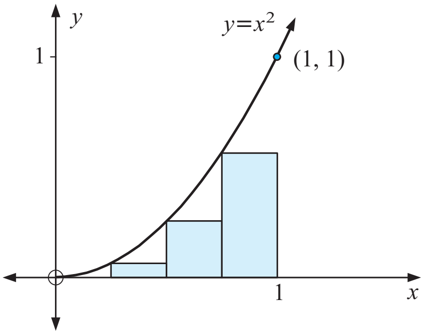
- **a** between $y = x^2$ and the x-axis from $x = 0$ to $x = 1$ if five vertical strips of equal width are drawn
- **b** between $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to $x = 2$ if five vertical strips of equal width are drawn.

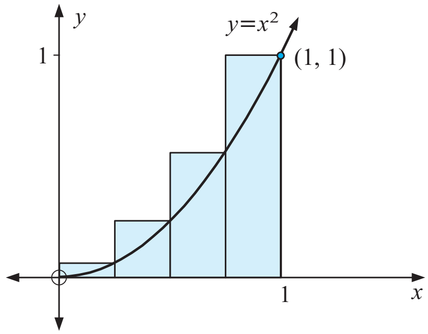
2 Repeat **1** but this time use ten vertical strips.

UPPER AND LOWER RECTANGLES

Another way of finding areas is to use only rectangles and find lower and upper sums of their areas. This gives us a lower and an upper bound for the actual area.

For example, consider $y = x^2$ again with four vertical strips.





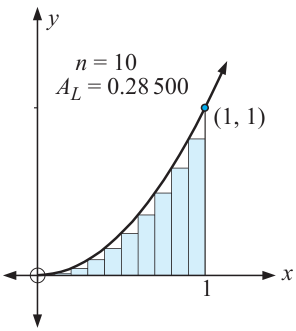
If A_L represents the lower area sum and A_U represents the upper sum, then

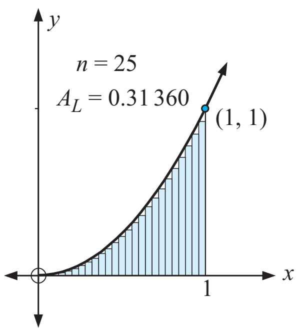
$$A_L = \tfrac{1}{4}(0)^2 + \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 = 0.218\,75$$

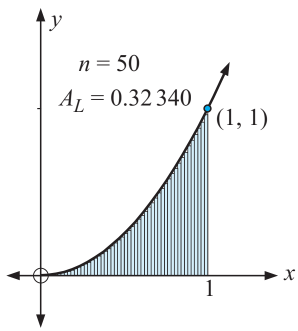
$$\text{and} \quad A_U = \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}(1)^2 = 0.468\,75$$

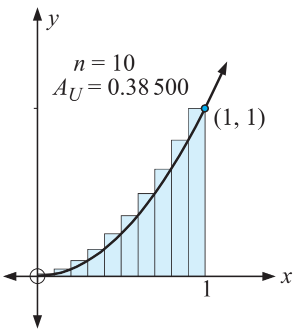
So, if A is the actual area then $\underset{\text{a lower bound}}{0.218\,75} < A < \underset{\text{an upper bound}}{0.468\,75}$

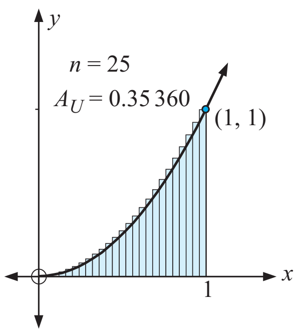
The following diagrams show lower and upper rectangles for n subdivisions where $n = 10$, 25 and 50.

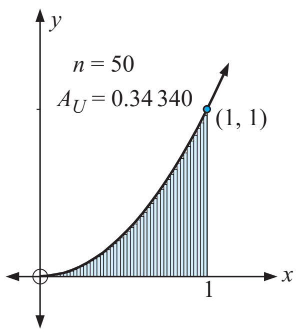












A summary of these results together with average of A_L and A_U are worth considering:

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
10	0.285 00	0.385 00	0.335 00
25	0.313 60	0.353 60	0.333 60
50	0.323 40	0.343 40	0.333 40

Now click on the icon to examine cases $n = 4, 10, 25, 50, 100, 1000$ and 10 000.

AREA FINDER

From the table it seems that as n gets larger A_L and A_U both approach or converge to the same number 0.333 333 3..... which we recognise as the decimal expansion of $\frac{1}{3}$.

INVESTIGATION 1 FINDING AREAS USING RECTANGLES

This investigation is about finding estimates of areas under simple curves by finding upper and lower rectangle sums by direct calculation and by using technology. **Note:** $[a, b]$ is interval notation for $a \leqslant x \leqslant b$.

What to do:

1 Consider finding the area between $y = x^3$ and the x-axis from $x = 0$ to $x = 1$.

a First graph the curve and shade the required area.

b Subdivide the interval $[0, 1]$ into five equal intervals and construct upper and lower rectangles. This case is $n = 5$.

AREA FINDER

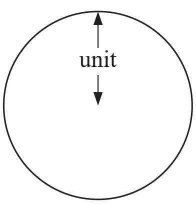
c Find the upper and lower area sums.

d Click on the icon and use the technology to find upper and lower area sums when $n = 5, 10, 50, 200, 1000$ and 10 000. Display your answers in table form.

e What do you suspect the actual area to be?

2 Repeat **1 a** to **d** for the function $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to $x = 2$.

RATIONAL APPROXIMATIONS FOR π



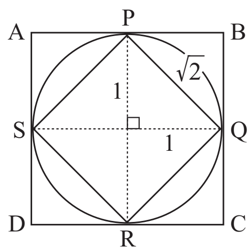
A circle of radius 1 unit has area $A = \pi \times 1^2 = \pi$ units^2.

Notice in the figure that the length of PQ is $\sqrt{2}$.

We can see that

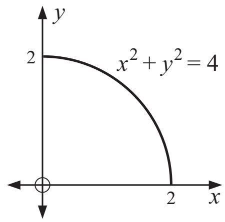
$$\text{area PQRS} < \pi < \text{area ABCD}$$
$$\therefore \quad \sqrt{2} \times \sqrt{2} < \pi < 2 \times 2$$
$$\therefore \quad 2 < \pi < 4$$

lower bound upper bound



It is clear that 2 and 4 are not good estimates of π, however they do provide us with lower and upper bounds between which the true value of π lies.

Consider the quarter circle of centre (0, 0) and radius 2 units as illustrated.



Its area is $\frac{1}{4}$ (full circle of radius 2)

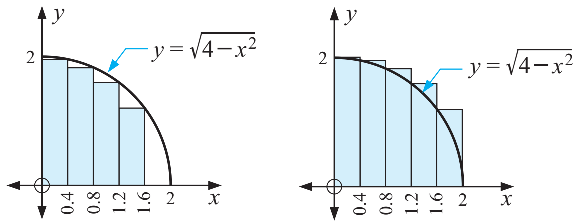
$= \frac{1}{4} \times \pi \times 2^2$

$= \pi$

Recall that π is an irrational number whose decimal expansion neither terminates nor recurs.

We will now employ the lower and upper rectangle technique to find **rational bounds** for π.

Consider the case $n = 5$.



$$A_L = (0.4)\sqrt{4-(0.4)^2} + (0.4)\sqrt{4-(0.8)^2} + (0.4)\sqrt{4-(1.2)^2} + (0.4)\sqrt{4-(1.6)^2}$$
$$= 2.637\,04......$$

and $$A_U = (0.4)\sqrt{4-0^2} + (0.4)\sqrt{4-(0.4)^2} + (0.4)\sqrt{4-(0.8)^2} + (0.4)\sqrt{4-(1.2)^2} + (0.4)\sqrt{4-(1.6)^2}$$
$$= 3.437\,04......$$

From this, $2.637 < \pi < 3.437$ provides us with rational bounds for π which are *better* than the earlier ones obtained geometrically.

As in the case of previous area finding, using lower and upper sums, we should be able to obtain even better bounds by increasing the number of vertical strips from $n = 5$ to 10 to 50 to 100, etc.

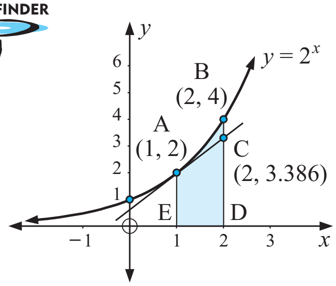
Click on the icon and experiment with increasing values of n to get better rational bounds for the actual value of π.

AREA FINDER

EXERCISE 25A.2

1 Alongside is the graph of $y = 2^x$. The tangent at (1, 2) is drawn.

AREA FINDER



a Use the figure to find rational lower and upper bounds for the shaded area.

b Now consider the original graph without the tangent.

Use five subdivisions of the interval [1, 2] to obtain lower and upper sums of rectangular areas.

c Click on the icon to check your answer to **b** ($n = 5$). Now obtain 'better' rational bounds for the area by considering $n = 10,\ 50,\ 100,\ 500,\ 5000$. Display your answers in table form.

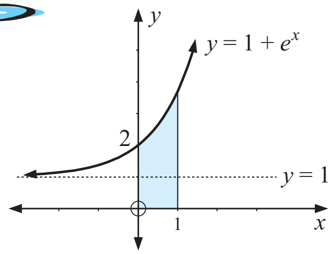
2 Consider the graph of $y = 1 + e^x$ where $e = 2.718\,281\,82.....$

a Use five subdivisions of [0, 1] and find the lower and upper area sums.

b Now use the area finder to find the lower and upper sums for $n = 100$, 1000, $10\,000$, $100\,000$.

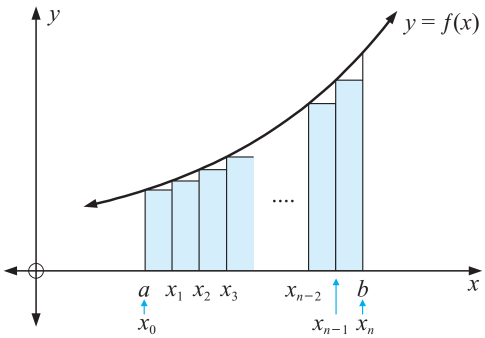
c What do you notice about your answers to **b**?

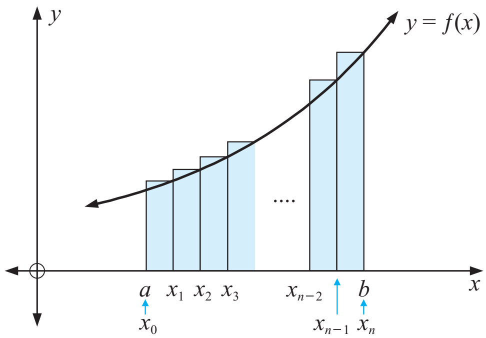
AREA FINDER



B DEFINITE INTEGRALS

We will now have a closer look at lower and upper rectangle sums for a function which is above the x-axis on the interval $[a, b]$, and is increasing.





Notice that the **lower sum** is

$$\begin{aligned} A_L &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x \\ &= \sum_{i=0}^{n-1} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}. \end{aligned}$$

Likewise the **upper sum** is

$$\begin{aligned} A_U &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + ... + f(x_{n-1})\Delta x + f(x_n)\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}. \end{aligned}$$

From the work of the previous two sections the following has been observed:

- As n gets larger, as $b - a$ is fixed, Δx gets smaller and closer to 0.
- There exists a unique number A, say, such that for any value of n $A_L < A < A_U$ and both A_L and A_U approach A as n gets very large.
- If $f(x) \geqslant 0$ on $[a, b]$, then A is the area between $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

Notation:

We talk about n getting very large and write $n \to \infty$.

$n \to \infty$ could be read as n *approaches* infinity or n *tends to* infinity.

Using this notation, as $n \to \infty$, $A_L \to A$ and $A_U \to A$.

THE DEFINITE INTEGRAL

We define the unique number between all lower and upper sums as $\int_a^b f(x)dx$ and call it "the **definite integral** of $f(x)$ from a to b",

$$\text{i.e.,}\quad \sum_{i=0}^{n-1} f(x_i)\Delta x < \int_a^b f(x)dx < \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}.$$

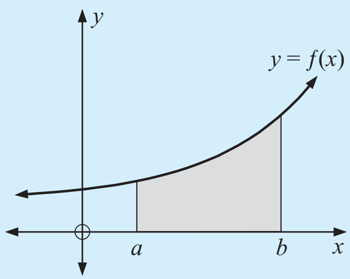
We note that as $n \to \infty$, $\displaystyle\sum_{i=0}^{n-1} f(x_i)\Delta x \to \int_a^b f(x)\,dx$ and

$$\sum_{i=1}^{n} f(x_i)\Delta x \to \int_a^b f(x)\,dx$$

We write $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx.$

We have observed that:

If $f(x) \geqslant 0$ for all x on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx$ is the shaded area.



The following exercise is best done with technology. Either use the **computer package** or a **graphics calculator**.

AREA FINDER

EXERCISE 25B.1

1 **a** Sketch the region between $y = x^4$ and the x-axis from $x = 0$ to $x = 1$.

b Use technology to find the lower and upper rectangle sums for n equal subdivisions where $n = 10, 100, 1000, 10\,000$.

c What do you suspect the exact area to be?

2 **a** Sketch the region between the curve $y = \sqrt{x}$ and the x-axis on the interval $[0, 4]$.

b Use technology to find the lower and upper rectangle sums for $n = 100$, and $n = 10\,000$ subdivisions.

c What do you suspect the exact area to be?

3 **a** Sketch the region between the curve $y = \sqrt{1+x^3}$ and the x-axis on the interval $[0,\ 2]$.
b Use technology to find the lower and upper rectangle sums for $n = 100$ and $n = 10\,000$ subdivisions.
c What is your best estimate of the exact area?

4 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis on $[0,\ 1]$.
b Use technology to find the lower and upper rectangle sums for $n = 100$, 1000, and $10\,000$.
c What is your best estimate of the exact area? Comment on the result.

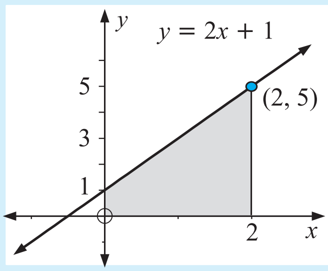
5 Find the area enclosed by the curve $y = \sqrt[3]{x^2+2}$ and the x-axis from $x = 0$ to $x = 5$ to the best accuracy you can.

Example 1

Use graphical evidence and known area facts to find:

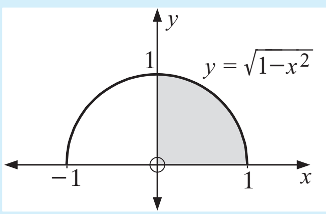
a $\displaystyle\int_0^2 (2x+1)\,dx$ **b** $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

a



$\int_0^2 (2x+1)\,dx$
$=$ shaded area
$= \left(\frac{1+5}{2}\right) \times 2$
$= 6$

b As $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2$ i.e., $x^2 + y^2 = 1$ which is the equation of the unit circle. $y = \sqrt{1-x^2}$ is the upper half.



$\int_0^1 \sqrt{1-x^2}\,dx$
$=$ shaded area
$= \frac{1}{4}(\pi r^2)$ where $r = 1$
$= \frac{\pi}{4}$

6 Use graphical evidence and known area facts to find:

a $\displaystyle\int_1^3 (1+4x)\,dx$ **b** $\displaystyle\int_{-1}^2 (2-x)\,dx$ **c** $\displaystyle\int_{-1}^1 |x|\,dx$

d $\displaystyle\int_0^2 |x-1|\,dx$ **e** $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$ **f** $\displaystyle\int_{-2}^0 (1+\sqrt{4-x^2})dx$

THE DEFINITE INTEGRAL WHEN $f(x) \leqslant 0$

So far our arguments have been restricted to $f(x) \geqslant 0$.

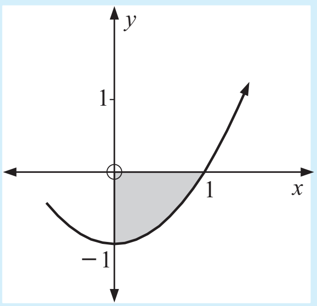
But, what is $\displaystyle\int_a^b f(x)\,dx$ for $f(x) \leqslant 0$ on the interval $[a, b]$?

Since $$\sum_{i=0}^{n-1} f(x_i)\Delta x < \int_a^b f(x)\,dx < \sum_{i=1}^{n} f(x_i)\Delta x,$$

$\displaystyle\int_a^b f(x)\,dx$ must be **negative** as clearly $\Delta x = \dfrac{b-a}{n}$ is always positive and $f(x_i)$ values are always negative.

Example 2

Find upper and lower bounds for $\displaystyle\int_0^1 (x^2-1)\,dx$ using upper and lower sums when $n = 5$.



$$\begin{aligned} a = x_0 &= 0 & f(0) &= -1 \\ x_1 &= 0.2 & f(0.2) &= -0.96 \\ x_2 &= 0.4 & f(0.4) &= -0.84 \\ x_3 &= 0.6 & f(0.6) &= -0.64 \\ x_4 &= 0.8 & f(0.8) &= -0.36 \\ x_5 &= 1.0 & f(1) &= 0 \end{aligned}$$

and $\Delta x = \frac{1-0}{5} = \frac{1}{5}$

$$\begin{aligned} \sum_{i=0}^{4} f(x_i)\Delta x &= \tfrac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \\ &= -0.76 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{5} f(x_i)\Delta x &= \tfrac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \\ &= -0.56 \end{aligned}$$

So, $\displaystyle -0.76 < \int_0^1 (x^2-1)dx < -0.56$

Click on the icon and use the integral finder to find *better* upper and lower bounds for $\displaystyle\int_0^1 (x^2-1)\,dx$.

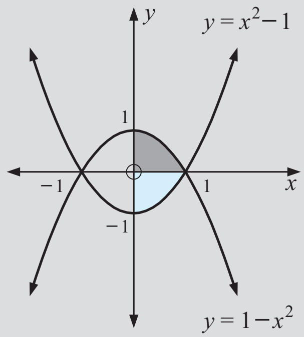
INTEGRAL FINDER

It seems that the actual value of $\displaystyle\int_0^1 (x^2-1)\,dx$ is $-\frac{2}{3}$.

INVESTIGATION 2 $\int_0^1 (1-x^2)\,dx$, $\int_0^1 (x^2-1)\,dx$

The graphs of $y = x^2 - 1$ and $y = 1 - x^2$ are reflections of one another in the x-axis.

Hence, the two shaded regions have equal area.



What to do:

1. Use the integral finder to find $\int_0^1 (x^2-1)\,dx$ using $n = 10\,000$.
2. Likewise find $\int_0^1 (1-x^2)\,dx$.
3. $\int_0^1 (x^2-1)dx$ is the upper shaded region's area.

   What does $\int_0^1 (1-x^2)\,dx$ represent?

INTEGRAL FINDER

EXERCISE 25B.2

1. Use technology to find:

   a $\int_1^4 \sqrt{x}\,dx$ and $\int_1^4 (-\sqrt{x})\,dx$ b $\int_0^1 x^7\,dx$ and $\int_0^1 (-x^7)\,dx$

2. Use technology to find:

   a $\int_0^1 x^2\,dx$ b $\int_1^2 x^2\,dx$ c $\int_0^2 x^2\,dx$ d $\int_0^1 3x^2\,dx$

3. Use technology to find:

   a $\int_0^2 (x^3-4x)\,dx$ b $\int_2^3 (x^3-4x)\,dx$ c $\int_0^3 (x^3-4x)\,dx$

4. What generalisation can be made from

   a question **1** b questions **2** and **3**?

5. Use technology to find:

   a $\int_0^1 x^2\,dx$ b $\int_0^1 \sqrt{x}\,dx$ c $\int_0^1 (x^2+\sqrt{x})\,dx$

   What do you notice?

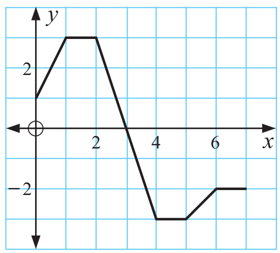
PROPERTIES OF DEFINITE INTEGRALS

From the previous exercise the following properties of definite integrals were observed:

- $\displaystyle\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx,$ c is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$
- $\displaystyle\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

EXERCISE 25B.3

1 The graph of $y = f(x)$ is illustrated:

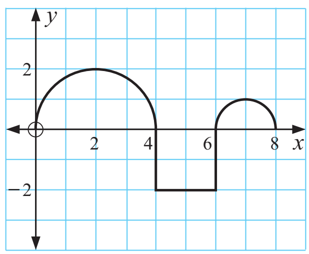


Evaluate the following integrals using area interpretation:

a $\displaystyle\int_0^3 f(x)\,dx$ **b** $\displaystyle\int_3^7 f(x)\,dx$

c $\displaystyle\int_2^4 f(x)\,dx$ **d** $\displaystyle\int_0^7 f(x)\,dx$

2 The graph of $y = f(x)$ is illustrated:



Evaluate the following using area interpretation:

a $\displaystyle\int_0^4 f(x)\,dx$ **b** $\displaystyle\int_4^6 f(x)\,dx$

c $\displaystyle\int_6^8 f(x)\,dx$ **d** $\displaystyle\int_0^8 f(x)\,dx$

3 Write as a single integral:

a $\displaystyle\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$ **b** $\displaystyle\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$

4 **a** If $\displaystyle\int_1^3 f(x)\,dx = 2$ and $\displaystyle\int_1^6 f(x)\,dx = -3,$ find $\displaystyle\int_3^6 f(x)\,dx.$

b If $\displaystyle\int_0^2 f(x)\,dx = 5,$ $\displaystyle\int_4^6 f(x)\,dx = -2$ and $\displaystyle\int_0^6 f(x)\,dx = 7,$

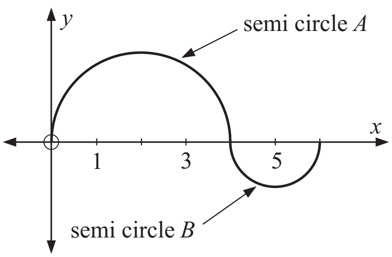
find $\displaystyle\int_2^4 f(x)\,dx.$

REVIEW SET 25

1 **a** Sketch the graph of $y = \sqrt{x}$ from $x = 1$ to $x = 4$.

b By using a lower sum and rectangular strips of width 0.5, estimate the area between $y = \sqrt{x}$, the x-axis, $x = 1$ and $x = 4$.

2 For the given function $y = f(x)$, $0 \leqslant x \leqslant 6$:



a Show that A has equation $y_A = \sqrt{4x - x^2}$.

b Show that B has equation

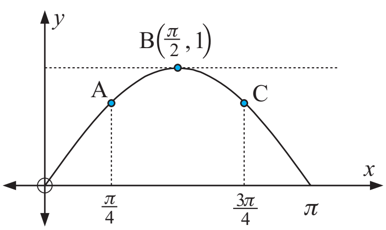
$$y_B = -\sqrt{10x - x^2 - 24}.$$

c Find $\int_0^4 y_A\,dx$ and $\int_4^6 y_B\,dx$.

d Hence, find $\int_0^6 f(x)\,dx$.

3 **a** The graph of $y = \sin x$ is drawn alongside. Use the graph to explain why

$$\frac{\pi}{2} < \int_0^\pi \sin x\,dx < \pi$$

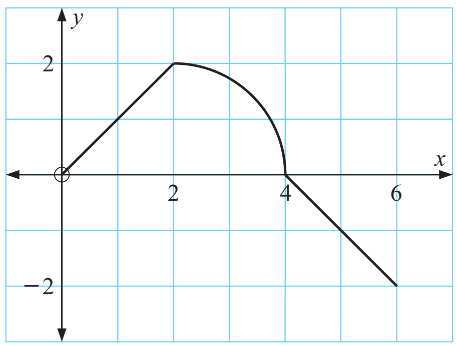


b A is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and C is $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Use the diagram to show that the area under one arch of $y = \sin x$ (as illustrated) is close to $\frac{\pi}{4}(1 + \sqrt{2})$ units^2, but less than it.

4 The function $y = f(x)$ is graphed.

Find:

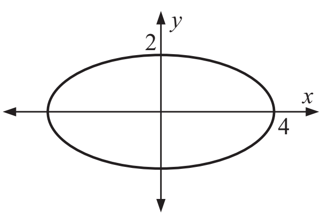


a $\int_0^4 f(x)\,dx$

b $\int_4^6 f(x)\,dx$

c $\int_0^6 f(x)\,dx$

5 The ellipse shown has equation $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.



a Sketch the graph again and mark on it the area represented by $\int_0^4 \frac{1}{2}\sqrt{16 - x^2}\,dx$.

b Explain from the graph why we can say $8 < \int_0^4 \sqrt{16 - x^2}\,dx < 16$.

6 Use technology to find:

a $\int_0^2 x^3\,dx$ **b** $\int_1^3 x^3\,dx$

Chapter 26

Integration

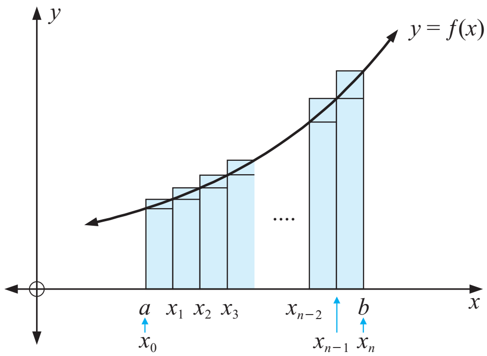
Contents:

A Reviewing the definite integral
B The area function
Investigation 1: The area function
C Antidifferentiation
D The fundamental theorem of calculus
E Integration
F Integrating e^{ax+b} and $(ax+b)^n$
G Integrating $f(u)\,u'(x)$ by substitution
H Distance from velocity
I Definite integrals
Investigation 2: $\int_a^b f(x)\,dx$ and areas
J Finding areas
K Problem solving by integration

Review set 26A
Review set 26B
Review set 26C

A REVIEWING THE DEFINITE INTEGRAL

Recall from **Chapter 25** that for a function $y = f(x)$ which is positive and continuous on the interval $[a, b]$, the **area under the curve** can be approximated using vertical rectangular strips.



If the strips go above the curve the area can be approximated by the **upper sum**,

$A_U = \sum_{i=1}^{n} f(x_i)\Delta x$ where Δx is the strip width.

If the strips remain below the curve the area can be approximated by the **lower sum**,

$$A_L = \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

We noticed that reducing the strip width Δx, improved the approximations.

So, $A_L \leqslant A \leqslant A_U$ where A_U and A_L approach A as $\Delta x \to 0$.

Now $\Delta x = \dfrac{b-a}{n}$, so as $\Delta x \to 0$, $n \to \infty$.

Also notice that $x_0 = a$ and $x_n = b$.

Consequently we wrote $$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

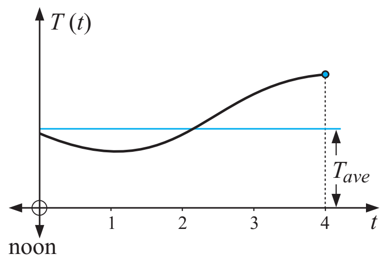
where the limit sum was called **the definite integral**.

We will now consider two other approaches which give meaning to the definite integral.

THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL $[a, b]$

Suppose we wish to find the average temperature during a four hour period from 12-noon to 4 pm.

Taking temperature readings every half hour and averaging them would give us an estimate of the average temperature over the 4-hour period,

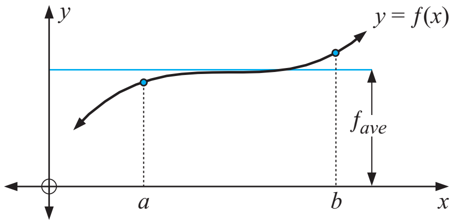


$$\text{i.e.,}\quad T_{ave} \doteqdot \frac{T(0) + T(\frac{1}{2}) + T(1) + T(1\frac{1}{2}) + \ldots\ldots + T(4)}{9}$$

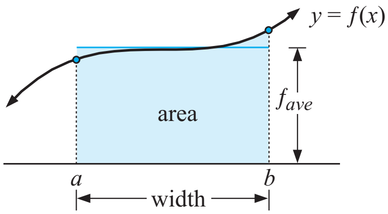
A closer approximation would occur if we averaged temperatures every 10 minutes. Better still would be an average every minute, etc. We see that the smaller the time increments the better the approximation.

Best of all, suppose we could take infinitely many temperature readings. Then averaging them we would get the average temperature, T_{ave}.

Now consider a general function $y = f(x)$ and the average value of $f(x)$ over the interval $[a, b]$.



f_{ave} could be interpreted as the average height above the x-axis.



As area $=$ average height $\times$ width

then,

$$\int_a^b f(x)\,dx = f_{ave} \times (b - a)$$

Consequently

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

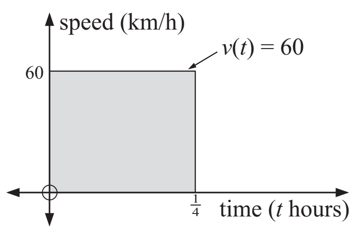
DISTANCES FROM VELOCITY-TIME GRAPHS

Suppose a car travels at a constant positive velocity of 60 km/h for 15 minutes.

Since the velocity is positive we can use speed instead of velocity.

The distance travelled is $60 \text{ km/h} \times \frac{1}{4} \text{ h}$
$= 15$ km

However, when we graph *speed* against *time*, the graph is a horizontal line and it is clear that the distance travelled is the area shaded.



[As average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

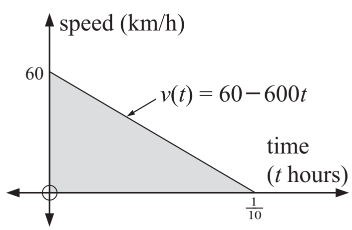
then distance $=$ speed $\times$ time.]

Now suppose the speed decreases at a constant rate so that the car, initially travelling at 60 km/h, stops in 6 minutes.

Now average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

then $\dfrac{60+0}{2} = \dfrac{\text{distance}}{\frac{1}{10}}$

$\therefore$ $30 \times \frac{1}{10} =$ distance i.e., distance $= 3$ km



But the triangle has area $= \frac{1}{2}$ base $\times$ altitude $= \frac{1}{2} \times \frac{1}{10} \times 60 = 3$

So, once again the shaded area gives us the distance travelled.

Using definite integral notation:

$$\text{distance travelled} = \int_0^{\frac{1}{4}} 60\,dt = 15 \quad \{\text{for the first example}\}$$

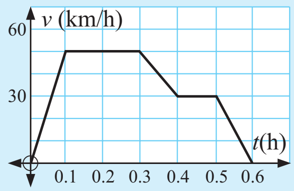
$$\text{and distance travelled} = \int_0^{\frac{1}{10}} (60 - 600t)\ dt = 3 \quad \{\text{for the second example}\}$$

These results suggest that: distance travelled $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt$ provided we don't change direction.

The area under the velocity-time graph gives distance travelled regardless of the shape of the velocity function. However, when the graph is made up of straight line segments the distance is usually easy to calculate.

Example 1

The velocity-time graph for a train journey is as illustrated in the graph alongside. Find the total distance travelled by the train.



Total distance travelled

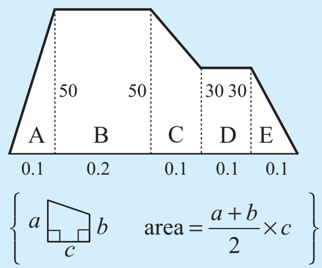
$=$ total area under the graph

$=$ area A $+$ area B $+$ area C $+$ area D $+$ area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30$
$\quad + \frac{1}{2}(0.1)30$

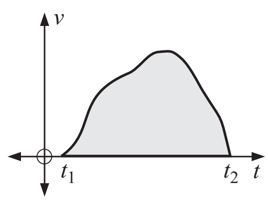
$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km



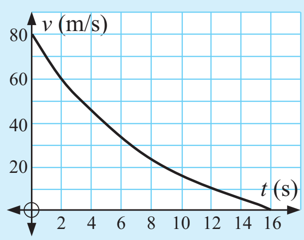
In general,

> for a velocity-time function $v(t)$ where $v(t) \geqslant 0$,
>
> distance travelled $= \displaystyle\int_{t_1}^{t_2} v(t)dt$



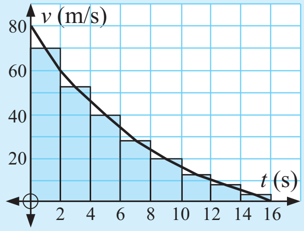
Example 2

The velocity-time graph of a braking car is shown. Use it to determine the total distance travelled by the car over this period.



The total distance travelled over the 16 seconds is $\int_0^{16} v(t)\,dt$

$\doteqdot$ the area of the 8 shaded rectangles where the height of each one is the 'midpoint' of the interval.



$\doteqdot 2\times 70 + 2\times 53 + 2\times 40 + 2\times 28 + 2\times 20 + 2\times 13 + 2\times 7 + 2\times 2$

$\doteqdot 466$ m

THE MIDPOINT RULE

The midpoint rule provides us with another way of estimating the value of a definite integral. It uses the midpoint of each interval $\overline{x_i}$, and the corresponding value of $f(\overline{x_i})$ gives an estimate of the height of the column under consideration.

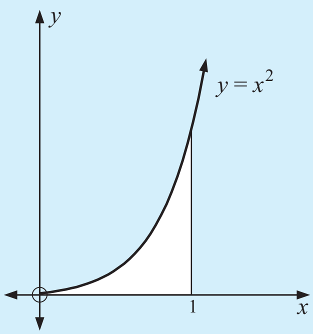
The **midpoint rule** for estimating definite integrals is:

$$\int_a^b f(x)\,dx \doteqdot [f(\overline{x_1}) + f(\overline{x_2}) + \ldots\ldots + f(\overline{x_n})]\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x_i} = \dfrac{x_{i-1}+x_i}{2}$, the midpoint of $[x_{i-1}, x_i]$.

Example 3

Estimate $\int_0^1 x^2\,dx$ using the midpoint rule and ten sub-intervals.



The endpoints of the intervals are:

0, 0.1, 0.2, 0.3, ..., 0.9, 1

$\therefore$ the midpoints are 0.05, 0.15, 0.25, ..., 0.95

So, $\int_0^1 x^2\,dx$

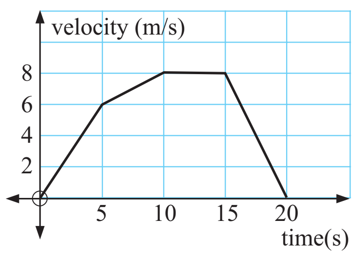
$= [f(0.05) + f(0.15) + \ldots\ldots + f(0.95)]\frac{1}{10}$

$= [(0.05)^2 + (0.15)^2 + \ldots\ldots + (0.95)^2] \times 0.1$

$\doteqdot 0.3325$

EXERCISE 26A

1 A runner has velocity-time graph as shown. Find the total distance travelled by the runner.



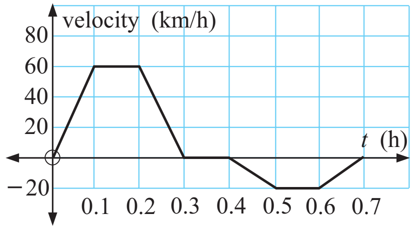
2 A car travels along a straight road and its velocity-time function is illustrated.

a What is the significance of the graph:

i above the t-axis

ii below the t-axis?

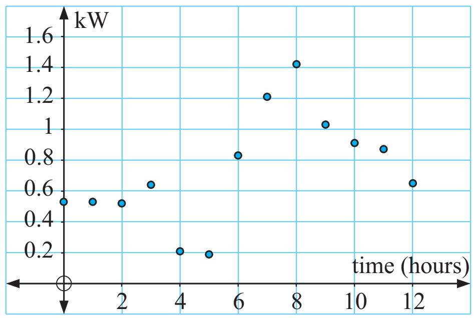
b Find the final displacement of the car.



3 A triathlete rides off from rest accelerating at a constant rate for 3 minutes until she reaches 40 km/h. She then maintains a constant speed for 4 minutes until reaching a section where she slows down at a constant rate to 30 km/h in one minute. She then continues at this rate for 10 minutes before reducing her speed uniformly and is stationary 2 minutes later. After drawing a graph, find how far she has travelled.

4 A household's rate of consumption of electricity in kWh (kilowatt hours) is shown in the graph alongside over a period of 12 hours from midnight.

Use the graph to estimate the total consumption of electricity over this period.



5 Estimate the area bounded by the curve $y = \frac{1}{x}$ and the x-axis from $x = 1$ to $x = 3$ by partitioning into 10 strips and using:

a upper rectangular sums **b** lower rectangular sums

c the midpoint rule.

6 Estimate the area bounded by the x-axis and the curve $y = e^x$ from $x = 0$ to $x = 2$ by partitioning into 10 strips and using the midpoint rule.

7 Find $\int_1^5 \ln x \, dx$ by partitioning into 20 strips and using the midpoint rule.

Note that a **graphics calculator** can be used to quickly find simple definite integrals.

For example, on a TI-83 using $\boxed{\text{MATH}}$ 9 brings up fnInt

TI
C

So to find $\int_0^1 x^2 \, dx$, use fnInt (X^2, X, 0, 1)

and the result is $\int_0^1 x^2 \, dx = 0.3333 \ldots\ldots$

the function $f(x)$
the variable
the upper limit
the lower limit

8 Use your graphics calculator to check your answers to:

a question **6** **b** question **7**.

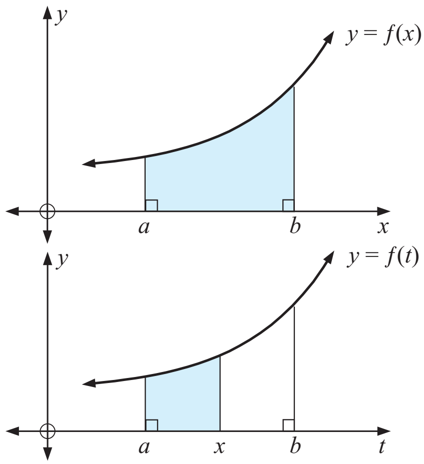
B THE AREA FUNCTION

We have seen that $\displaystyle\int_a^b f(x)\,dx$ is the area between the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

This is, of course, provided that $f(x) \geqslant 0$ over the interval $[a, b]$. The result still holds if we change the variable from x to t say.

Consider $y = f(t)$ as shown alongside.

Let $A(x)$ be the area between $y = f(t)$ and the t-axis from $t = a$ to $t = x$ then $A(x)$ varies as x varies and is therefore an **area function**.



Now $\displaystyle A(x) = \int_a^x f(t)\,dt$

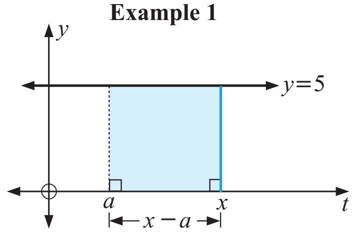
We will apply the area function to constant and linear functions.

USING THE AREA FUNCTION

Consider a constant function $f(t) = 5$, say.

$$\begin{aligned}\int_a^x 5\,dt &= \text{the shaded area}\\ &= (x-a)\times 5\\ &= 5x - 5a\end{aligned}$$

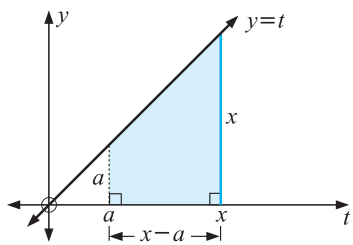
Example 1



Now consider the simplest linear function, $f(t) = t$.

$$\begin{aligned}\int_a^x t\,dt &= \text{the shaded area}\\ &= \left(\tfrac{x+a}{2}\right)(x-a)\\ &= \frac{x^2-a^2}{2}\\ &= \frac{x^2}{2} - \frac{a^2}{2}\end{aligned}$$

Example 2



Can you see the relationship between the original functions and the shaded functions in the answers?

EXERCISE 26B

1 Consider the function $f(t) = 2$. Draw the graph of the function and shade the area between the graph and the t-axis from $t = a$ to $t = x$. Show that $\displaystyle\int_a^x 2\,dt = 2x - 2a$.

2 Consider the function $f(t) = 3t$. Draw the graph of the function and shade the area between the graph and the t-axis from $t = a$ to $t = x$.

Show that $\displaystyle\int_a^x 3t\,dt = \tfrac{3}{2}x^2 - \tfrac{3}{2}a^2$.

3 Consider the function $f(t) = 2t + 3$. Draw the graph of the function and shade the area between the graph and the t-axis from $t = a$ to $t = x$.

Show that $\displaystyle\int_a^x (2t+3)\,dt = (x^2+3x) - (a^2+3a)$.

INVESTIGATION 1 THE AREA FUNCTION

Suppose the shaded part of the answer in the two examples above is $F(x)$, i.e., in the first example $F(x) = 5x$.

What to do:

1 What is the form of each final answer in terms of F?

2 What is the connection between $F(x)$ and the graph connected to it?

3 Repeat the procedure of area finding with: **a** $f(t) = \frac{1}{2}t + 3$ **b** $f(t) = 5 - 2t$
Do your results fit your observations in **1** and **2**?

4 If $f(t) = 3t^2 + 4t + 5$, predict what $F(x)$ would be without attempting to go through the graphical procedure.

From the investigation you should have discovered that for $f(t) \geqslant 0$

- $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$ where $F'(x) = f(x)$
- geometrically $f(x)$ is the leading edge (right side) of the shaded area.

$F(x)$ is called the **antiderivative** of $f(x)$.

C ANTIDIFFERENTIATION

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

We have already seen the usefulness in problem solving of the derivative. The antiderivative has a large number of useful applications. These include:

- finding areas where curved boundaries are involved
- finding volumes of revolution
- finding distances travelled from velocity functions
- finding hydrostatic pressure
- finding work done by a force
- finding centres of mass and moments of inertia
- solving problems in economics and biology
- solving problems in statistics.

In many problems in calculus we know the rate of change of one variable with respect to another, i.e., $\dfrac{dy}{dx}$, but we need to know y in terms of x.

For example,

- The slope function $f'(x)$, of a curve is $2x+3$, and the curve passes through the origin. What is the function $y=f(x)$?
- The rate of change in temperature (in ^{o}C) $\dfrac{dT}{dt}=10e^{-t}$ where t is the time in minutes, $t \geqslant 0$. What is the temperature function given that initially the temperature was 11^{o}C?

The process of finding y from $\dfrac{dy}{dx}$ (or $f(x)$ from $f'(x)$) is the reverse process of differentiation and is called **antidifferentiation**.

Consider the following problem: If $\dfrac{dy}{dx}=x^2$, what is y in terms of x?

From our work on differentiation we know that y must involve x^3 as when we differentiate power functions the index reduces by 1.

If $y=x^3$ then $\dfrac{dy}{dx}=3x^2$, so if we start with $y=\frac{1}{3}x^3$ then $\dfrac{dy}{dx}=x^2$.

However, if $y=\frac{1}{3}x^3+2$ then $\dfrac{dy}{dx}=x^2$, if $y=\frac{1}{3}x^3+100$ then $\dfrac{dy}{dx}=x^2$

and if $y=\frac{1}{3}x^3-7$ then $\dfrac{dy}{dx}=x^2$.

In fact, there are infinitely many such functions of the form $y=\frac{1}{3}x^3+c$ where c is an arbitrary constant.

Ignoring the arbitrary constant we say that: $\frac{1}{3}x^3$ is the antiderivative of x^2 as it is the simplest function which when differentiated gives x^2.

Example 4

Find the antiderivative of: **a** x^3 **b** e^{2x} **c** $\dfrac{1}{\sqrt{x}}$

a We know that the derivative of x^4 involves x^3

i.e., $\dfrac{d}{dx}\left(x^4\right)=4x^3$ and $\therefore$ $\dfrac{d}{dx}\left(\frac{1}{4}x^4\right)=x^3$

So, the antiderivative of x^3 is $\frac{1}{4}x^4$.

b As $\dfrac{d}{dx}\left(e^{2x}\right)=e^{2x}\times 2$

then $\dfrac{d}{dx}\left(\frac{1}{2}e^{2x}\right)=\frac{1}{2}\times e^{2x}\times 2=e^{2x}$

So, the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c $\dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ Now $\dfrac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \quad \dfrac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\dfrac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 26C

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2} **v** x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{2}}$

b From your answers in **a**, predict a general rule for the antiderivative of x^n.

2 **a** Find the antiderivative of:

i e^{2x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b From your answers in **a**, predict a general rule for the anti-derivative of e^{kx} where k is a constant.

3 Find the antiderivative of:

a $6x^2+4x$ by differentiating x^3+x^2 **b** e^{3x+1} by differentiating e^{3x+1}

c $\sqrt{x}$ by differentiating $x\sqrt{x}$ **d** $(2x+1)^3$ by differentiating $(2x+1)^4$

4 Find y if: (Do not forget the integrating constant c.)

a $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = 5x - x^2$

d $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **e** $\dfrac{dy}{dx} = e^{-3x}$ **f** $\dfrac{dy}{dx} = 4x^3 + 3x^2$

Summary

Function	*Antiderivative*
k	kx
x^n	$\dfrac{x^{n+1}}{n+1}$
e^x	e^x

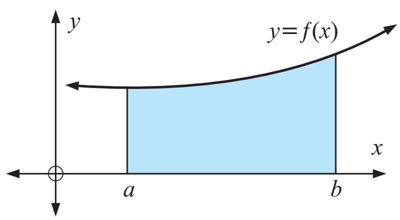
$\{k$ is a constant$\}$

D THE FUNDAMENTAL THEOREM OF CALCULUS

Isaac Newton showed the link between differential calculus and the definite integral (the limit of an area sum). This link is called the **Fundamental theorem of calculus**. The beauty of this theorem is that it enables us to evaluate complicated summations.

We have already established that:

"if $f(x)$ is a continuous positive function on an interval $[a, b]$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b f(x)\,dx$ ".

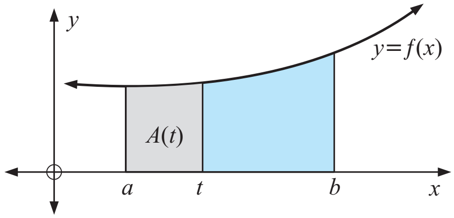


Consider a function $y = f(x)$ which has antiderivative $F(x)$ and an area function $A(t)$, where $A(t)$ is the area from $x = a$ to $x = t$,

i.e., $A(t) = \int_a^t f(x)\,dx$.

$A(t)$ is clearly an increasing function and

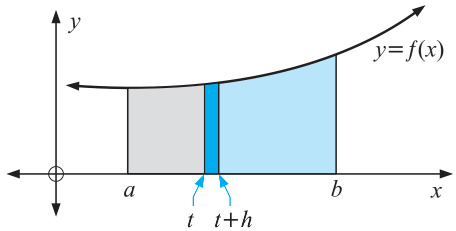
$A(a) = 0$ and $A(b) = \int_a^b f(x)\,dx$ ..... (1)



Now consider a narrow strip of the region between $x = t$ and $x = t + h$.

The area of this strip is $A(t+h) - A(t)$.

Since the narrow strip is contained within two rectangles then



area of smaller rectangle $\leqslant A(t+h) - A(t) \leqslant$ area of larger rectangle

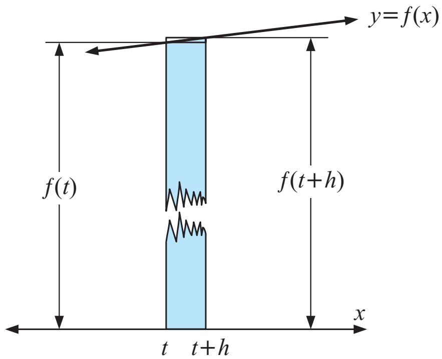
$\therefore \quad hf(t) \leqslant A(t+h) - A(t) \leqslant hf(t+h)$

$\therefore \quad f(t) \leqslant \dfrac{A(t+h) - A(t)}{h} \leqslant f(t+h)$

Now taking limits as $h \to 0$ gives

$$f(t) \leqslant A'(t) \leqslant f(t)$$

Consequently $A'(t) = f(t)$



enlarged strip

i.e., $A(t)$, the area function, is an antiderivative of $f(t)$.

So, $A(t)$ and $F(t)$ differ by a constant

i.e., $A(t) = F(t) + c$ ..... (2)

$$\text{Now} \quad \int_a^b f(x)\,dx = A(b) \qquad \{\text{from (1)}\}$$
$$= F(b) + c \qquad \{\text{from (2)}\}$$

However $A(a) = F(a) + c$ on letting $t = a$ in (2)

$\therefore \quad 0 = F(a) + c$

$\therefore \quad c = -F(a)$

$$\text{Thus,} \quad \int_a^b f(x)\,dx = F(b) - F(a)$$

Hence,

The Fundamental theorem of calculus is:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

Note: Considering a velocity-time function $v(t)$ we know that $\dfrac{ds}{dt} = v$.

So, $s(t)$ is the antiderivative of $v(t)$ and by the Fundamental theorem of calculus,

$\displaystyle\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ gives the **displacement** over the time interval $[t_1, t_2]$.

The following properties of the definite integral can all be deduced from the Fundamental theorem of calculus and some can be easily demonstrated graphically.

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b c\,dx = c(b-a)$ {c is a constant}
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

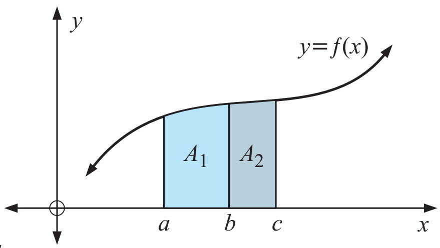
For example, consider $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Proof: $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

$= F(b) - F(a) + F(c) - F(b)$

$= F(c) - F(a)$

$\displaystyle= \int_a^c f(x)\,dx$



i.e., $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$

EXERCISE 26D

1 Use the Fundamental theorem of calculus to show that:

a $\displaystyle\int_a^a f(x)\,dx = 0$ and explain the result graphically

b $\displaystyle\int_a^b c\,dx = c(b-a)$, c is a constant

c $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

d $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

e $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

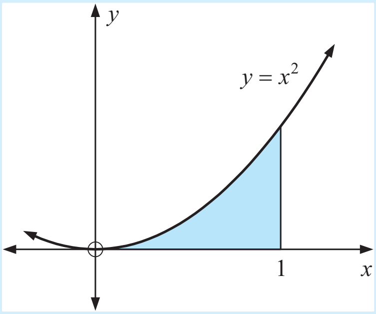
Example 5

Use the Fundamental theorem of calculus to find the area:

a between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b between the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$

a



$f(x) = x^2$ has antiderivative

$F(x) = \dfrac{x^3}{3}$

So the area

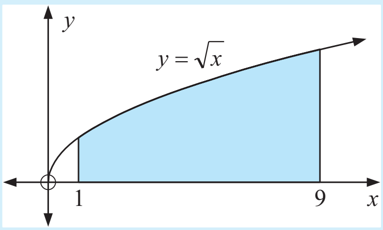
$= \int_0^1 x^2\, dx$

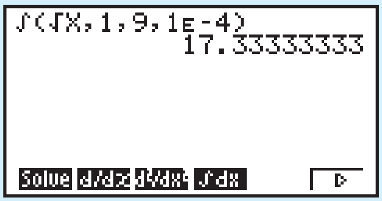
$= F(1) - F(0)$

$= \frac{1}{3} - 0$

$= \frac{1}{3}$ units^2

b





$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

So the area

$= \int_1^9 x^{\frac{1}{2}}\, dx$

$= F(9) - F(1)$

$= \frac{2}{3} \times 27 - \frac{2}{3} \times 1$

$= 17\frac{1}{3}$ units^2

TI
C

2 Use the Fundamental theorem of calculus to find the area between the x-axis and:

a $y = x^3$ from $x = 0$ to $x = 1$

b $y = x^3$ from $x = 1$ to $x = 2$

c $y = x^2 + 3x + 2$ from $x = 1$ to $x = 3$

d $y = \sqrt{x}$ from $x = 0$ to $x = 2$

e $y = e^x$ from $x = 0$ to $x = 1.5$

f $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$

g $y = x^3 + 2x^2 + 7x + 4$ from $x = 1$ to $x = 1.25$

Check each answer using technology.

E INTEGRATION

Earlier we showed that the **antiderivative** of x^2 was $\frac{1}{3}x^3$

i.e., if $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3$.

We also showed that $\frac{1}{3}x^3 + c$ where c is any constant has derivative x^2.

We say that "the **integral** of x^2 is $\frac{1}{3}x^3 + c$" and write

$$\int x^2 dx = \tfrac{1}{3}x^3 + c$$

this reads "the integral of x^2 with respect to x"

In general, if $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

DISCOVERING INTEGRALS

Since integration or finding antiderivatives is the reverse process of differentiating we can discover integrals by differentiation.

For example,

- if $F(x) = x^4$, then $F'(x) = 4x^3$

  $\therefore \int 4x^3 dx = x^4 + c$

- if $F(x) = \sqrt{x} = x^{\frac{1}{2}}$, then $F'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

  $$\therefore \int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + c$$

The rules

- $\int k\,f(x)\,dx = k\int f(x)\,dx$, k is a constant

and

- $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ may prove useful.

The first tells us that a constant k may be written before the integral sign and the second tells us that the integral of a sum is the sum of the separate integrals.

This rule enables us to **integrate term-by-term**.

To prove the first of these rules we consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\text{Now} \quad \frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$

$$\therefore \quad \int k\,f(x)\,dx = k\,F(x)$$
$$= k\int f(x)\,dx$$

Example 6

If $y = x^4 + 2x^3$, find $\dfrac{dy}{dx}$ and hence find $\displaystyle\int (2x^3 + 3x^2)\,dx$.

If $y = x^4 + 2x^3$, then $\dfrac{dy}{dx} = 4x^3 + 6x^2$

$$\therefore \int 4x^3 + 6x^2\,dx = x^4 + 2x^3 + c_1$$
$$\therefore \int 2(2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c_1$$
$$\therefore 2\int (2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c_1$$
$$\therefore \int (2x^3 + 3x^2)\,dx = \tfrac{1}{2}x^4 + x^3 + c$$

EXERCISE 26E.1

1 If $y = x^7$, find $\dfrac{dy}{dx}$ and hence find $\int x^6\,dx$.

2 If $y = x^3 + x^2$, find $\dfrac{dy}{dx}$ and hence find $\int (3x^2 + 2x)\,dx$.

3 If $y = e^{2x+1}$, find $\dfrac{dy}{dx}$ and hence find $\int e^{2x+1}\,dx$.

4 If $y = (2x+1)^4$, find $\dfrac{dy}{dx}$ and hence find $\int (2x+1)^3\,dx$.

5 If $y = x\sqrt{x}$, find $\dfrac{dy}{dx}$ and hence find $\int \sqrt{x}\,dx$.

6 If $y = \dfrac{1}{\sqrt{x}}$, find $\dfrac{dy}{dx}$ and hence find $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx$.

7 Prove the rule $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

[**Hint:** Suppose $F(x)$ is the antiderivative of $f(x)$ and
$G(x)$ is the antiderivative of $g(x)$.

Find $\dfrac{d}{dx}[F(x) + G(x)]$.]

8 **a** Find $\dfrac{dy}{dx}$ if $y = (2x-1)^6$ and hence find $\int (2x-1)^5\,dx$.

b Find $\dfrac{dy}{dx}$ if $y = \sqrt{1-4x}$ and hence find $\displaystyle\int \frac{1}{\sqrt{1-4x}}\,dx$.

c Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{\sqrt{3x+1}}$ and hence find $\displaystyle\int \frac{1}{(3x+1)^{\frac{3}{2}}}\,dx$.

9 **a** If $y = e^{1-3x}$, find $\dfrac{dy}{dx}$ and hence find $\int e^{1-3x}\,dx$.

b If $y = \ln(4x+1)$, find $\dfrac{dy}{dx}$ and hence find $\displaystyle\int \frac{1}{4x+1}\,dx$, for $4x+1 > 0$.

10 **a** By considering $\dfrac{d}{dx}(e^{x-x^2})$, find $\int e^{x-x^2}(1-2x)dx$.

b By considering $\dfrac{d}{dx}\ln(5-3x+x^2)$, find $\displaystyle\int \frac{4x-6}{5-3x+x^2}\,dx$.

c By considering $\dfrac{d}{dx}(x^2-5x+1)^{-2}$, find $\displaystyle\int \frac{2x-5}{(x^2-5x+1)^3}\,dx$.

d By considering $\dfrac{d}{dx}(xe^x)$, find $\int xe^x\,dx$.

e By considering $\dfrac{d}{dx}(2^x)$, find $\int 2^x\,dx$. [**Hint:** $2^x = (e^{\ln 2})^x$.]

f By considering $\dfrac{d}{dx}(x\ln x)$, find $\int \ln x\,dx$.

In earlier chapters we developed rules to help us differentiate functions more efficiently. Following is a summary of these rules:

Function	*Derivative*	*Name*
c, a constant	0	
$mx+c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x)+v(x)$	$u'(x)+v'(x)$	**sum rule**
$u(x)v(x)$	$u'(x)v(x)+u(x)v'(x)$	**product rule**
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y=f(u)$ where $u=u(x)$	$\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)}f'(x)$	
$\ln x$	$\dfrac{1}{x}$	
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1}f'(x)$	

These rules or combinations of them can be used to differentiate almost all functions.

However, the task of finding **antiderivatives** is not so easy and cannot be contained by listing a set of rules as we did above. In fact huge books of different types of functions and their integrals have been written. Fortunately our course is restricted to a few special cases.

SIMPLE INTEGRALS

Notice that $\frac{d}{dx}(kx+c) = k$ $\qquad \therefore \quad \int k\,dx = kx + c$

if $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{(n+1)x^n}{n+1} = x^n$ $\qquad \therefore \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$\frac{d}{dx}(e^x + c) = e^x$ $\qquad \therefore \quad \int e^x\,dx = e^x + c$

if $x > 0$, $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$

if $x < 0$, $\frac{d}{dx}(\ln(-x) + c) = \frac{-1}{-x} = \frac{1}{x}$ $\qquad \therefore \quad \int \frac{1}{x}\,dx = \ln|x| + c$

Summary

Function	*Integral*
k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln\|x\| + c$

k is a constant

c is always an arbitrary constant called "the **integrating constant**" or "the **constant of integration**".

Example 7

Find **a** $\int (x^3 - 2x^2 + 5)\,dx$ **b** $\int \left(\frac{1}{x^3} - \sqrt{x}\right)\,dx$

a $\int (x^3 - 2x^2 + 5)\,dx$

$$= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c$$

b $\int \left(\frac{1}{x^3} - \sqrt{x}\right)\,dx$

$$= \int (x^{-3} - x^{\frac{1}{2}})\,dx$$

$$= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= -\frac{1}{2x^2} - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

Example 8

Find **a** $\displaystyle\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\displaystyle\int \left(\frac{x^2-2}{\sqrt{x}}\right) dx$

a $\displaystyle\int \left(3x + \frac{2}{x}\right)^2 dx$

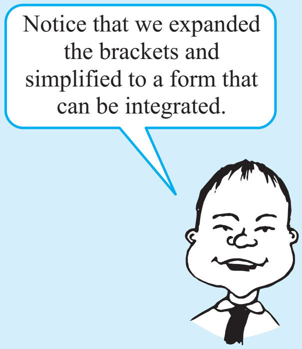
$$= \int \left((3x)^2 + 2(3x)\left(\frac{2}{x}\right) + \left(\frac{2}{x}\right)^2\right) dx$$

$$= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx \qquad \{(a+b)^2 = a^2 + 2ab + b^2\}$$

$$= \int (9x^2 + 12 + 4x^{-2})\, dx$$

$$= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c$$

$$= 3x^3 + 12x - \frac{4}{x} + c$$



b $\displaystyle\int \left(\frac{x^2-2}{\sqrt{x}}\right) dx$

$$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx \qquad \left\{\text{splitting into two fractions as } \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right\}$$

$$= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\, dx \qquad \{\text{index laws}\}$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left\{\int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right\}$$

$$= \tfrac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c \qquad \{\text{simplifying}\}$$

Note:
There is no product or quotient rule for integration. Consequently we often have to carry out multiplication or division before we integrate.

EXERCISE 26E.2

1 Find:

a $\int (x^4 - x^2 - x + 2)\, dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int 2e^x - \frac{1}{x^2}\, dx$

d $\displaystyle\int x\sqrt{x} - \frac{1}{x}\, dx$ **e** $\int (2x+1)^2\, dx$ **f** $\displaystyle\int \frac{x^2+x-3}{x}\, dx$

g $\displaystyle\int \frac{2x-1}{\sqrt{x}}\, dx$ **h** $\displaystyle\int \frac{1}{x\sqrt{x}} - \frac{4}{x}\, dx$ **i** $\int (x+1)^3\, dx$

2 Find y if:

a $\dfrac{dy}{dx} = (1-2x)^2$ **b** $\dfrac{dy}{dx} = \sqrt{x} - \dfrac{2}{\sqrt{x}}$ **c** $\dfrac{dy}{dx} = \dfrac{x^2+2x-5}{x^2}$

3 Find $f(x)$ if:

a $f'(x) = x^3 - 5x + 3$ **b** $f'(x) = 2\sqrt{x}(1-3x)$ **c** $f'(x) = 3e^x - \dfrac{4}{x}$

The constant of integration can be found if we are given a point on the curve.

Example 9

Find $f(x)$ given that $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$.

Since $f'(x) = x^3 - 2x^2 + 3$, then

$$f(x) = \int (x^3 - 2x^2 + 3)\,dx$$

i.e., $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + c$

But $f(0) = 2$, $\therefore$ $0 - 0 + 0 + c = 2$

i.e., $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$

4 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$ **b** $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = e^x + \dfrac{1}{\sqrt{x}}$ and $f(1) = 1$ **d** $f'(x) = x - \dfrac{2}{\sqrt{x}}$ and $f(1) = 2$.

If we are given the second derivative we need to integrate twice to find the function. This creates two integrating constants and so we need two facts about the function in order to find them.

Example 10

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$ and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

$f'(x) = \dfrac{12x^3}{3} - 4x + c$ {integrating with respect to x}

i.e., $f'(x) = 4x^3 - 4x + c$

But $f'(0) = -1$ $\therefore$ $0 - 0 + c = -1$

$\therefore$ $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$\therefore$ $f(x) = \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$ {integrating again}

i.e., $f(x) = x^4 - 2x^2 - x + d$

But $f(1) = 4$ $\therefore$ $1 - 2 - 1 + d = 4$

$\therefore$ $d = 6$

Thus $f(x) = x^4 - 2x - x + 6$

5 Find $f(x)$ given that:

a $f''(x) = 2x + 1$, $f'(1) = 3$ and $f(2) = 7$

b $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$, $f'(1) = 12$ and $f(0) = 5$

c $f''(x) = 2x$ and the points (1, 0) and (0, 5) lie on the curve.

F INTEGRATING e^{ax+b} AND $(ax+b)^n$

Since $\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = \frac{1}{a}e^{ax+b} \times a = e^{ax+b}$

then $$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Likewise if $n \neq -1$,

since $\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \times a,$

$= (ax+b)^n$

then $$\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c, \quad n \neq -1$$

Also, since $\frac{d}{dx}\left(\frac{1}{a}\ln(ax+b)\right) = \frac{1}{a}\left(\frac{a}{ax+b}\right) = \frac{1}{ax+b}$ for $ax+b > 0$

then $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$

In fact $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

Example 11

Find: **a** $\int (2x+3)^4\,dx$ **b** $\displaystyle\int \frac{1}{\sqrt{1-2x}}\,dx$

a $\int (2x+3)^4\,dx$

$= \frac{1}{2} \times \dfrac{(2x+3)^5}{5} + c$

$= \frac{1}{10}(2x+3)^5 + c$

b $\displaystyle\int \frac{1}{\sqrt{1-2x}}\,dx = \int (1-2x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{-2} \times \dfrac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\sqrt{1-2x} + c$

EXERCISE 26F

1 Find:

a $\int (2x+5)^3\,dx$ **b** $\displaystyle\int \frac{1}{(3-2x)^2}\,dx$ **c** $\displaystyle\int \frac{4}{(2x-1)^4}\,dx$

d $\int (4x-3)^7\,dx$ **e** $\int \sqrt{3x-4}\,dx$ **f** $\displaystyle\int \frac{10}{\sqrt{1-5x}}$

g $\int 3(1-x)^4\,dx$ **h** $\displaystyle\int \frac{4}{\sqrt{3-4x}}\,dx$ **i** $\int \sqrt[3]{2x-1}\,dx$

2 **a** If $\dfrac{dy}{dx} = \sqrt{2x-7}$, find $y = f(x)$ given that $y = 11$ when $x = 8$.

b Function $f(x)$ has slope function $\dfrac{4}{\sqrt{1-x}}$, and passes through the point $(-3, -11)$.
Find the point on the graph of the function $y = f(x)$ with x-coordinate -8.

3 Find:

a $\int 3(2x-1)^2\,dx$ **b** $\int (x^2-x)^2\,dx$ **c** $\int (1-3x)^3\,dx$

d $\int (1-x^2)^2\,dx$ **e** $\int 4\sqrt{5-x}\,dx$ **f** $\int (x^2+1)^3\,dx$

Example 12

Find: **a** $\int e^{2x-1}\,dx$ **b** $\int (2e^{2x} - e^{-3x})\,dx$ **c** $\displaystyle\int \frac{4}{1-2x}\,dx$

a $\int e^{2x-1}\,dx$

$= \frac{1}{2}e^{2x-1} + c$

b $\int (2e^{2x} - e^{-3x})\,dx$

$= 2(\frac{1}{2})e^{2x} - (\frac{1}{-3})e^{-3x} + c$

$= e^{2x} + \frac{1}{3}e^{-3x} + c$

c $\displaystyle\int \frac{4}{1-2x}\,dx = 4\int \frac{1}{1-2x}\,dx$

$= 4\left(\frac{1}{-2}\right)\ln|1-2x| + c$

$= -2\ln|1-2x| + c$

4 Find:

a $\int \left(2e^x + 5e^{2x}\right) dx$ **b** $\int \left(x^2 - 2e^{-3x}\right) dx$ **c** $\int \left(\sqrt{x} + 4e^{2x} - e^{-x}\right) dx$

d $\int \frac{1}{2x-1}\, dx$ **e** $\int \frac{5}{1-3x}\, dx$ **f** $\int \left(e^{-x} - \frac{4}{2x+1}\right) dx$

g $\int (e^x + e^{-x})^2\, dx$ **h** $\int (e^{-x} + 2)^2\, dx$ **i** $\int \left(x - \frac{5}{1-x}\right) dx$

5 Find y given that:

a $\frac{dy}{dx} = (1 - e^x)^2$ **b** $\frac{dy}{dx} = 1 - 2x + \frac{3}{x+2}$ **c** $\frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1}$

6 To find $\int \frac{1}{4x}\, dx$, Tracy's answer was $\int \frac{1}{4x}\, dx = \frac{1}{4}\ln|4x| + c$

and Nadine's answer was $\int \frac{1}{4x} dx = \frac{1}{4}\int \frac{1}{x}\, dx = \frac{1}{4}\ln|x| + c$

Which of them has found the correct answer? Prove your statement.

7 **a** If $f'(x) = 2e^{-2x}$ and $f(0) = 3$, find $f(x)$.

b If $f'(x) = 2x - \frac{2}{1-x}$ and $f(-1) = 3$, find $f(x)$.

c If a curve has slope function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through (1, 0), find the equation of the function.

8 Show that $\frac{3}{x+2} - \frac{1}{x-2} = \frac{2x-8}{x^2-4}$, and hence find $\int \frac{2x-8}{x^2-4}\, dx$.

9 Show that $\frac{1}{2x-1} - \frac{1}{2x+1} = \frac{2}{4x^2-1}$, and hence find $\int \frac{2}{4x^2-1}\, dx$.

G INTEGRATING $f(u)u'(x)$ BY SUBSTITUTION

$\int (x^2 + 3x)^4 (2x+3)\, dx$ is of the form $\int f(u)\, u'(x)\, dx$

where $f(u) = u^4$, $u = x^2 + 3x$ and $u'(x) = 2x + 3$

Likewise, $\int e^{x^2 - x}(2x - 1)\, dx$ is of the form $\int f(u)\, u'(x)\, dx$

where $f(u) = e^u$, $u = x^2 - x$ and $u'(x) = 2x - 1$

Also, $\int \frac{3x^2+2}{x^3+2x}\, dx$ is of the form $\int f(u)\, u'(x)\, dx$

where $f(u) = \frac{1}{u}$, $u = x^3 + 2x$ and $u'(x) = 3x^2 + 2$

How do we integrate such functions? It is clear that making a **substitution** involving u makes it easier to understand. Let us examine the first example more closely.

$$\int (x^2+3x)^4(2x+3)\,dx = \int u^4 \frac{du}{dx}\,dx \qquad \{\text{if} \quad u = x^2+3x \quad \text{then} \quad \frac{du}{dx} = 2x+3\}$$

What do we do now? Fortunately we can apply the following **theorem** (special result):

$$\boxed{\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du}$$

This theorem enables us to replace $\dfrac{du}{dx}\,dx$ by du.

Proof: Suppose $F(u)$ is the antiderivative of $f(u)$, i.e., $F'(u) = f(u)$

$\therefore \quad \int f(u)\,du = F(u)+c \quad \ldots\ldots \text{(1)}$

$$\text{But} \quad \frac{d}{dx}F(u) = \frac{d}{du}F(u)\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$= F'(u)\frac{du}{dx}$$
$$= f(u)\frac{du}{dx}$$

$$\therefore \quad \int f(u)\frac{du}{dx}\,dx = F(u)+c$$
$$= \int f(u)\,du \qquad \{\text{from (1)}\}$$

So, $$\int (x^2+3x)^4(2x+3)\,dx = \int u^4\frac{du}{dx}\,dx \qquad \{u = x^2+3x\}$$
$$= \int u^4\,du \qquad \{\text{replacing } \frac{du}{dx}\,dx \text{ by } du\}$$
$$= \frac{u^5}{5}+c$$
$$= \tfrac{1}{5}(x^2+3x)^5+c$$

Example 13

Use substitution to find:

$\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$

$$\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$$
$$= \int \sqrt{u}\,\frac{du}{dx}\,dx \quad \text{where} \quad u = x^3+2x$$
$$= \int \sqrt{u}\,du \qquad \{\text{theorem}\}$$
$$= \int u^{\frac{1}{2}}\,du$$
$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c$$
$$= \tfrac{2}{3}(x^3+2x)^{\frac{3}{2}}+c \quad \{\text{substituting } u = x^3+2x\}$$

Example 14

Use substitution to find: **a** $\displaystyle\int \frac{3x^2+2}{x^3+2x}\,dx$ **b** $\int xe^{1-x^2}\,dx$

a
$$\begin{aligned}\int \frac{3x^2+2}{x^3+2x}\,dx &= \int \frac{1}{x^3+2x}(3x^2+2)\,dx\\ &= \int \frac{1}{u}\frac{du}{dx}\,dx \qquad \{\text{letting } u = x^3+2x\}\\ &= \int \frac{1}{u}\,du \qquad \{\text{theorem}\}\\ &= \ln|u|+c\\ &= \ln\left|x^3+2x\right|+c\end{aligned}$$

b
$$\begin{aligned}\int xe^{1-x^2}\,dx &= \int e^u\left(\tfrac{1}{-2}\frac{du}{dx}\right)dx \qquad \left\{\text{letting } u = 1-x^2 \;\therefore\; \frac{du}{dx} = -2x\right\}\\ &= -\tfrac{1}{2}\int e^u\,du \qquad \{\text{theorem}\}\\ &= -\tfrac{1}{2}e^u+c\\ &= -\tfrac{1}{2}e^{1-x^2}+c\end{aligned}$$

EXERCISE 26G

1 Integrate with respect to x:

a $3x^2(x^3+1)^4$ **b** $\dfrac{2x}{\sqrt{x^2+3}}$ **c** $\sqrt{x^3+x}\,(3x^2+1)$

d $4x^3(2+x^4)^3$ **e** $(x^3+2x+1)^4(3x^2+2)$ **f** $\dfrac{x^2}{(3x^3-1)^4}$

g $\dfrac{x}{(1-x^2)^5}$ **h** $\dfrac{x+2}{(x^2+4x-3)^2}$ **i** $x^4(x+1)^4(2x+1)$

2 Find:

a $\displaystyle\int -2e^{1-2x}\,dx$ **b** $\displaystyle\int 2xe^{x^2}\,dx$ **c** $\displaystyle\int x^2e^{x^3+1}\,dx$

d $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **e** $\displaystyle\int (2x-1)e^{x-x^2}\,dx$ **f** $\displaystyle\int \frac{e^{\frac{x-1}{x}}}{x^2}\,dx$

3 Find:

a $\displaystyle\int \frac{2x}{x^2+1}\,dx$ **b** $\displaystyle\int \frac{x}{2-x^2}\,dx$ **c** $\displaystyle\int \frac{2x-3}{x^2-3x}\,dx$

d $\displaystyle\int \frac{6x^2-2}{x^3-x}\,dx$ **e** $\displaystyle\int \frac{4x-10}{5x-x^2}\,dx$ **f** $\displaystyle\int \frac{1-x^2}{x^3-3x}\,dx$

4 Find $f(x)$ if $f'(x)$ is:

a $x^2(3-x^3)^2$ **b** $\dfrac{3x}{x^2-2}$ **c** $x\sqrt{1-x^2}$

d xe^{1-x^2} **e** $\dfrac{1-3x^2}{x^3-x}$ **f** $\dfrac{(\ln x)^3}{x}$

g $\dfrac{4x+3x^2}{x^3+2x^2-1}$ **h** $\dfrac{4}{x\ln x}$ **i** $\dfrac{1}{x(\ln x)^2}$

H DISTANCE FROM VELOCITY

In this exercise we are concerned with **motion in a straight line**, i.e., **linear motion.**

Given a **velocity function** we can determine the **displacement function** by integration.

From the displacement function we can determine **total distances travelled** in some time interval $a \leqslant t \leqslant b$.

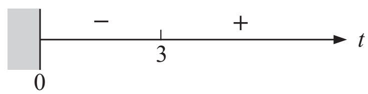
Recall that for some displacement function $s(t)$ the velocity function is $s'(t)$ and that $t \geqslant 0$ in all situations.

Consider the following example:

A particle moves in a straight line with velocity function $v(t) = t - 3$ cm s^{-1}.

How far does it travel in the first 4 seconds of motion?

We notice that $v(t) = s'(t) = t - 3$ which has sign diagram:



Since the velocity function changes sign at $t = 3$ seconds, the particle **reverses direction** at this time.

Now $s(t) = \int (t-3)\,dt$

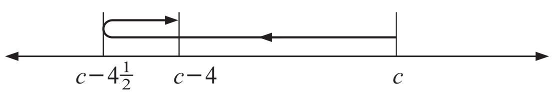
$\therefore$ $s(t) = \dfrac{t^2}{2} - 3t + c$ cm and we are given no information to determine the value of c.

Also the total distance travelled is not $s(4) - s(0)$ because of the reversal of direction at $t = 3$ seconds.

We find the position of the particle at $t = 0$, $t = 3$ and $t = 4$.

$$s(0) = c, \qquad s(3) = c - 4\tfrac{1}{2}, \qquad s(4) = c - 4.$$

Hence, we can draw a diagram of the motion:



Thus the total distance travelled is $(4\frac{1}{2} + \frac{1}{2})$ cm $= 5$ cm.

Notice that $s(4) - s(0) = c - 4 - c = -4$ does not give the total distance travelled.

Summary

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $a \leqslant t \leqslant b$:

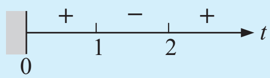
- Draw a sign diagram for $v(t)$ so that we can determine directional changes, if they exist.
- Determine $s(t)$ by integration, with integrating constant c, say.
- Find $s(a)$ and $s(b)$. Also find $s(t)$ at every point where there is a direction reversal.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

Example 15

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ ms^{-1}. How far does P travel in the first 4 seconds of motion?

$$v(t) = s'(t) = t^2 - 3t + 2 = (t-1)(t-2)$$

$\therefore$ sign diagram of v is:



Since the signs change, P reverses direction at $t = 1$ and $t = 2$ secs.

Now $s(t) = \int (t^2 - 3t + 2)\, dt$

$$= \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$$

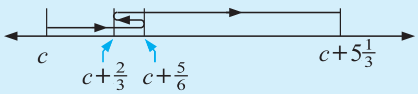
Now $s(0) = c$

$$s(1) = \tfrac{1}{3} - \tfrac{3}{2} + 2 + c = c + \tfrac{5}{6}$$

$$s(2) = \tfrac{8}{3} - 6 + 4 + c = c + \tfrac{2}{3}$$

$$s(4) = \tfrac{64}{3} - 24 + 8 + c = c + 5\tfrac{1}{3}$$

Motion diagram:



$\therefore$ total distance $= \left(c + \tfrac{5}{6} - c\right) + \left(c + \tfrac{5}{6} - [c + \tfrac{2}{3}]\right) + \left(c + 5\tfrac{1}{3} - [c + \tfrac{2}{3}]\right)$

$$= \tfrac{5}{6} + \tfrac{5}{6} - \tfrac{2}{3} + 5\tfrac{1}{3} - \tfrac{2}{3}$$

$$= 5\tfrac{2}{3} \text{ m}$$

EXERCISE 26H

1 A particle has velocity function $v(t) = 1 - 2t$ cms^{-1} as it moves in a straight line. Find the total distance travelled in the first second of motion.

2 Particle P has velocity $v(t) = t^2 - t - 2$ cms^{-1}. Find the total distance travelled in the first 3 seconds of motion.

3 A particle moves along the x-axis with velocity function $x'(t) = 16t - 4t^3$ units/s. Find the total distance travelled in the time interval:

a $0 \leqslant t \leqslant 3$ seconds **b** $1 \leqslant t \leqslant 3$ seconds.

4 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ ms^{-1}, where $t \geqslant 0$, t in seconds.

- **a** State the initial velocity of the particle.
- **b** Find the velocity of the particle after 3 seconds.
- **c** How long would it take for the particle's velocity to increase to 45 ms^{-1}?
- **d** Discuss $v(t)$ as $t \to \infty$.
- **e** Show that the particle's acceleration is always positive.
- **f** Draw the graph of $v(t)$ against t.
- **g** Find the total distance travelled by the particle in the first 3 seconds of motion.

5 A train moves along a straight track with acceleration $\frac{t}{10} - 3$ ms^{-2}. If the initial velocity of the train is 45 ms^{-1}, determine the total distance travelled in the first minute.

6 A body has initial velocity 20 ms^{-1} as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ ms^{-2}.

- **a** Show that as t increases the body approaches a limiting velocity.
- **b** Find the total distance travelled in the first 10 seconds of motion.

I DEFINITE INTEGRALS

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$ then the **definite integral** of $f(x)$ on this interval is

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Note: $\int_a^b f(x)\,dx$ reads "the integral of $f(x)$ from $x = a$ to $x = b$, with respect to x".

Notation: We write $F(b) - F(a) = [F(x)]_a^b$.

Example 16

Find $\int_1^3 (x^2 + 2)\,dx$

$$\int_1^3 (x^2+2)\,dx$$
$$= \left[\frac{x^3}{3} + 2x\right]_1^3$$
$$= \left(\tfrac{3^3}{3} + 2(3)\right) - \left(\tfrac{1^3}{3} + 2(1)\right)$$
$$= (9+6) - (\tfrac{1}{3} + 2)$$
$$= 12\tfrac{2}{3}$$

Check:

```
fnInt(X²+2,X,1,3
)
        12.66666667
```

EXERCISE 26I

1 Evaluate the following and check with your graphics calculator:

a $\int_0^1 x^3\,dx$ **b** $\int_0^2 (x^2-x)\,dx$ **c** $\int_0^1 e^x\,dx$

d $\int_1^4 \left(x-\frac{3}{\sqrt{x}}\right)dx$ **e** $\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$ **f** $\int_1^3 \frac{1}{x}\,dx$

g $\int_1^2 (e^{-x}+1)^2\,dx$ **h** $\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **i** $\int_0^1 e^{1-x}\,dx$

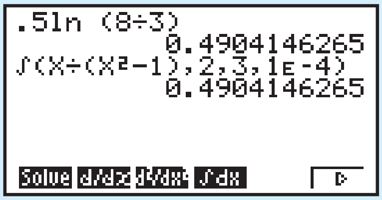
Example 17

Evaluate: **a** $\int_2^3 \frac{x}{x^2-1}\,dx$ **b** $\int_0^1 \frac{6x}{(x^2+1)^3}\,dx$

a In $\int_2^3 \frac{x}{x^2-1}\,dx$, We let $u=x^2-1$ $\therefore$ $du=\frac{du}{dx}\,dx = 2x\,dx$

and when $x=2$, $u=2^2-1=3$
when $x=3$, $u=3^2-1=8$

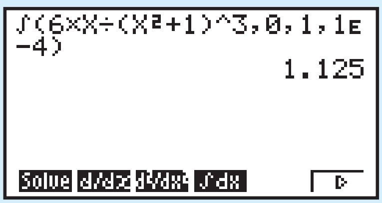
$$\therefore \int_2^3 \frac{x}{x^2-1}\,dx = \int_3^8 \frac{1}{u}\left(\tfrac{1}{2}\,du\right)$$
$$= \tfrac{1}{2}\int_3^8 \frac{1}{u}\,du$$
$$= \tfrac{1}{2}\left[\ln|u|\right]_3^8$$
$$= \tfrac{1}{2}(\ln 8 - \ln 3)$$
$$= \tfrac{1}{2}\ln\left(\tfrac{8}{3}\right)$$



b In $\int_0^1 \frac{6x}{(x^2+1)^3}\,dx$, we let $u=x^2+1$ $\therefore$ $du=\frac{du}{dx}\,dx = 2x\,dx$

So $\int_0^1 \frac{6x}{(x^2+1)^3}\,dx = \int_1^2 \frac{1}{u^3}(3du)$ and when $x=0$, $u=1$
when $x=1$, $u=2$

$$= 3\int_1^2 u^{-3}\,du$$
$$= 3\left[\frac{u^{-2}}{-2}\right]_1^2$$
$$= 3\left(\frac{2^{-2}}{-2}-\frac{1^{-2}}{-2}\right)$$
$$= \tfrac{9}{8}$$



2 Evaluate the following and check with your graphics calculator:

a $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\,dx$ **b** $\displaystyle\int_0^1 x^2 e^{x^3+1}\,dx$ **c** $\displaystyle\int_0^3 x\sqrt{x^2+16}\,dx$

d $\displaystyle\int_1^2 xe^{-2x^2}\,dx$ **e** $\displaystyle\int_2^3 \frac{x}{2-x^2}\,dx$ **f** $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

g $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx$ **h** $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$ **i** $\displaystyle\int_0^1 (x^2+2x)^n(x+1)\,dx$

[**Careful!**]

3 Show that $\dfrac{3}{x+4} - \dfrac{2}{x-1} = \dfrac{x-11}{x^2+3x-4}$.

Hence show that $\displaystyle\int_{-2}^{-1} \frac{x-11}{x^2+3x-4}\,dx = 5\ln\left(\tfrac{3}{2}\right)$.

INVESTIGATION 2 $\int_a^b f(x)\,dx$ AND AREAS

Does $\displaystyle\int_a^b f(x)\,dx$ always give us an area?

What to do:

1 Find $\displaystyle\int_0^1 x^3\,dx$ and $\displaystyle\int_{-1}^1 x^3\,dx$.

2 Explain why the first integral in **1** gives an area whereas the second integral does not. Graphical evidence is essential.

3 Find $\displaystyle\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.

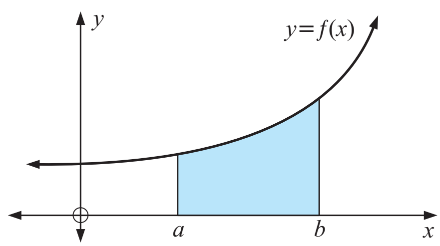
4 Check that $\displaystyle\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

J FINDING AREAS

We have already established that:

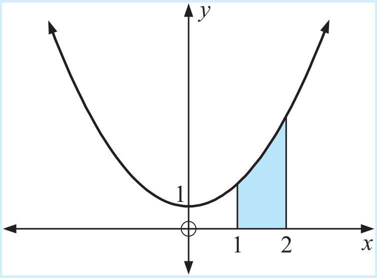
If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx.$$



Example 18

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$.



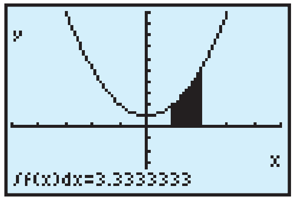
$$\begin{aligned}\text{Area} &= \int_1^2 (x^2+1)\,dx \\ &= \left[\frac{x^3}{3} + x\right]_1^2 \\ &= \left(\tfrac{8}{3}+2\right) - \left(\tfrac{1}{3}+1\right) \\ &= \tfrac{7}{3} + 1 \\ &= 3\tfrac{1}{3}\ \text{units}^2\end{aligned}$$

To check your result on a graphics calculator, (e.g., TI-83)

enter $y = x^2 + 1$ on **Y=** then **GRAPH** , **CALC** 7

asks for lower and upper limits 1 **ENTER** 2 **ENTER**

or fnInt ($X^2 + 1$, X, 1, 2)



EXERCISE 26J

1 Find the area of the region bounded by:

a $y = x^2$, the x-axis and $x = 1$

b $y = x^3$, the x-axis, $x = 1$ and $x = 2$

c $y = e^x$, the x-axis, the y-axis and $x = 1$

d the x-axis and the part of $y = 6 + x - x^2$ above the x-axis

e the axes and $y = \sqrt{9 - x}$

f $y = \dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$

g $y = \dfrac{1}{x}$, the x-axis, $x = -1$ and $x = -3$

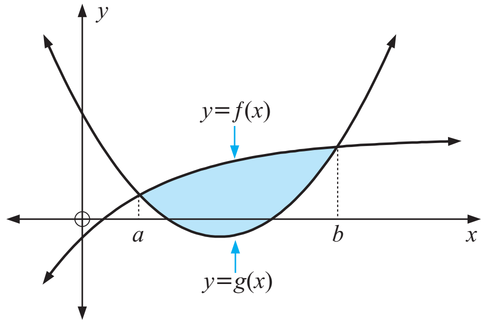
h $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis and $x = 4$

i $y = e^x + e^{-x}$, the x-axis, $x = -1$ and $x = 1$.

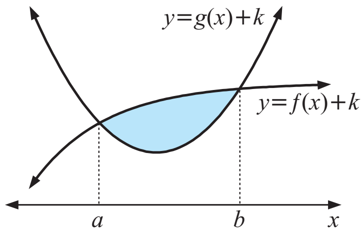
AREA BETWEEN TWO FUNCTIONS

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$ and $f(x) \geqslant g(x)$ for all x in the interval $a \leqslant x \leqslant b$, regardless of the position of the x-axis, then the area of the shaded region between their points of intersection is given by

$$\int_a^b [f(x) - g(x)]\,dx.$$



Proof: If we translate each curve vertically through $[0, k]$ until it is completely above the x-axis, the area is preserved (i.e., does not change).



Area of shaded region

$$= \int_a^b [f(x) + k]\,dx - \int_a^b [g(x) + k]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

Example 19

Find the area of the region bounded by the x-axis and the curve $y = x^2 - 2x$.

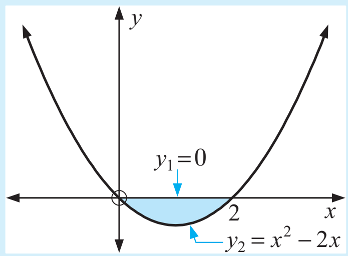
The curve cuts the x-axis when $y = 0$

$\therefore \quad x^2 - 2x = 0$

$\therefore \quad x(x-2) = 0$

$\therefore \quad x = 0$ or 2

i.e., x intercepts are 0 and 2.



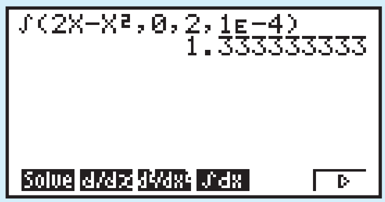
$$\text{Area} = \int_0^2 [y_1 - y_2]\,dx$$

$$= \int_0^2 [0 - (x^2 - 2x)]\,dx$$

$$= \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= \left(4 - \tfrac{8}{3}\right) - (0)$$



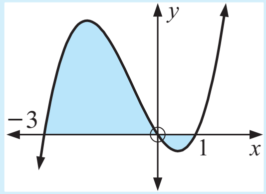
$\therefore$ the area is $\frac{4}{3}$ units^2.

2 Find the area bounded by:

- **a** the x-axis and the curve $y = x^2 + x - 2$
- **b** the x-axis, $y = e^{-x} - 1$ and $x = 2$
- **c** the x-axis and the part of $y = 3x^2 - 8x + 4$ below the x-axis
- **d** $y = x^3 - 4x$, the x-axis, $x = 1$, and $x = 2$.

Example 20

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.



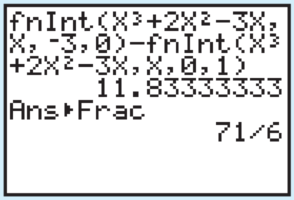
$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x-1)(x+3) \end{aligned}$$

$\therefore \quad y = f(x)$ cuts the x-axis at 0, 1, -3.

Total area

$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\, dx + \int_0^1 [0 - (x^3 + 2x^2 - 3x)]\, dx$

$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\, dx - \int_0^1 (x^3 + 2x^2 - 3x)\, dx$



$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_0^1$

$= (0 - -11\frac{1}{4}) - (-\frac{7}{12} - 0)$

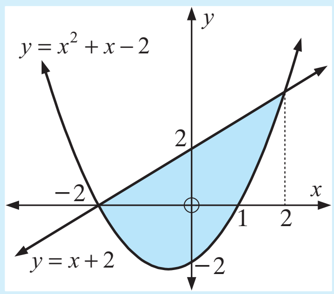
$= 11\frac{5}{6}$ units^2

3 Find the area enclosed by the function $y = f(x)$ and the x-axis for:

a $f(x) = x^3 - 9x$ **b** $f(x) = -x(x-2)(x-4)$ **c** $f(x) = x^4 - 5x^2 + 4$.

Example 21

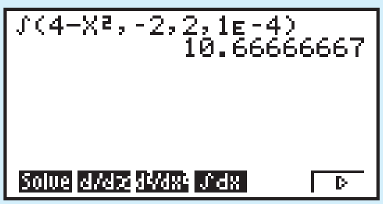
Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.



$y = x + 2$ meets $y = x^2 + x - 2$ where

$$\begin{aligned} x^2 + x - 2 &= x + 2 \\ \therefore \quad x^2 - 4 &= 0 \\ \therefore \quad (x+2)(x-2) &= 0 \\ \therefore \quad x &= \pm 2 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_{-2}^{2} [(x+2) - (x^2 + x - 2)]\, dx \\ &= \int_{-2}^{2} (4 - x^2)\, dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3}) \\ &= 16 - \tfrac{16}{3} \\ &= 10\tfrac{2}{3} \text{ units}^2 \end{aligned}$$



4 **a** Find the area of the region enclosed by $y = x^2 - 2x$ and $y = 3$.

b Consider the graphs of $y = x - 3$ and $y = x^2 - 3x$.

i Sketch each graph on the same set of axes.

ii Find the coordinates of the points where the graphs meet. Check algebraically.

iii Find the area of the region enclosed by the two graphs.

c Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

d On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$, showing axis intercepts and asymptotes.

Find algebraically, the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.

Find the area of the region enclosed by the two curves.

e Determine the area of the region bounded by $y = 2e^x$, $y = e^{2x}$ and $x = 0$.

5 On the same set of axes, draw the graphs of the relations $y = 2x$ and $y^2 = 4x$. Determine the area of the region enclosed by these relations.

6 Sketch the circle with equation $x^2 + y^2 = 9$.

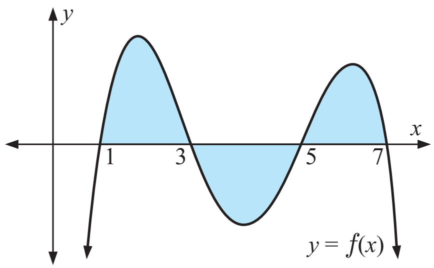
a Explain why the 'upper half' of the circle has equation $y = \sqrt{9 - x^2}$.

b Hence, determine $\int_0^3 \sqrt{9 - x^2}\,dx$ without actually integrating the function.

c Check your answer using technology.

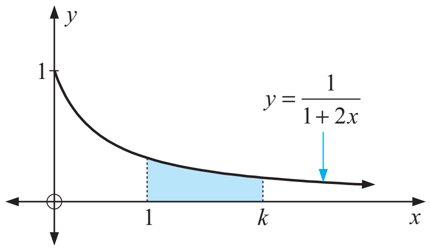
7 **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\,dx$.

b What is the total shaded area equal to in terms of integrals?



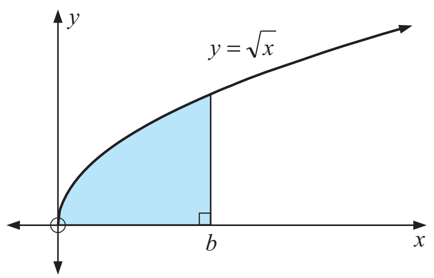
8 The shaded area is 0.2 units^2.

Find k, correct to 4 decimal places.



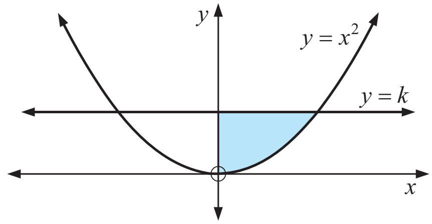
9 The shaded area is 1 unit^2.

Find b, correct to 4 decimal places.



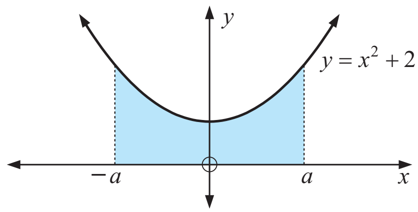
10 The shaded area is 2.4 units^2.

Find k, correct to 4 decimal places.



11 The shaded area is $3a$ units^2.

Find a.



K PROBLEM SOLVING BY INTEGRATION

Example 22

The marginal cost of producing x urns per week is given by $2.15 - 0.02x + 0.000\,36x^2$ dollars per urn provided that $0 \leqslant x \leqslant 120$.
The set up costs before production starts are \$185. Find the total cost of producing 100 urns per day.

The marginal cost is $\dfrac{dC}{dx}$ and $\dfrac{dC}{dx} = 2.15 - 0.02x + 0.000\,36x^2$ \$/urn

$$\therefore \quad C(x) = \int (2.15 - 0.02x + 0.000\,36x^2)\,dx$$
$$= 2.15x - 0.02\frac{x^2}{2} + 0.000\,36\frac{x^3}{3} + c$$
$$= 2.15x - 0.01x^2 + 0.000\,12x^3 + c$$

But $C(0) = 185 \qquad \therefore \quad c = 185$

$$\therefore \quad C(x) = 2.15x - 0.01x^2 + 0.000\,12x^3 + 185$$
$$C(100) = 2.15(100) - 0.01(100)^2 + 0.00012(100)^3 + 185$$
$$= 420$$

$\therefore$ the total cost is \$420.

EXERCISE 26K

1 The marginal cost per day of producing x gadgets is $C'(x) = 3.15 + 0.004x$ dollars per gadget. What is the total cost of daily production of 800 gadgets given that the fixed costs before production commences are \$450 a day?

2 The marginal cost of producing x items is given by $C'(x) = 10 - \dfrac{4}{\sqrt{x+1}}$ dollars per item. If the fixed cost of production is \$200 (i.e., the cost before any items are produced), find the cost of producing 100 items.

3 Swiftflight Pty Ltd makes aeroplanes. The initial cost of designing a new model and setting up to manufacture them will be \$275 million dollars. The cost of manufacturing each additional plane is modelled by $25x - 4x^{0.8} + 0.0024x^2$ million dollars where x is the number of aeroplanes made. Find the total cost of manufacturing the first 20 aeroplanes.

4 The rate of change in sales S, with respect to advertising expenditure x, is given by

$$\frac{dS}{dx} = 10 + \frac{3}{x} + x^2 \quad \text{for} \quad 2 \leqslant x \leqslant 10.$$

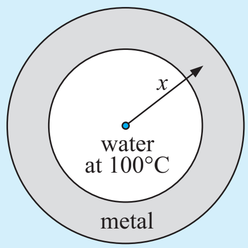
Determine the total change in sales as advertising expenditure increases from 3 to 6.

5 The marginal profit for producing x dinner plates per week is given by $15 - 0.03x$ dollars per plate. If no plates are made a loss of \$650 each week occurs.

a Find the profit function.
b What is the maximum profit and when does it occur?
c What production levels enable a profit to be made?

Example 23

A metal tube has an annulus cross-section as shown. The outer radius is 4 cm and the inner radius is 2 cm. Within the tube, water is maintained at a temperature of 100^{o}C. Within the metal the temperature drops off from inside to outside according to $\dfrac{dT}{dx} = -\dfrac{10}{x}$ where x is the distance from the central axis O, and $2 \leqslant x \leqslant 4$.
Find the temperature of the outer surface of the tube.



tube cross-section

$$\frac{dT}{dx} = \frac{-10}{x} \qquad \therefore \quad T = \int \frac{-10}{x}\,dx$$

$$\therefore \quad T = -10 \ln|x| + c$$

But when $x = 2$, $T = 100$

$$\therefore \quad 100 = -10 \ln 2 + c$$
$$\therefore \quad c = 100 + 10 \ln 2$$

Thus $T = -10 \ln x + 100 + 10 \ln 2$

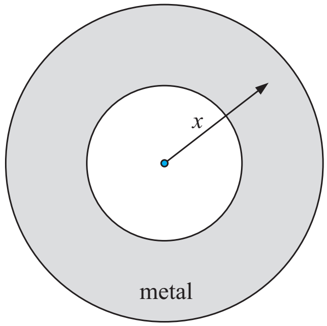
i.e., $T = 100 + 10 \ln\left(\dfrac{2}{x}\right)$

and when $x = 4$, $T = 100 + 10 \ln\left(\frac{1}{2}\right) \doteqdot 93.07$

$\therefore$ the outer surface temperature is 93.07^{o}C.

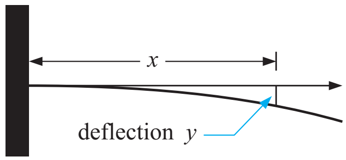
6 Jon needs to bulk-up for the next AFL season. His energy needs t days after starting his weight gain program are given by $E(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories/day. Find Jon's total energy needs over the first week of the program.

7 The tube cross-section shown has inner radius of 3 cm and outer radius 6 cm and within the tube water is maintained at a temperature of 100^{o}C. Within the metal the temperature falls off at a rate according to $\dfrac{dT}{dx} = \dfrac{-20}{x^{0.63}}$ where x is the distance from the central axis O and $3 \leqslant x \leqslant 6$.



Find the temperature of the outer surface of the tube.

8 A thin horizontal cantilever of length 1 metre has a deflection of y metres at a distance of x m from the fixed end.



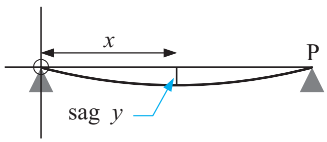
It is known that $\dfrac{d^2y}{dx^2} = -\frac{1}{10}(1 - x)^2$.

a Find the equation for measuring the deflection from the horizontal at any point on the beam.

[**Hint:** When $x = 0$, what are y and $\dfrac{dy}{dx}$?]

b Determine the greatest deflection of the beam.

9



A plank of wood is supported only at its ends, O and P 4 metres from O. The plank sags under its own weight, a distance of y metres, x metres from end O.

The differential equation $\dfrac{d^2y}{dx^2} = 0.01\left(2x - \dfrac{x^2}{2}\right)$ relates the variables x and y.

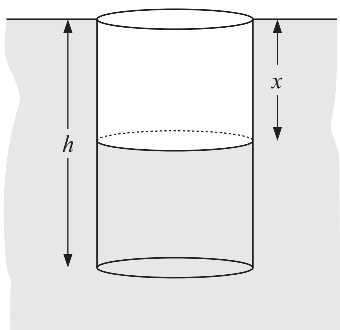
a Find the equation for measuring the sag from the horizontal at any point along the plank.

b Find the maximum sag from the horizontal.

c Find the sag from the horizontal at a distance 1 m from P.

d Find the angle the plank makes with the horizontal 1 m from P.

10 A contractor digs roughly cylindrical wells to a depth of h metres and estimates that the cost of digging a well x metres deep is $\frac{1}{2}x^2 + 4$ dollars per m^3.



If a well is to have a radius r cm, show that the total cost of digging a well is given by

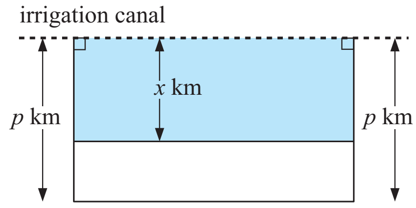
$$C(h) = \pi r^2 \left(\frac{h^3 + 24h}{6}\right) + C_0 \quad \text{dollars.}$$

$$\left[\textbf{Hint:} \quad \frac{dC}{dx} = \frac{dC}{dV} \times \frac{dV}{dx}\right]$$

11 A farmer with a large property plans a rectangular fruit orchard with one boundary being an irrigation canal. He has 4 km of fencing to fence the orchard.

The farmer knows that the yield per unit of area changes the further you are away from the canal. This yield, Y per unit of area is proportional to

$\dfrac{1}{\sqrt{x+4}}$ where x is as shown in the figure.



a Explain why $\dfrac{dY}{dA} = \dfrac{k}{\sqrt{x+4}}$ where k is a constant.

b Show that $\dfrac{dY}{dx} = \dfrac{k(4-2p)}{\sqrt{x+4}}$ by using the chain rule.

c Explain why $Y = \displaystyle\int_0^p \frac{k(4-2p)}{\sqrt{x+4}}\,dx.$

d Show that $Y = 4k(2-p)[\sqrt{p+4}-2]$

e What dimensions should the orchard be if yield is to be a maximum?

REVIEW SET 26A

1 Integrate with respect to x: **a** $\dfrac{4}{\sqrt{x}}$ **b** $\dfrac{3}{1-2x}$ **c** xe^{1-x^2}

2 Evaluate: **a** $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\,dx$ **b** $\displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\,dx$

3 By differentiating $y = \sqrt{x^2-4}$, find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx.$

4 A particle moves in a straight line with velocity $v(t) = 2t - 3t^2$ m s^{-1}. Find the distance travelled in the first second of motion.

5 Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

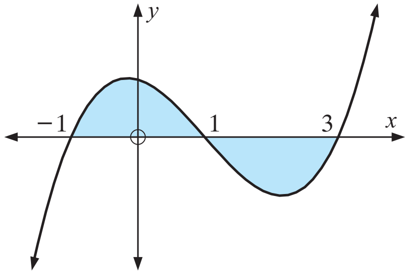
6 The current $I(t)$ amps, in a circuit falls off in accordance with $\dfrac{dI}{dt} = \dfrac{-100}{t^2}$ where t is the time in seconds, provided that $t \geqslant 0.2$ seconds.

It is known that when $t = 2$, the current is 150 amps. Find a formula for the current at any time $(t \geqslant 0.2)$, and hence find:

a the current after 20 seconds **b** what happens to the current as $t \to \infty$.

7 Determine $\int_0^2 \sqrt{4-x^2}\,dx$ by considering the graph of $y = \sqrt{4-x^2}$.

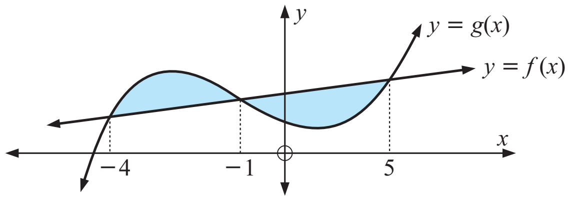
8 Is it true that $\int_{-1}^{3} f(x)\,dx$ represents the area of the shaded region?
Explain your answer briefly.



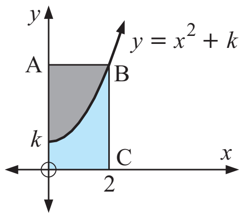
9 Consider $f(x) = \dfrac{x}{1+x^2}$.

- **a** Find the position and nature of all turning points of $y = f(x)$.
- **b** Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **c** Sketch the graph of $y = f(x)$.
- **d** Find the area enclosed by $y = f(x)$, the x-axis and the vertical line $x = -2$.

10 Write down an expression for the shaded area:



11



OABC is a rectangle and the two shaded regions are equal in area. Find k.

REVIEW SET 26B

1 Find: **a** $\displaystyle\int \left(2e^{-x} - \frac{1}{x} + 3\right) dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

2 Evaluate: **a** $\int_1^2 (x^2-1)^2\,dx$ **b** $\int_1^2 x(x^2-1)^2\,dx$

3 By differentiating $(3x^2+x)^3$, find $\int (3x^2+x)^2(6x+1)\,dx$.

4 A particle moves in a straight line with velocity given by $v(t) = t^2 - 6t + 8$ m s^{-1} for $t \geqslant 0$.

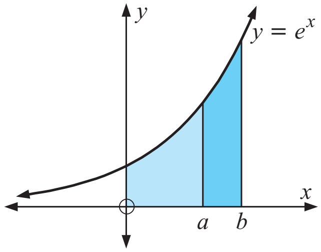
- **a** Draw a sign diagram for $v(t)$.
- **b** Explain exactly what is happening to the particle in the first 5 seconds of motion.
- **c** After 5 seconds, how far is the particle from its original position?
- **d** Find the total distance travelled in the first 5 seconds of motion.

5 Determine the area enclosed by the axes and $y = 4e^x - 1$.

6 A curve $y = f(x)$ has $f''(x) = 18x + 10$. Find $f(x)$ given that $f(0) = -1$ and $f(1) = 13$.

7 Find a given that the area of the region between $y = e^x$ and the x-axis from $x = 0$ to $x = a$ is 2 units^2.

Hence determine b, given that the area of the region between $x = a$ and $x = b$ is also 2 units^2.

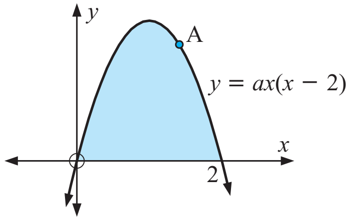


8 $\dfrac{4x-3}{2x+1}$ can be written in the form $A + \dfrac{B}{2x+1}$.

a Find the value of A and B.

b Hence find $\displaystyle\int_0^2 \frac{4x-3}{2x+1}\,dx$.

9 Find a given that the shaded area is 4 units^2. Find the x-coordinate of A if OA divides the shaded region into equal areas.



10 Consider $f(x) = \dfrac{3x-5}{(x-2)^2}$.

a State the axis intercepts.

b State the equation of any vertical asymptotes.

c Find the position and nature of any turning points.

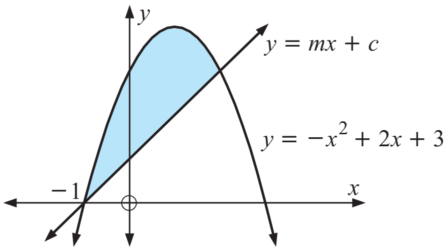
d Using a graphics calculator to help, sketch the graph of $y = f(x)$.

e Find constants A and B such that $\dfrac{3x-5}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$.

f Find the area of the region defined by $y = f(x)$, the x-axis and the vertical line $x = -1$.

11 Determine m and c if the enclosed region has area $4\frac{1}{2}$ units^2.

Hint: You may need your graphics calculator to do the algebra.



REVIEW SET 26C

1 Find y if: **a** $\dfrac{dy}{dx} = (x^2-1)^2$ **b** $\dfrac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

2 Evaluate: **a** $\displaystyle\int_{-2}^{0} \frac{4}{2x-1}\,dx$ **b** $\displaystyle\int_0^1 \frac{10x}{\sqrt{3x^2+1}}\,dx$

3 By differentiating $\sqrt{3x^2+1}$, find $\displaystyle\int \frac{x}{\sqrt{3x^2+1}}\,dx$.

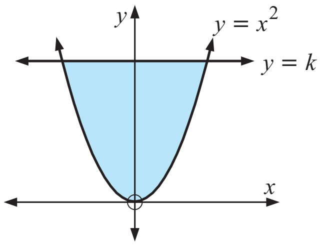
4 O is a point on a straight line. A particle moving on this straight line has a velocity of 27 cm s^{-1} as it passes through O. Its acceleration t seconds later is $6t - 30$ cm s^{-2}. Find the total distance (from O) that the particle has travelled when it momentarily comes to rest for the **second** time.

5 Draw the graphs of $y^2 = x - 1$ and $y = x - 3$.

a Determine the coordinates where the graphs meet.

b Determine the area of the enclosed region.

6 Determine k if the enclosed region has area $5\frac{1}{3}$ units^2.

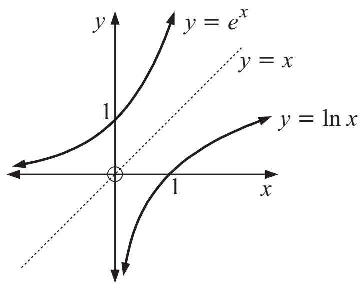


7 A function has slope function $2\sqrt{x} + \dfrac{a}{\sqrt{x}}$ and passes through the points (0, 2) and (1, 4). Find a and hence explain why the function $y = f(x)$ has no stationary points.

8 Write $\dfrac{-2x}{4 - x^2}$ as $\dfrac{A}{x+2} + \dfrac{B}{2-x}$ and hence show that $\displaystyle\int_3^4 \frac{-2x}{4-x^2}\,dx = \ln\left(\tfrac{12}{5}\right)$.

9 By appealing only to geometrical evidence, explain why:

$$\int_0^1 e^x\,dx + \int_1^e \ln x\,dx = e.$$



10 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity in metres per second at time t seconds is then given by

$$v(t) = \frac{100}{(t+2)^2} \text{ m s}^{-1}, \quad t \geqslant 0.$$

a Find the initial velocity of the boat, and its velocity after 3 seconds.

b Discuss $v(t)$ as $t \to \infty$.

c Sketch the graph of $v(t)$ against t.

d Find how long it takes for the boat to travel 30 metres.

e Find the acceleration of the boat at any time t.

f Show that $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$, and find the value of the constant k.

11 Find $\dfrac{d}{dx}(\ln x)^2$ and hence find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Chapter 27

Trigonometric integration

Contents:

A Basic trigonometric integrals

B Integrals of trigonometric functions of the form $f(ax+b)$

C Definite integrals

D Area determination

Review set 27A

Review set 27B

A BASIC TRIGONOMETRIC INTEGRALS

Earlier we discovered that integrals of functions can be found by appropriate differentiation. Observe the following:

$$\frac{d}{dx}(\sin x + c) = \cos x + 0 = \cos x$$

$$\frac{d}{dx}(-\cos x + c) = -(-\sin x) + 0 = \sin x$$

$$\frac{d}{dx}(\tan x + c) = \frac{1}{\cos^2 x}$$

$$\therefore \int \cos x\, dx = \sin x + c \qquad \therefore \int \sin x\, dx = -\cos x + c \qquad \therefore \int \frac{1}{\cos^2 x}\, dx = \tan x + c$$

We can now extend our basic list of integrals to include these discoveries.

Thus, the integrals of basic functions are:

Function	*Integral*	*Function*	*Integral*
k (a constant)	$kx + c$	e^x	$e^x + c$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$	$\cos x$	$\sin x + c$
		$\sin x$	$-\cos x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$	$\frac{1}{\cos^2 x}$	$\tan x + c$

Reminder:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kf(x)dx = k\int f(x)dx \qquad (k \text{ is a constant})$$

Example 1

Integrate with respect to x:

a $2\sin x - \cos x$ **b** $\frac{3}{\cos^2 x} - \frac{2}{x} + \sqrt{x}$

a
$$\int [2\sin x - \cos x]dx$$
$$= 2(-\cos x) - \sin x + c$$
$$= -2\cos x - \sin x + c$$

b
$$\int \left[\frac{3}{\cos^2 x} - \frac{2}{x} + x^{\frac{1}{2}}\right] dx$$
$$= 3\tan x - 2\ln\lvert x\rvert + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= 3\tan x - 2\ln\lvert x\rvert + \tfrac{2}{3}x^{\frac{3}{2}} + c$$

EXERCISE 27A

1 Integrate with respect to x:

a $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $2\sqrt{x} + \frac{4}{\cos^2 x}$

d $\frac{1}{\cos^2 x} + 2\sin x$ **e** $\frac{x}{2} - \frac{1}{\cos^2 x}$ **f** $\sin x - 2\cos x + e^x$

2 Find:

a $\int \left(\sqrt{x} + \frac{1}{2}\cos x\right)\, dx$ **b** $\int (\theta - \sin\theta)\, d\theta$ **c** $\int \left(t\sqrt{t} + \dfrac{2}{\cos^2 t}\right) dt$

d $\int (2e^t - 4\sin t)\, dt$ **e** $\int \left(3\cos t - \dfrac{1}{t}\right) dt$ **f** $\int \left(3 - \dfrac{2}{\theta} + \dfrac{1}{\cos^2\theta}\right) d\theta$

Example 2

Find $\dfrac{d}{dx}(x\sin x)$ and hence deduce $\int x\cos x\, dx$.

$$\frac{d}{dx}(x\sin x) = (1)\sin x + (x)\cos x \qquad \{\text{product rule of differentiation}\}$$
$$= \sin x + x\cos x$$

Thus $\int (\sin x + x\cos x)dx = x\sin x + c_1$ {antidifferentiation}

$\therefore$ $\int \sin x\, dx + \int x\cos x\, dx = x\sin x + c_1$

$\therefore$ $-\cos x + c_2 + \int x\cos x\, dx = x\sin x + c_1$

$\therefore$ $\int x\cos x\, dx = x\sin x + \cos x + c$

Example 3

By considering $\dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)$, determine $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\sin x}\right) &= \frac{d}{dx}[\sin x]^{-1}\\ &= -[\sin x]^{-2} \times \frac{d}{dx}(\sin x)\\ &= -\frac{1}{\sin^2 x} \times \cos x\\ &= -\frac{\cos x}{\sin^2 x}\end{aligned}$$

Hence $\displaystyle\int -\frac{\cos x}{\sin^2 x}\, dx = \frac{1}{\sin x} + c_1$

$\therefore$ $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx = -\frac{1}{\sin x} - c_1$

3 **a** Find $\dfrac{d}{dx}(e^x \sin x)$ and hence find $\int e^x(\sin x + \cos x)dx$.

b By considering $\dfrac{d}{dx}(e^{-x}\sin x)$, deduce $\displaystyle\int \frac{\cos x - \sin x}{e^x}\, dx$.

c Find $\dfrac{d}{dx}(x\cos x)$ and hence deduce $\int x\sin x\, dx$.

d By considering $\dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$, determine $\displaystyle\int \frac{\tan x}{\cos x}\,dx$.

Example 4

Find $f(x)$ given that $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

$$\begin{aligned}
\text{Since} \quad f'(x) &= 2\sin x - \sqrt{x} \\
&= 2\sin x - x^{\frac{1}{2}} \\
\text{then} \quad f(x) &= \int \left[2\sin x - x^{\frac{1}{2}}\right] dx \\
\therefore \quad f(x) &= 2 \times (-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
\therefore \quad f(x) &= -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad f(0) &= -2\cos 0 - 0 + c \\
\therefore \quad 4 &= -2 + c \\
\therefore \quad 6 &= c
\end{aligned}$$

$$\text{Thus} \quad f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + 6.$$

4 Find $f(x)$ given that:

a $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

b $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

c $f'(x) = \sqrt{x} - \dfrac{2}{\cos^2 x}$ and $f(\pi) = 0$.

B INTEGRALS OF TRIGONOMETRIC FUNCTIONS OF THE FORM $f(ax+b)$

Observe the following: As $\dfrac{d}{dx}(\sin(ax+b)) = \cos(ax+b) \times a$,

$$\begin{aligned}
\text{then} \quad \int a\cos(ax+b)dx &= \sin(ax+b) + c_1 \\
\text{i.e.,} \quad a\int \cos(ax+b)dx &= \sin(ax+b) + c_1 \\
\therefore \quad \int \cos(ax+b)dx &= \frac{1}{a}\sin(ax+b) + c
\end{aligned}$$

Likewise we can show $\displaystyle\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$ and

$$\int \frac{1}{\cos^2(ax+b)}\,dx = \frac{1}{a}\tan(ax+b) + c$$

Summary:

Function	*Integral*
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$
$\dfrac{1}{\cos^2(ax+b)}$	$\frac{1}{a}\tan(ax+b)+c$

Earlier we showed that:

Function	*Integral*
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, \quad n\neq -1$
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$\dfrac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert+c$

Example 5

Integrate with respect to x:

a $e^{-2x}-\dfrac{4}{\cos^2(2x)}$

b $2\sin(3x)+\cos(4x)$

a

$$\int\left(e^{-2x}-\frac{4}{\cos^2(2x)}\right)dx$$

$$=\tfrac{1}{-2}\times e^{-2x}-4\times\tfrac{1}{2}\tan(2x)+c$$

$$-\tfrac{1}{2}e^{-2x}-2\tan(2x)+c$$

b

$$\int(2\sin(3x)+\cos(4x))\,dx$$

$$=2\times\tfrac{1}{3}(-\cos(3x))+\tfrac{1}{4}\sin(4x)+c$$

$$=-\tfrac{2}{3}\cos(3x)+\tfrac{1}{4}\sin(4x)+c$$

EXERCISE 27B

1 Integrate with respect to x:

a $\sin(3x)$

b $2\cos(4x)$

c $\dfrac{1}{\cos^2(2x)}$

d $3\cos\left(\dfrac{x}{2}\right)$

e $3\sin(2x)-e^{-x}$

f $e^{2x}-\dfrac{2}{\cos^2\left(\frac{x}{2}\right)}$

g $2\sin\left(2x+\frac{\pi}{6}\right)$

h $-3\cos\left(\frac{\pi}{4}-x\right)$

i $\dfrac{4}{\cos^2\left(\frac{\pi}{3}-2x\right)}$

j $\cos(2x)+\sin(2x)$

k $2\sin(3x)+5\cos(4x)$

l $\frac{1}{2}\cos(8x)-3\sin x$

Integrals involving $\sin^2(ax+b)$ and $\cos^2(ax+b)$ can be found by first using

$$\sin^2\theta=\tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta) \quad \text{or} \quad \cos^2\theta=\tfrac{1}{2}+\tfrac{1}{2}\cos(2\theta).$$

These formulae are simply rearrangements of $\cos(2\theta)$ formulae.

For example,

- $\sin^2(3x)$ becomes $\frac{1}{2}-\frac{1}{2}\cos(6x)$
- $\cos^2(\dfrac{x}{2})$ becomes $\frac{1}{2}+\frac{1}{2}\cos 2(\dfrac{x}{2})=\frac{1}{2}+\frac{1}{2}\cos x$

etc

Example 6

Integrate $(2-\sin x)^2$.

$$\begin{aligned}&\int (2-\sin x)^2 dx\\ &= \int (4 - 4\sin x + \sin^2 x) dx\\ &= \int \left(4 - 4\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) dx\\ &= \int \left(\tfrac{9}{2} - 4\sin x - \tfrac{1}{2}\cos(2x)\right) dx\\ &= \tfrac{9}{2}x + 4\cos x - \tfrac{1}{2} \times \tfrac{1}{2}\sin(2x) + c\\ &= \tfrac{9}{2}x + 4\cos x - \tfrac{1}{4}\sin(2x) + c\end{aligned}$$

2 Integrate with respect to x:

a $\cos^2 x$ **b** $\sin^2 x$ **c** $1+\cos^2(2x)$

d $3-\sin^2(3x)$ **e** $\frac{1}{2}\cos^2(4x)$ **f** $(1+\cos x)^2$

3 Use the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ to show that
$\cos^4 x = \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}$ and hence find $\int \cos^4 x\, dx$.

Example 7

Integrate with respect to x: **a** $\cos^3 x \sin x$ **b** $\dfrac{\cos x}{\sin x}$

a $\int \cos^3 x \sin x$

$= \int [\cos x]^3 \sin x\, dx$

We let $u = \cos x$, $\therefore \dfrac{du}{dx} = -\sin x$

$$\begin{aligned}&= \int u^3 \left(-\frac{du}{dx}\right) dx\\ &= -\int u^3\, du\\ &= -\frac{u^4}{4} + c\\ &= -\tfrac{1}{4}[\cos x]^4 + c\end{aligned}$$

b $\displaystyle\int \frac{\cos x}{\sin x} dx$

We let $u = \sin x$, $\therefore \dfrac{du}{dx} = \cos x$

$$\begin{aligned}&= \int \frac{1}{u}\frac{du}{dx} dx\\ &= \int \frac{1}{u} du\\ &= \ln|u| + c\\ &= \ln|\sin x| + c\end{aligned}$$

Note: In **b** if we let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$ and $\displaystyle\int \frac{\cos x}{\sin x} dx = \int \frac{u}{-\frac{du}{dx}} dx$.

However, this substitution leads nowhere as simplification to a form where the integration can be performed does not occur.

4 Integrate by substitution:

a $\sin^4 x \cos x$ **b** $\dfrac{\sin x}{\sqrt{\cos x}}$ **c** $\tan x$

d $\sqrt{\sin x}\cos x$ **e** $\dfrac{\cos x}{(2+\sin x)^2}$ **f** $\dfrac{\sin x}{\cos^3 x}$

g $\dfrac{\sin x}{1-\cos x}$ **h** $\dfrac{\cos(2x)}{\sin(2x)-3}$ **i** $\dfrac{\cos(3x)}{\sin(3x)}$

Example 8

Find $\int \sin^3 x\,dx$.

$$\begin{aligned}
&\int \sin^3 x\,dx\\
&= \int \sin^2 x\,\sin x\,dx\\
&= \int (1-\cos^2 x)\sin x\,dx \qquad \text{let } u=\cos x \quad \therefore \frac{du}{dx} = -\sin x\\
&= \int (1-u^2)\left(-\frac{du}{dx}\right)dx\\
&= \int (u^2-1)\,du\\
&= \frac{u^3}{3} - u + c\\
&= \tfrac{1}{3}\cos^3 x - \cos x + c
\end{aligned}$$

5 Find:

a $\int \cos^3 x\,dx$ **b** $\int \sin^5 x\,dx$ **c** $\int \sin^4 x\cos^3 x\,dx$

Hint: In **b** $\sin^5 x = \sin^4 x\sin x = (1-\cos^2 x)^2\sin x$, etc.

6 Find $f(x)$ if $f'(x)$ is:

a $\sin x\, e^{\cos x}$ **b** $\sin^3(2x)\cos(2x)$ **c** $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ **d** $\dfrac{e^{\tan x}}{\cos^2 x}$

C DEFINITE INTEGRALS

Recall that: the **Fundamental Theorem of Calculus** is:

If $F(x)$ is the integral of $f(x)$, then providing $f(x)$ is continuous on $a \leqslant x \leqslant b$,

$$\int_a^b f(x)dx = F(b) - F(a).$$

Notes:
- $F(b)-F(a)$ is usually denoted by $[F(x)]_a^b$.
- $\int_a^b f(x)\,dx$ reads "the integral from a to b of $f(x)$ with respect to x".
- The function $f(x)$ is called the **integrand**.
- a and b are called the **lower limit** and **upper limit** of the integral, respectively.

- $\int f(x)dx$ is said to be an **indefinite integral** whereas
  $\int_a^b f(x)dx$ is said to be a **definite integral**.

PROPERTIES OF DEFINITE INTEGRALS

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx$ for $a < m < b$.

Example 9

Evaluate: **a** $\int_0^{\frac{\pi}{3}} \sin x\,dx$ **b** $\int_0^{\frac{\pi}{8}} \frac{1}{\cos^2(2x)}$

a

$$\int_0^{\frac{\pi}{3}} \sin x\,dx$$
$$= [-\cos x]_0^{\frac{\pi}{3}}$$
$$= \left(-\cos\tfrac{\pi}{3}\right) - (-\cos 0)$$
$$= -\tfrac{1}{2} + 1$$
$$= \tfrac{1}{2}$$

```
fnInt(sin(X),X,0
,π/3)
                .50000
```

b

$$\int_0^{\frac{\pi}{8}} \frac{1}{\cos^2(2x)}\,dx$$
$$= \left[\tfrac{1}{2}\tan(2x)\right]_0^{\frac{\pi}{8}}$$
$$= \left(\tfrac{1}{2}\tan\tfrac{\pi}{4}\right) - \left(\tfrac{1}{2}\tan 0\right)$$
$$= \tfrac{1}{2}\times 1 - \tfrac{1}{2}\times 0$$
$$= \tfrac{1}{2}$$

EXERCISE 27C

1 Evaluate:

a $\int_0^{\frac{\pi}{6}} \cos x\,dx$ **b** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx$

d $\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$ **e** $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$ **f** $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

Example 10

Evaluate: **a** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{8}} \tan 2x\,dx$

a $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx$

$\displaystyle= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{u}\,\frac{du}{dx}\,dx$

$\displaystyle= \int_{\frac{1}{2}}^{1} u^{\frac{1}{2}}\,du$

$\displaystyle= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{2}}^{1}$

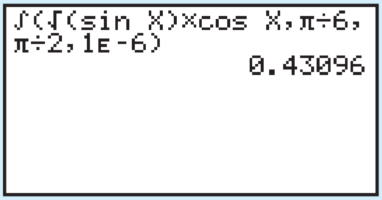
$= \frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}$

$= \frac{2}{3} - \frac{1}{3\sqrt{2}}$

Let $u = \sin x$ $\therefore$ $\dfrac{du}{dx} = \cos x$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

when $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$



b $\displaystyle\int_0^{\frac{\pi}{8}} \tan(2x)\,dx$

$\displaystyle= \int_0^{\frac{\pi}{8}} \frac{\sin(2x)}{\cos(2x)}\,dx$

$\displaystyle= \int_0^{\frac{\pi}{8}} \frac{1}{u}\left(\frac{1}{-2}\frac{du}{dx}\right)dx$

$\displaystyle= -\frac{1}{2}\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du$

$= -\frac{1}{2}\left[\ln|u|\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\frac{1}{2}\left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right)$

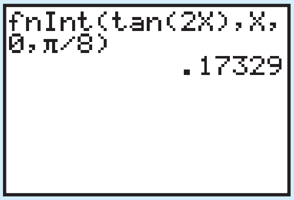
$= -\frac{1}{2}(\ln 2^{-\frac{1}{2}} - 0)$ $\{\ln 1 = 0\}$

$= \frac{1}{4}\ln 2$ $\{\ln a^n = n\ln a\}$

Let $u = \cos(2x)$ $\therefore$ $\dfrac{du}{dx} = -2\sin(2x)$

when $x = \frac{\pi}{8}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

when $x = 0$, $u = \cos 0 = 1$



2 Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 x\cos x\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\,dx$

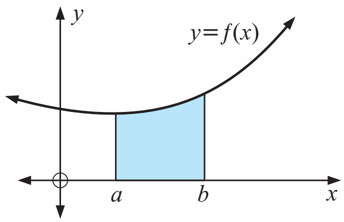
d $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x}\,dx$ **e** $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin x}\,dx$

D AREA DETERMINATION

Reminders:

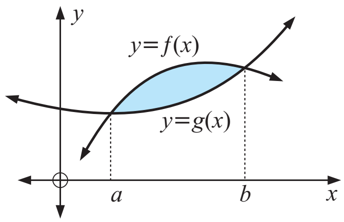
- If $f(x)$ is **positive** and **continuous** on the interval $a \leqslant x \leqslant b$ then the shaded area is given by

$$\int_a^b f(x)\,dx.$$

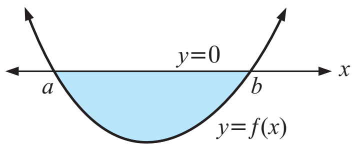


- If $f(x)$ and $g(x)$ are **continuous** functions on the interval $a \leqslant x \leqslant b$ then if $f(x) \geqslant g(x)$ on this interval the shaded area is given by

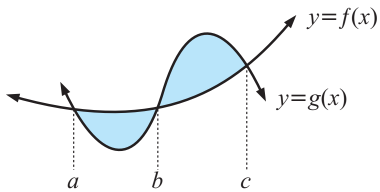
$$\int_a^b [f(x) - g(x)]\,dx.$$



Examples:



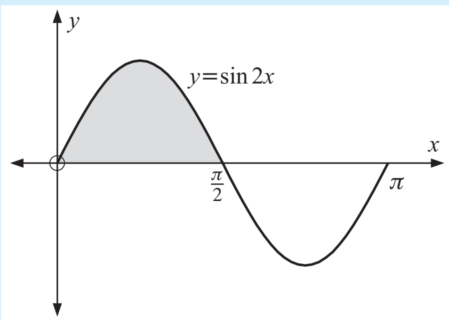
$$\text{Area} = \int_a^b [0 - f(x)]\,dx$$
$$= -\int_a^b f(x)\,dx$$



$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx$$

Example 11

Find the area enclosed by one arch of $y = \sin(2x)$.



period is $\frac{2\pi}{2} = \pi$

$\therefore$ first positive x-intercept is $\frac{\pi}{2}$

required area

$$= \int_0^{\frac{\pi}{2}} \sin(2x)\,dx$$
$$= \left[\tfrac{1}{2}(-\cos(2x))\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\left[\cos(2x)\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}(\cos\pi - \cos 0)$$
$$= 1 \text{ unit}^2$$

EXERCISE 27D

1 **a** Show that the area enclosed by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ is 2 units^2.

b Find the area enclosed by $y = \sin^2 x$ and the x-axis from $x = 0$ to $x = \pi$.

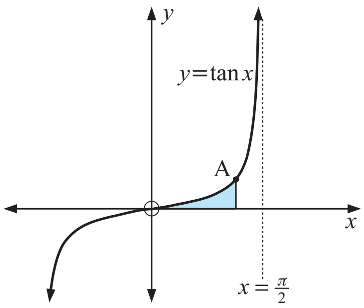
2 A region has boundaries defined by $y = \sin x$, $y = \cos x$ and the y-axis. Find the area of the region.

3 The graph alongside shows a small portion of the graph of $y = \tan x$.

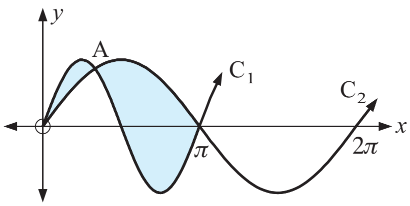
A is a point on the graph with a y-coordinate of 1.

a Find the coordinates of A.

b Find the shaded area.



4



The illustrated curves are those of $y = \sin x$ and $y = \sin(2x)$.

a Identify each curve.

b Find algebraically the coordinates of A.

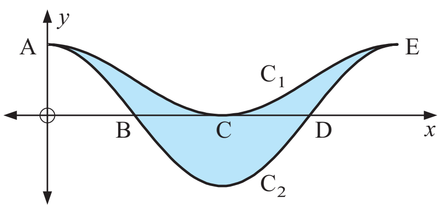
c Find the total area enclosed by C_1 and C_2 for $0 \leqslant x \leqslant \pi$.

5 The illustrated curves are those of $y = \cos(2x)$ and $y = \cos^2 x$.

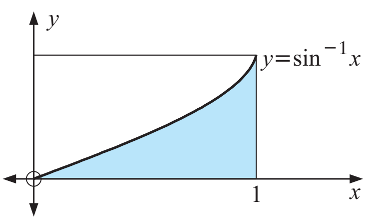
a Identify each curve.

b Determine the coordinates of A, B, C, D and E.

c Show that the area of the shaded region is $\frac{\pi}{2}$ units^2.



6



Find $\displaystyle\int_0^1 \sin^{-1} x \, dx$.

[**Hint:** $y = \sin^{-1} x$ is the inverse function of $y = \sin x$, i.e., its reflection in the line $y = x$.]

REVIEW SET 27A

1 Integrate with respect to x:

a $\sin^7 x \cos x$ **b** $\tan(2x)$ **c** $e^{\sin x} \cos x$

2 Find the derivative of $x \tan x$ and hence determine $\displaystyle\int \frac{x}{\cos^2 x} dx$.

3 Determine the area enclosed by $y = \dfrac{2x}{\pi}$ and $y = \sin x$.

4 Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx$ **b** $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx$

5 Differentiate $\ln \sec x$, given that $\sec x > 0$.
What integral can be deduced from this derivative?

6 Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} d\theta$

7 Determine the area of the region enclosed by $y = x$, $y = \sin x$ and $x = \pi$.

REVIEW SET 27B

1 Integrate:

a $4\sin^2\left(\dfrac{x}{2}\right)$ **b** $(2 - \cos x)^2$

2 Find $\displaystyle\int \frac{\sin x}{\cos^4 x} dx$

3 Differentiate $\sin(x^2)$ and hence find $\int x\cos(x^2) dx$.

4 Find the area of the region enclosed by $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$.

5 A particle moves in a straight line with velocity given by $v(t) = \sin t$ metres per second. Find the total distance travelled by the particle in the first 4 seconds of motion. [Give your answer correct to 2 decimal places.]

6 Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right) dx$ **b** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x \tan x} dx$

7 Without actually integrating $\sin^3 x$, prove that $\int_0^\pi \sin^3 x \, dx < 4$.
[**Hint:** Graph $y = \sin^3 x$ for $0 \leqslant x \leqslant \pi$.]

Chapter 28

Volumes of revolution

Contents:

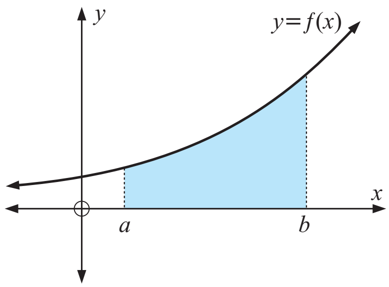
A Solids of revolution

B Volumes for two defining functions

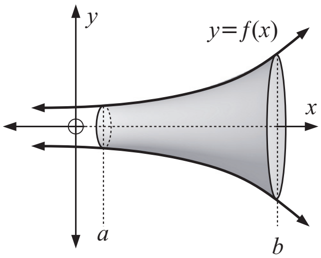
Review set 28

A SOLIDS OF REVOLUTION

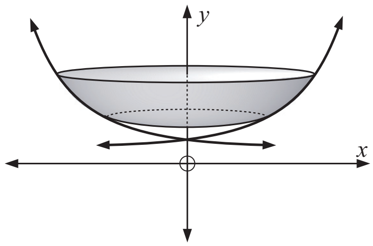
Consider the curve $y = f(x)$ for $a \leqslant x \leqslant b$. If the shaded part is rotated about the x-axis a 3-dimensional solid will be formed.



This solid is called a **solid of revolution**.



Similarly if the part of the curve is rotated about the y-axis a solid of revolution will also be formed.



In this chapter we will mainly be concerned with volumes of solids formed by a **revolution about the x-axis**.

VOLUME OF REVOLUTION

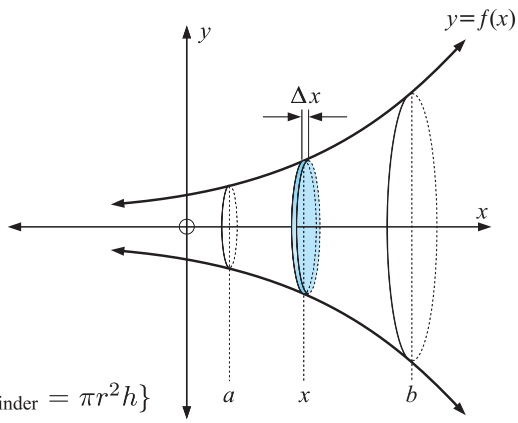
We can use integration to find volumes of revolution between $x = a$ and $x = b$.

The solid can be thought to be made up of an infinite number of thin cylindrical discs.

The left-most disc has approximate volume $\pi[f(a)]^2\Delta x$.

The right-most disc has approximate volume $\pi[f(b)]^2\Delta x$.

$\{V_{\text{cylinder}} = \pi r^2 h\}$

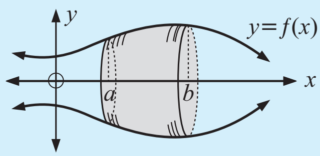


In general, $\pi[f(x)]^2\Delta x$ is the approximate volume for the illustrated disc.

As there are infinitely many discs, $\Delta x \to 0$ and

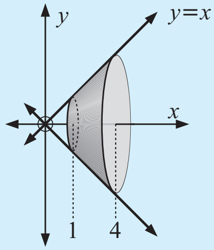
$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \pi[f(x)]^2\Delta x = \int_a^b \pi[f(x)]^2\,dx$$

When the region enclosed by $y = f(x)$, the x-axis and the vertical lines $x = a$, $x = b$ is rotated about the x-axis to generate a solid, the volume of the solid is given by $\pi \int_a^b [f(x)]^2\,dx$.



Example 1

Use integration to find the volume of the solid generated when the line $y = x$ for $1 \leqslant x \leqslant 4$ is revolved around the x-axis.



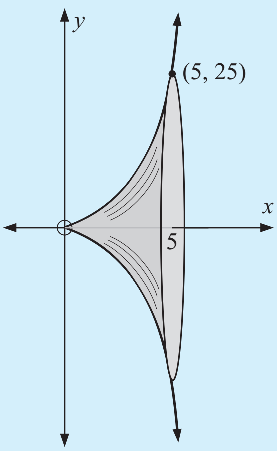
$$\begin{aligned}\text{Volume of revolution} &= \pi \int_a^b [f(x)]^2\, dx \\ &= \pi \int_1^4 x^2\, dx \\ &= \pi \left[\frac{x^3}{3}\right]_1^4 \\ &= \pi\left(\tfrac{64}{3} - \tfrac{1}{3}\right) \\ &= 21\pi \quad \text{cubic units}\end{aligned}$$

Note: The volume of a cone can be calculated using $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$

So, in **Example 1**,
$$\begin{aligned} V &= \tfrac{1}{3}\pi 4^2(4) - \tfrac{1}{3}\pi 1^2(1) \\ &= \tfrac{64\pi}{3} - \tfrac{\pi}{3} \\ &= 21\pi \quad \text{which checks} \quad \checkmark\end{aligned}$$

Example 2

Find the volume of the solid formed when the graph of the function $y = x^2$ for $0 \leqslant x \leqslant 5$ is revolved about the x-axis.



Volume of revolution
$$\begin{aligned} &= \pi \int_a^b [f(x)]^2\, dx \\ &= \pi \int_0^5 (x^2)^2\, dx \\ &= \pi \int_0^5 x^4\, dx \\ &= \pi \left[\frac{x^5}{5}\right]_0^5 \\ &= \pi(625 - 0) \\ &= 625\pi \quad \text{cubic units}\end{aligned}$$

Note:
enter $Y_1 = X^2$
$fnInt(\pi{*}Y_1^2,\, X,\, 0,\, 5)$
gives this volume.

TI
C

EXERCISE 28A.1

1 Find the volume of the solid formed when the following are revolved about the x-axis:

a $y = 2x$ for $0 \leqslant x \leqslant 3$

b $y = \sqrt{x}$ for $0 \leqslant x \leqslant 4$

c $y = x^3$ for $1 \leqslant x \leqslant 2$

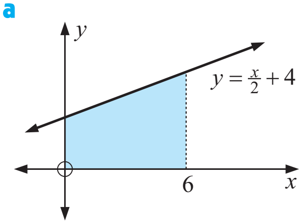
d $y = x^{\frac{3}{2}}$ for $1 \leqslant x \leqslant 4$

e $y = x^2$ for $2 \leqslant x \leqslant 4$

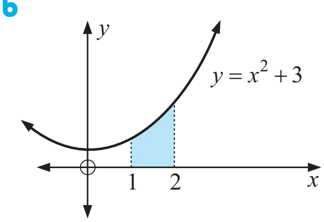
f $y = \sqrt{25 - x^2}$ for $0 \leqslant x \leqslant 5$

2 Find the volume of revolution when the shaded region is revolved about the x-axis.

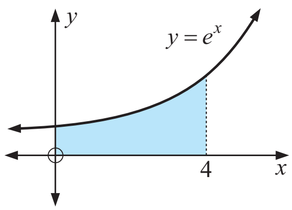
a



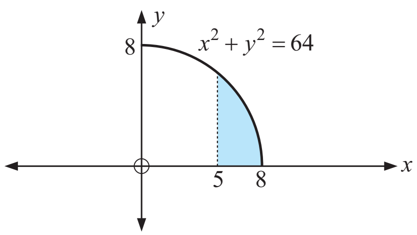
b



c



3



The shaded region is rotated about the x-axis.

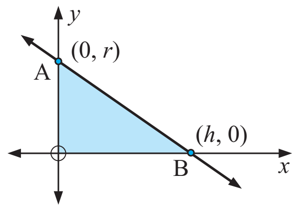
a Find the volume of revolution.

b A hemispherical bowl of radius 8 cm contains water to a depth of 3 cm. What is the volume of water?

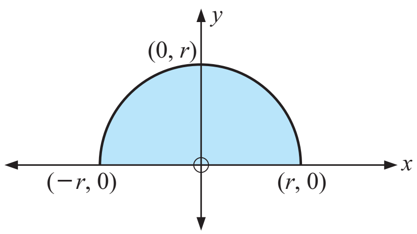
4 **a** What is the name of the solid of revolution when the shaded region is revolved about the x-axis?

b Find in the form $y = ax + b$, the equation of the line segment AB.

c Find a formula for the volume of the solid using $\pi \int_a^b [f(x)]^2\, dx$.



5

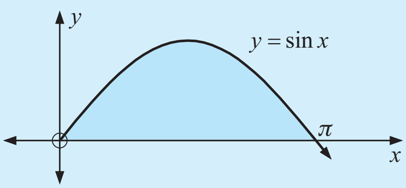


A circle, centre $(0, 0)$ and radius r units has equation $x^2 + y^2 = r^2$.

a If the shaded region is revolved about the x-axis, what solid is formed?

b Use integration to show that the volume of revolution is $\frac{4}{3}\pi r^3$.

Example 3



One arch of $y = \sin x$ is rotated about the x-axis.

What is the volume of revolution?

$$\begin{aligned}\text{Volume} &= \pi\int_a^b [f(x)]^2\,dx\\ &= \pi\int_0^\pi \sin^2 x\,dx\\ &= \pi\int_0^\pi \left[\tfrac{1}{2}-\tfrac{1}{2}\cos(2x)\right]\,dx\\ &= \pi\left[\tfrac{x}{2}-\tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^\pi\\ &= \pi\left[\left(\tfrac{\pi}{2}-\tfrac{1}{4}\sin(2\pi)\right)-\left(0-\tfrac{1}{4}\sin 0\right)\right]\\ &= \pi\times\tfrac{\pi}{2}\\ &= \frac{\pi^2}{2}\quad\text{cubic units}\end{aligned}$$

EXERCISE 28A.2

1 Find the volume of revolution when these regions are rotated about the x-axis:

a $y=\cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $y=\cos(2x)$ for $0 \leqslant x \leqslant \frac{\pi}{4}$

c $y=\sqrt{\sin x}$ for $0 \leqslant x \leqslant \pi$ **d** $y=\dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant \frac{\pi}{3}$

2 **a** Sketch the graph of $y=\sin x+\cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$

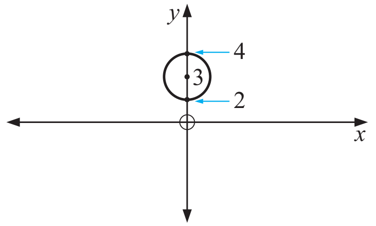
b Hence, find the volume of revolution of the shape bounded by $y=\sin x+\cos x$, the x-axis, $x=0$ and $x=\frac{\pi}{4}$ when it is rotated about the x-axis.

3 **a** Sketch the graph of $y=4\sin(2x)$ from $x=0$ to $x=\frac{\pi}{4}$

b Hence, find the volume of the revolution of the shape bounded by $y=4\sin(2x)$, the x-axis, $x=0$ and $x=\frac{\pi}{4}$ when it is rotated about the x-axis.

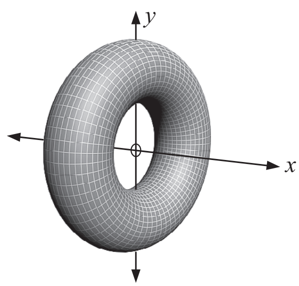
B VOLUMES FOR TWO DEFINING FUNCTIONS

Consider the circle, centre (0, 3), radius 1 unit.



What solid of revolution will be obtained if this circle is revolved about the x-axis?

Yes, a doughnut is formed! (*Torus* is its proper mathematical name). We can use integration to find its volume.

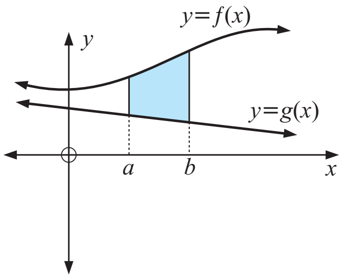


In general, if the region bounded by $y = f(x)$ (on top) and $y = g(x)$ and the lines $x = a$, $x = b$ is revolved about the x-axis, then its volume of revolution is given by:

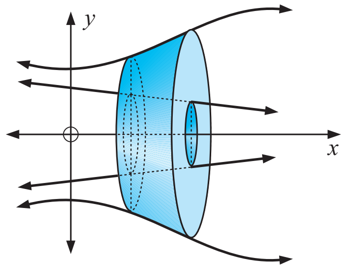
$$V = \pi \int_a^b [f(x)]^2\,dx - \pi \int_a^b [g(x)]^2\,dx$$

So, $$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)\,dx$$

Rotating

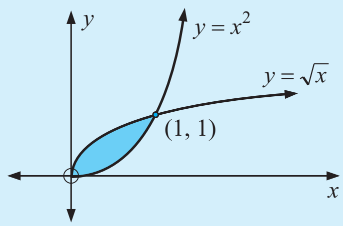


about the x-axis gives



Example 4

Find the volume of revolution generated by revolving the region between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.



The required volume $= \pi \displaystyle\int_0^1 \left([f(x)]^2 - [g(x)]^2\right)\,dx$

$= \pi \displaystyle\int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right)\,dx$

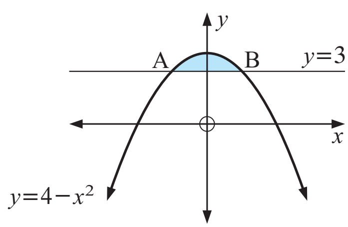
$= \pi \displaystyle\int_0^1 (x - x^4)\,dx$

$= \pi \left[\dfrac{x^2}{2} - \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\left(\frac{1}{2} - \frac{1}{5}\right) - (0)\right)$

$= \frac{3\pi}{10}$

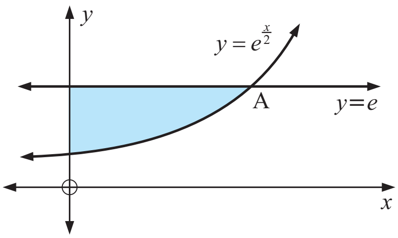
EXERCISE 28B



1 The shaded region (between $y = 4 - x^2$ and $y = 3$) is revolved about the x-axis.

  a What are the coordinates of A and B?

  b Find the volume of revolution.

2 

The shaded region is revolved about the x-axis.

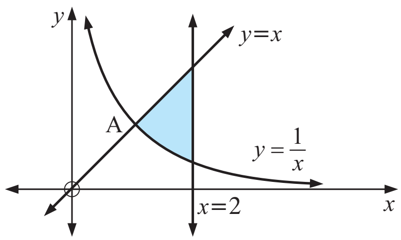
a Find the coordinates of A.

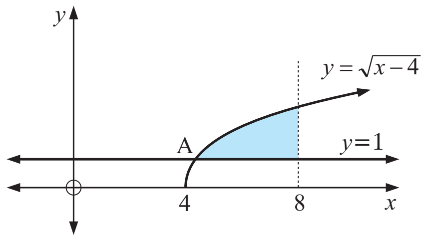
b Find the volume of revolution.

3 The shaded region (between $y = x$, $y = \dfrac{1}{x}$ and $x = 2$) is revolved about the x-axis.

a What are the coordinates of A?

b Find the volume of revolution.



4 

The shaded region (between $y = \sqrt{x-4}$, $y = 1$ and $x = 8$) is revolved about the x-axis.

a What are the coordinates of A?

b Find the volume of revolution.

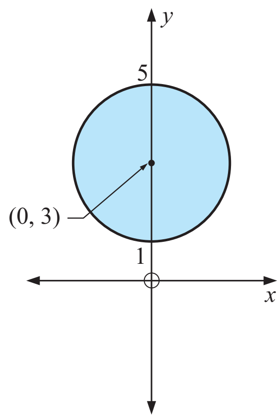
5 The illustrated circle has equation $x^2 + (y-3)^2 = 4$.

a Show that $y = 3 \pm \sqrt{4 - x^2}$.

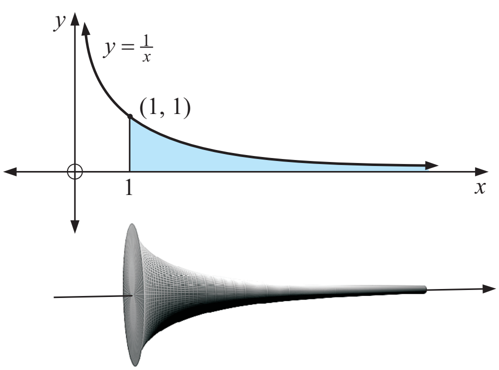
b Draw a diagram and show on it what part of the circle is represented by $y = 3 + \sqrt{4-x^2}$ and what part by $y = 3 - \sqrt{4-x^2}$.

c Find the volume of revolution of the shaded region about the x-axis.

Hint: Use your calculator to evaluate the integral.



A REMARKABLE FACT



The shaded area from $x = 1$ to infinity is **infinite** whereas its volume of revolution is **finite**.

You can use integration to prove this fact. Try it!

REVIEW SET 28

1 Find the volume of the solid of revolution when the following are revolved about the x-axis:

a $y = x$ between $x = 4$ and $x = 10$

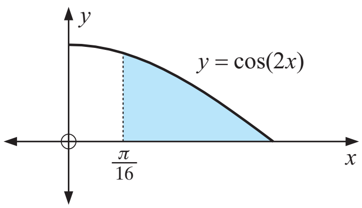
b $y = x + 1$ between $x = 4$ and $x = 10$

c $y = \sin x$ between $x = 0$ and $x = \pi$

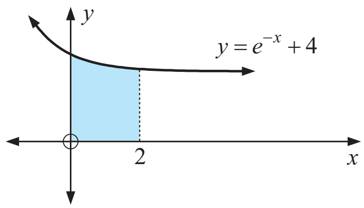
d $y = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$.

2 Find the volume of the solid of revolution when the shaded region is revolved about the x-axis:

a

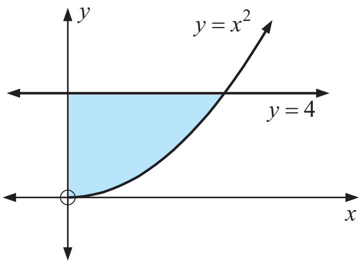


b

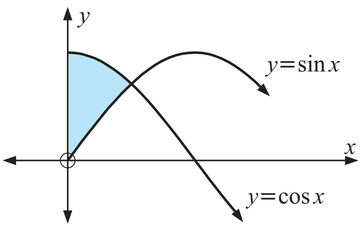


3 Find the volume of revolution when $y = \dfrac{1}{\sin x}$ is revolved about the x-axis for $\frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4}$.

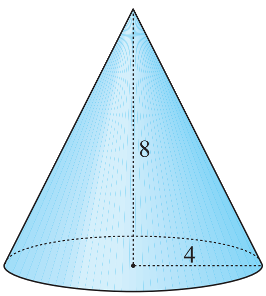
4 Find the volume of revolution generated by revolving the shaded region about the x-axis:



5 Find the volume of revolution if the shaded region is revolved about the x-axis:



6



a Use $V = \frac{1}{3}\pi r^2 h$ to find the volume of this cone.

b Check your answer to **a** by using integration.

Chapter 29

Statistical distributions

Contents:

A Discrete random variables
B Discrete probability distributions
C Expectation
Investigation 1: Concealed number tickets
D The mean and standard deviation of a discrete random variable
E The binomial distribution
F Mean and standard deviation of a binomial random variable
Investigation 2: The mean and standard deviation of a binomial random variable
G Normal distributions
Investigation 3: Standard deviation significance
Investigation 4: Mean and standard deviation of $z = \frac{x - \overline{x}}{s}$
H The standard normal distribution (z distribution)
I Applications of the normal distribution

Review sets 29A, B, C

A DISCRETE RANDOM VARIABLES

RANDOM VARIABLES

In previous work we have described events mainly using words. It is far more convenient where possible, to use numbers.

A **random variable** represents in number form the possible outcomes, which could occur for some random experiment.

Note: A **discrete random variable** X has possible values $x_1, x_2, x_3, ...$

For example,
- the number of houses in your suburb which have a 'power safety switch'
- the number of new bicycles sold each year by a bicycle store
- the number of defective light bulbs in the purchase order of a city store.

A **continuous random variable** X has all possible values in some interval (on the number line).

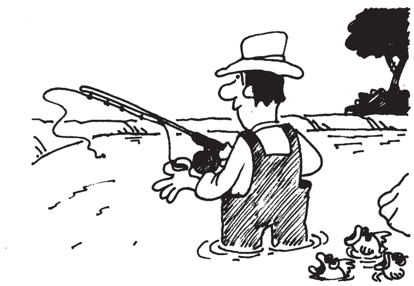
For example,
- the heights of men could all lie in the interval $50 < x < 250$ cm
- the volume of water in a rainwater tank during a given month.

Note: A discrete random variable involves a *count* whereas a continuous random variable involves *measurements*.

EXERCISE 29A

1 Classify the following random variables as continuous or discrete.

a The quantity of fat in a lamb chop.
b The mark out of 50 for a Geography test.
c The weight of a seventeen year old student.
d The volume of water in a cup of coffee.
e The number of trout in a lake.
f The number of hairs on a cat.
g The length of hairs on a horse.
h The height of a sky-scraper.



2 For each of the following:

i identify the random variable being considered
ii give possible values for the random variable
iii indicate whether the variable is continuous or discrete.

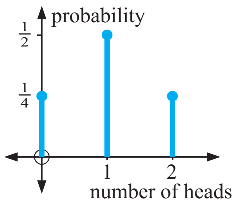
a To measure the rainfall over a 24-hour period in Singapore, the height of water collected in a rain gauge (up to 200 mm) is used.
b To investigate the stopping distance for a tyre with a new tread pattern a braking experiment is carried out.
c To check the reliability of a new type of light switch, switches are repeatedly turned off and on until they fail.

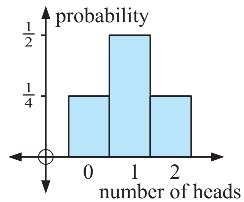
3 A supermarket has four checkouts A, B, C and D. Management checks the weighing devices at each checkout. If a weighing device is accurate a yes (Y) is recorded; otherwise, no (N) is recorded. The random variable being considered is the number of weighing devices which are accurate.

- **a** Suppose X is the random variable. What values can x have?
- **b** Tabulate the possible outcomes and the corresponding values for x.
- **c** Describe, using x, the events of:
  - **i** 2 being accurate
  - **ii** at least two being accurate.

For any random variable there is a **probability distribution** associated with it.

For example, when tossing two coins, the random variable could be 0 heads, 1 head or 2 heads, i.e., $x = 0$, 1 or 2. The associated probability distribution being $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{4}$ and graph:





Note: We will write $\mathrm{P}(X = x)$ as $P(x)$ or p_x.

Example 1

A supermarket has three checkout points A, B and C. Consumer affairs checks for accuracy of weighing scales at each checkout. If a weighing scale is accurate yes (Y) is recorded, and if not, no (N). Suppose the random variable is:

X is the number of accurate weighing scales at the supermarket.

- **a** List the possible outcomes.
- **b** Describe using x the events of there being:
  - **i** one accurate scale
  - **ii** at least one accurate scale.

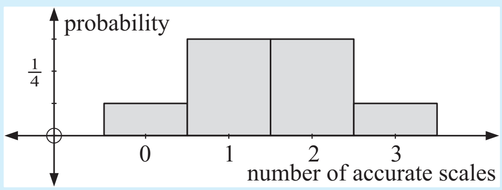
a Possible outcomes:

A	B	C	x
N	N	N	0
Y	N	N	1
N	Y	N	1
N	N	Y	1
N	Y	Y	2
Y	N	Y	2
Y	Y	N	2
Y	Y	Y	3

b **i** $x = 1$

ii $x = 1$, 2 or 3

Notice:



4 Consider tossing three coins simultaneously where the random variable under consideration is the number of heads that could result.

- **a** List the possible values of x.
- **b** Tabulate the possible outcomes and the corresponding values of x.
- **c** Find the values of $\mathrm{P}(X = x)$, the probability of each x value occurring.
- **d** Graph the probability distribution $\mathrm{P}(X = x)$ against x as a probability histogram.

B DISCRETE PROBABILITY DISTRIBUTIONS

For each random variable there is a **probability distribution**. The probability of any given event

p_i lies between 0 and 1 (inclusive), i.e., $0 \leqslant p_i \leqslant 1$

and $\sum_{i=1}^{n} p_i = 1$ i.e., $p_1 + p_2 + p_3 + ... + p_n = 1$.

The **probability distribution** of a **discrete random variable** can be given

- in table form
- in graphical form
- in functional form

and provides us with all possible values of the variable and the probability of the occurrence of each value.

Example 2

A magazine store recorded the number of magazines purchased by its customers in one day. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four and 5% purchased five.

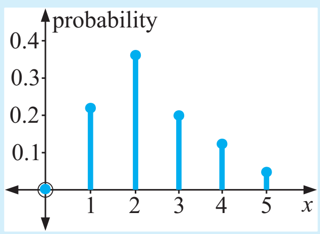
a What is the random variable?
b Make a random variable probability table.
c Graph the probability distribution.

a The random variable X is the number of magazines sold.
So $x = 0, 1, 2, 3, 4$ or 5.

b

x_i	0	1	2	3	4	5
p_i	0.00	0.23	0.38	0.21	0.13	0.05

c



EXERCISE 29B

1 Jason's home run probabilities per game of baseball during his great career are given in the following table where X is the number of home runs per game:

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a What is the value of a, i.e., $P(0)$? Explain what this figure means in real terms.
b What is the value of $P(2)$?
c What is the value of $P(1) + P(2) + P(3) + P(4) + P(5)$? Explain what this represents.
d Draw a probability distribution spike graph of $P(x)$ against x.

2 Explain why the following are not valid probability distributions:

a

x	0	1	2	3
$P(x)$	0.2	0.3	0.4	0.2

b

x	2	3	4	5
$P(x)$	0.3	0.4	0.5	-0.2

3 Sally's number of hits each softball match has the following probability distribution:

x	0	1	2	3	4	5
$P(x)$	0.07	0.14	k	0.46	0.08	0.02

a State clearly what the random variable represents.

b Find k.

c Find **i** $P(x \geqslant 2)$ **ii** $P(1 \leqslant x \leqslant 3)$

4 A die is rolled twice.

a Draw a grid which shows the sample space.

b Suppose X denotes the sum of the dots uppermost on the die for the two rolls and find the probability distribution of X.

c Draw a probability distribution histogram for this situation.

Consider the following example where a probability distribution is given in *functional form*.

Example 3

Show that the following are probability distribution functions:

a $P(x) = \dfrac{x^2+1}{34}$ for $x = 1, 2, 3, 4$

b $P(x) = C_x^3\,(0.6)^x(0.4)^{3-x}$ for $x = 0, 1, 2, 3$

a $P(1) = \frac{2}{34}$ $P(2) = \frac{5}{34}$ $P(3) = \frac{10}{34}$ $P(4) = \frac{17}{34}$

all of which lie in $0 \leqslant P(i) \leqslant 1$ and $\sum P(i) = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = 1$

$\therefore$ $P(x)$ is a probability distribution function.

b For $P(x) = C_x^3\,(0.6)^x(0.4)^{3-x}$

$$\begin{aligned}
P(0) &= C_0^3\,(0.6)^0(0.4)^3 && = 1 \times 1 \times (0.4)^3 && = 0.064\\
P(1) &= C_1^3\,(0.6)^1(0.4)^2 && = 3 \times (0.6) \times (0.4)^2 && = 0.288\\
P(2) &= C_2^3\,(0.6)^2(0.4)^1 && = 3 \times (0.6)^2 \times (0.4) && = 0.432\\
P(3) &= C_3^3\,(0.6)^3(0.4)^0 && = 1 \times (0.6)^3 \times 1 && = \underline{0.216}\\
& && \text{Total} && 1.000
\end{aligned}$$

Since all probabilities lie between 0 and 1 and $\sum P(x_i) = 1$ then $P(x)$ is a probability distribution function.

5 Find k for the following probability distributions:

a $P(x) = k(x+2)$ for $x = 1,\ 2,\ 3$ **b** $P(x) = \dfrac{k}{x+1}$ for $x = 0,\ 1,\ 2,\ 3$

6 A discrete random variable X has probability distribution given by:

$P(x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$ where $x = 0,\ 1,\ 2,\ 3,\ 4.$

a Find $P(x)$ for $x = 0,\ 1,\ 2,\ 3$ and 4. **b** Find k and hence find $P(x \geqslant 2)$.

7 Electrical components are produced and packed into boxes of 10. It is known that 4% of the components may be faulty. The random variable X denotes the number of faulty items in the box and has a probability distribution

$P(x) = C_x^{10}\,(0.04)^x\,(0.96)^{10-x},\ \ x = 0,\ 1,\ 2,\ ....,\ 10.$

a Find the probability that a randomly selected box will contain no faulty component.

b Find the probability that a randomly selected box will contain at least one faulty component.

C EXPECTATION

Consider the following problem:

A die is to be rolled 120 times. On how many occasions would you expect the result to be a "six"?

In order to answer this question we must first consider all the possible outcomes of rolling the die. The possibilities are 1, 2, 3, 4, 5 and 6 and each of these is equally likely to occur.

Therefore, we would expect $\frac{1}{6}$ of them to be a "six" and $\frac{1}{6}$ of 120 is 20.

That is, we expect 20 of the 120 rolls of the die to yield a "six".

In general: If there are n members of a sample and the probability of an event occurring is p for each member, then the **expectation** of the occurrence of that event is $n \times p$.

Example 4

Each time a footballer kicks for goal he has a $\frac{3}{4}$ chance of being successful. If, in a particular game, he has 12 kicks for goal, how many goals would you expect him to kick?

$p = \text{P(goal)} = \frac{3}{4}$

For a sample space of $n = 12$, the expected

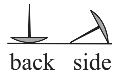
number of goals is $n \times p = 12 \times \frac{3}{4} = 9$

EXERCISE 29C

1 In a particular region, the probability that it will rain on any one day is 0.28. On how many days of the year would you expect it to rain?

2 **a** If 3 coins are tossed what is the chance that they all fall heads?
b If the 3 coins are tossed 200 times, on how many occasions would you expect them all to fall heads?

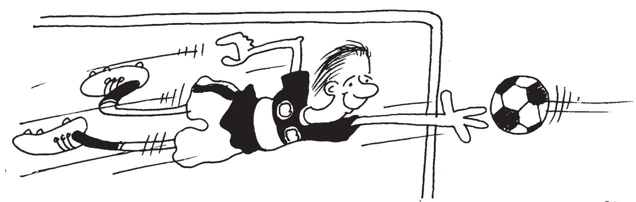
3 A certain type of drawing pin, when tossed 325 times, landed on its back 125 times.
a Estimate the probability that it will land on its back if it is tossed once.
b If the pin was tossed 50 times, how many "backs" would you expect?



4 A charity fundraiser gets a licence to run the following gambling game: A die is rolled and the returns to the player are given in the 'pay table' alongside. To play the game $4 is needed. A result of getting a 6 wins $10, so in fact you are ahead by $6 if you get a 6 on the first roll.

Result	*Wins*
6	$10
4, 5	$4
1, 2, 3	$1

a What are your chances of playing one game and winning:
i $10 **ii** $4 **iii** $1?
b Your expected return from throwing a 6 is $\frac{1}{6} \times \$10$. What is your expected return from throwing:
i a 4 or 5 **ii** a 1, 2 or 3 **iii** a 1, 2, 3, 4, 5 or 6?
c What is your expected result at the end of one game?
d What is your expected result at the end of 100 games?

5 

A soccer goalkeeper has a probability of $\frac{3}{10}$ of saving a penalty attempt. How many goals would he expect to save out of 90 penalty shots?

6 During the snow season there is a $\frac{3}{7}$ probability of snow falling on any particular day. If Udo skis for five weeks, on how many days could he expect to see snow falling?

7 If two dice are rolled simultaneously 180 times, on how many occasions would you expect to get a double?

8 A hat contains three yellow discs and four green discs. Two discs are drawn simultaneously from the hat. If these discs are then returned to the hat and the procedure is repeated 300 times, on how many occasions would you expect two green discs to be drawn?

9 In a random survey of her electorate, a politician (A) discovered the residents' voting intentions in relation to herself and her two opponents B and C. The results are indicated alongside:

A	B	C
165	87	48

a Estimate the probability that a randomly chosen voter in the electorate will vote for:
i A **ii** B **iii** C.

b If there are 7500 people in the electorate, how many of these would you expect to vote for: **i** A **ii** B **iii** C?

Example 5

In a game of chance, the player spins a square spinner labelled 1, 2, 3, 4, and wins the amount of money shown in the table alongside depending on which number comes up. Determine:

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a the expected return for one spin of the spinner

b whether you would recommend a person to play this game if it costs \$5 to play one game.

a As each number is equally likely, the probability for each number is $\frac{1}{4}$

$\therefore$ expected return $= \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4$.

b As the expected return is \$4 whereas it costs \$5 to play the game, you would not recommend that a person play the game.

EXPECTATION BY FORMULAE

For examples like **Example 5** part **a** we can define the expectation $E(x)$ of a random variable to be

$$E(X) = \sum_{i=1}^{n} p_i x_i$$

where

the x_i represents particular outcomes $(x_1 = \$1,\ x_2 = \$2,$ etc.)

p_i represents the probability of x_i occurring

X is the random variable we are concerned about (in this case the 'return').

10 A person rolls a normal six-sided die and wins the number of dollars shown on the face.

a How much does the person expect to win for one roll of the die?

b If it costs \$4 to play the game, would you advise the person to play several games?

11 A person plays a game with a pair of coins. If a double head is spun, \$10 is won. If a head and a tail appear, \$3 is won. If a double tail appears \$5 is lost.

a How much would a person expect to win playing this game once?

b If the organiser of the game is allowed to make an average of \$1 per game, how much should be charged to play the game once?

12 A single coin is tossed once. If a head appears you win \$2 and if a tail appears you lose \$1. How much would you expect to win when playing this game three times?

13 Boban has developed a new gambling game in which you roll a die and the following payouts are made: 1 - \$1, 2 - \$2, 3 - \$3, 4 - \$5, 5 - \$10, 6 - \$25.

- **a** If you play one game, what is your expected return?
- **b** If you play 100 games paying \$10 to play each game, how much would you expect to win/lose?

14 A die has 3 red, 2 blue and one white face. When the die is rolled a red result wins \$1, a blue result \$2 and a white result \$5. What is the "expected return" for one roll of this die?

INVESTIGATION 1 CONCEALED NUMBER TICKETS

Many clubs have ticket machines which contain sets of consecutive numbers from 0001, 0002, 0003, ... up to 2000. These 'games' are relatively inexpensive to play and are used as fund-raisers for the club (e.g., football club, golf club, etc).

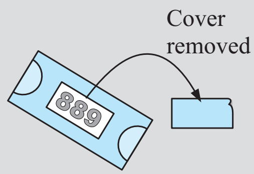
Tickets are ejected at random at a cost of 20 cents each. A small cardboard cover is removed to reveal the concealed number.

Suppose a golf club can buy golf balls as prizes for \$2.50 each and a set of 2000 tickets for the machine at \$30. One club shows the following winners table:

Winning Numbers	Prize
777	4 golf balls
1000, 2000	2 golf balls
any multiple of 25	1 golf ball

What to do:

1. If all tickets are sold, how many balls are paid out as prizes?
2. Determine the total cost to the club for a complete round of 2000 tickets going through the machine.
3. What percentage profit is made by the club?
4. If you purchase one ticket, what is your chance of winning at least one ball?



D THE MEAN AND STANDARD DEVIATION OF A DISCRETE RANDOM VARIABLE

Consider the table alongside:

x_i are the possible values of the random variable X, and f_i are the corresponding frequencies.

x_i	x_1	x_2	x_3	x_4	x_5	$\cdots$	x_n
f_i	f_1	f_2	f_3	f_4	f_5	$\cdots$	f_n

We define the **population mean** as $\mu = \dfrac{\sum f_i x_i}{\sum f_i}$ and

the **population standard deviation** as $\sigma = \sqrt{\dfrac{\sum f_i(x_i - \mu)^2}{\sum f_i}}$.

Suppose we have 10 counters, one with a 1 written on it, four with 2 written on them, three with a 3 and two with a 4. One counter is to be randomly selected from a hat.

If the random variable is the number on a counter then it has

possible values, x_i	1	2	3	4
with frequencies, f_i	1	4	3	2
and probabilities, p_i	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

the population mean $\mu = \dfrac{\sum f_i x_i}{\sum f_i}$ {mean for tabled values}

$$\therefore \quad \mu = \frac{1\times 1 \ + \ 2\times 4 \ + \ 3\times 3 \ + \ 4\times 2}{1+4+3+2}$$

$$\therefore \quad \mu = \frac{1\times 1 \ + \ 2\times 4 \ + \ 3\times 3 \ + \ 4\times 2}{10}$$

$$\therefore \quad \mu = 1\times \tfrac{1}{10} \ + \ 2\times \tfrac{4}{10} \ + \ 3\times \tfrac{3}{10} \ + \ 4\times \tfrac{2}{10}$$

This suggests that, $\mu = \sum x_i p_i$.

Likewise $\sigma^2 = \dfrac{\sum f_i(x_i-\mu)^2}{\sum f_i}$

$$\therefore \quad \sigma^2 = \frac{1(x_1-\mu)^2}{10} + \frac{4(x_2-\mu)^2}{10} + \frac{3(x_3-\mu)^2}{10} + \frac{2(x_4-\mu)^2}{10}$$

$$\therefore \quad \sigma^2 = \tfrac{1}{10}(x_1-\mu)^2 \ + \ \tfrac{4}{10}(x_2-\mu)^2 \ + \ \tfrac{3}{10}(x_3-\mu)^2 \ + \ \tfrac{2}{10}(x_4-\mu)^2$$

So, $\sigma^2 = \sum (x_i-\mu)^2 p_i$

Now let us consider the **general case**.

For a random variable having possible values $x_1, x_2, x_3, \ldots\ldots, x_n$
with frequencies $f_1, f_2, f_3, \ldots\ldots, f_n$:

the population mean $\mu = \dfrac{\sum f_i x_i}{\sum f_i}$ {mean for tabled values}

$$= \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_nx_n}{N} \qquad \{\text{if } \textstyle\sum f_i = N, \text{ say}\}$$

$$= x_1\left(\frac{f_1}{N}\right) + x_2\left(\frac{f_2}{N}\right) + x_3\left(\frac{f_3}{N}\right) + \ldots\ldots + x_n\left(\frac{f_n}{N}\right)$$

$$= x_1p_1 + x_2p_2 + x_3p_3 + \ldots\ldots + x_np_n$$

$$= \sum x_i p_i$$

Likewise $\sigma^2 = \dfrac{\sum f_i(x_i-\mu)^2}{\sum f_i}$

$$= \frac{f_1(x_1-\mu)^2}{N} + \frac{f_2(x_2-\mu)^2}{N} + \frac{f_3(x_3-\mu)^2}{N} + \ldots\ldots + \frac{f_n(x_n-\mu)^2}{N}$$

$$= p_1(x_1-\mu)^2 + p_2(x_2-\mu)^2 + p_3(x_3-\mu)^2 + \ldots\ldots + p_n(x_n-\mu)^2$$

So, $\sigma^2 = \sum (x_i-\mu)^2 p_i$

We observe that:

If a discrete random variable has possible values $x_1, x_2, x_3, \ldots\ldots, x_n$
with probabilities $p_1, p_2, p_3, \ldots\ldots, p_n$ of occurring,

then
- the population **mean** is $\mu = \sum x_i p_i$ and
- the population **standard deviation** is $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$.

  Note:

  We can also show that $\sigma = \sqrt{\sum x_i^2 p_i - \mu^2}$

  This formula is easier to use than the first one.

Note that for a die:

$$\mu = \sum x_i p_i = 1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + 5(\tfrac{1}{6}) + 6(\tfrac{1}{6}) = 3.5$$

and
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - \mu^2 \\ &= 1^2(\tfrac{1}{6}) + 2^2(\tfrac{1}{6}) + 3^2(\tfrac{1}{6}) + 4^2(\tfrac{1}{6}) + 5^2(\tfrac{1}{6}) + 6^2(\tfrac{1}{6}) - (3.5)^2 \\ &= 2.91666.....\end{aligned}$$

Consequently, $\sigma \doteqdot 1.708$ and this can be checked from your calculator using RandInt(1, 6, 800) or something similar to obtain 800 random digits from 1 to 6. Then find $\overline{x}$ and σ_x. You should get a good approximation to the theoretical values obtained above.

Example 6

Find the mean and standard deviation of the magazine store random variable of **Example 2**.

The probability table is:

x_i	0	1	2	3	4	5
p_i	0.00	0.23	0.38	0.21	0.13	0.05

Now
$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 0(0.00) + (0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05) \\ &= 2.39\end{aligned}$$

i.e., in the long run, the average number of magazines purchased per customer is 2.39 .

and
$$\begin{aligned}\sigma &= \sqrt{(x_i - \mu)^2 p_i} \\ &= \sqrt{(1-2.39)^2 \times 0.23 + (2-2.39)^2 \times 0.38 + \ldots\ldots + (5-2.39)^2 \times 0.05} \\ &\doteqdot 1.122\end{aligned}$$

Note: The mean of a random variable is often referred to as '*the expected value of* x' or sometimes as '*the long run average value of* x'.

Example 7

'Cheap Car Insurance' insures used cars valued at \$6000 under these conditions.

A \$6000 will be paid to the owner for total loss
B for damage between \$3000 and \$5999, \$3500 will be paid
C for damage between \$1500 and \$2999, \$1000 will be paid
D for damage less than \$1500, nothing will be paid.

From statistical information the insurance company knows that in any year the probabilities of **A**, **B**, **C** and **D** are 0.03, 0.12, 0.35 and 0.50 respectively.
If the company wishes to receive \$80 more on each policy than its expected payout, what should it charge for the policy?

Let x be the payout values, then $x = 6000, 3500, 1000$ or 0.

The probability distribution is:

x_i	0	1000	3500	6000
p_i	0.50	0.35	0.12	0.03

The expectation, $\mu = \sum x_i p_i$
$= 0(0.50) + 1000(0.35) + 3500(0.12) + 6000(0.03)$
$= 950$

i.e., the company expects to payout \$950 on average in the long run.
So, the company should charge \$950 + \$80 = \$1030.

EXERCISE 29D

1 New Zealand crayfish are exported to Asian markets. The buyers are prepared to pay high prices when the crayfish arrive still alive. If X is the number of deaths per dozen crayfish, the probability distribution for X is given by:

x_i	0	1	2	3	4	5	> 5
$P(x_i)$	0.54	0.26	0.15	0.03	0.01	0.01	0.00

a Over a long period, what is the mean number of deaths per dozen crayfish?

b Find σ, the standard deviation for the probability distribution.

2 A random variable X has probability distribution given by

$P(x) = \dfrac{x^2 + x}{20}$ for $x = 1, 2, 3$. Calculate μ and σ for this distribution.

3 A random variable X has probability distribution given by

$$P(x) = C_x^3 \, (0.4)^x (0.6)^{3-x} \quad \text{for} \quad x = 0, 1, 2, 3.$$

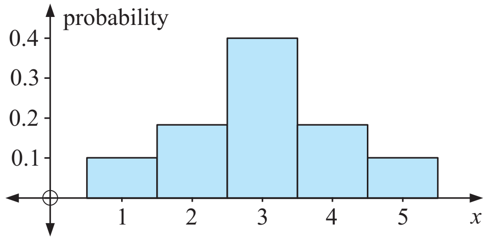
a Find $P(x)$ for $x = 0, 1, 2$ and 3 and display the results in table form.

b Find the mean and standard deviation for the distribution.

4 Using $\sigma^2 = \sum (x_i - \mu)^2 p_i$ show that $\sigma^2 = \sum x_i^2 p_i - \mu^2$.

(Hint: $\sigma^2 = \sum (x_i - \mu)^2 p_i = (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \ldots\ldots + (x_n - \mu)^2 p_n$.
Expand and regroup the terms.)

5 A random variable X has probability distribution shown alongside.



a Copy and complete:

x_i	1	2	3	4	5
$P(x_i)$					

b Find the mean μ and standard deviation of the X-distribution.

c Determine **i** $P(\mu - \sigma < x < \mu + \sigma)$ **ii** $P(\mu - 2\sigma < x < \mu + 2\sigma)$

6 An insurance policy covers a \$20 000 sapphire ring against theft and loss. If it is stolen the insurance company will pay the policy owner in full. If it is lost they will pay the owner \$8000. From past experience the insurance company knows that the probability of theft is 0.0025 and of being lost is 0.03. How much should the company charge to cover the ring if they want a \$100 expected return?

7 A pair of dice is rolled and the random variable is M, the larger of the two numbers that are shown uppermost.

a In table form obtain the probability distribution of M.

b Find the mean and standard deviation of the M-distribution.

8 A fair coin is tossed (at most) three times until either one tail or three heads occurs. Let X be the number of tosses needed and find μ_x, the mean of the X-distribution.

E THE BINOMIAL DISTRIBUTION

In the previous section we considered the properties of discrete random variables.

We now examine a special discrete random variable which is applied to **sampling with replacement**. The probability distribution associated with this variable is known as the **binomial probability distribution**.

For **sampling without replacement** the **hypergeometric probability distribution** is the model used but is not appropriate in this course.

BINOMIAL EXPERIMENTS

When we have a repetition of a number of **independent trials** in which there are two possible results, **success** (the event occurs) or **failure** (the event does not occur), we have a **binomial experiment**.

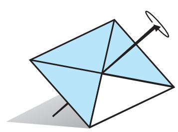
The probability of a success p, must be constant for all trials.

Let q be the probability of a failure. So, $q = 1 - p$ (since $p + q = 1$).

The random variable X is the total number of successes in n trials.

THE BINOMIAL PROBABILITY DISTRIBUTION

Suppose a spinner has three blue edges and one white edge. Then, on each occasion it is spun, we will get a blue or a white.



The chance of finishing on blue is $\frac{3}{4}$ and on white is $\frac{1}{4}$.

If p is the probability of getting a blue, and q is the probability of getting a white then
$p = \frac{3}{4}$ and $q = \frac{1}{4}$ (the chance of failing to get a blue).

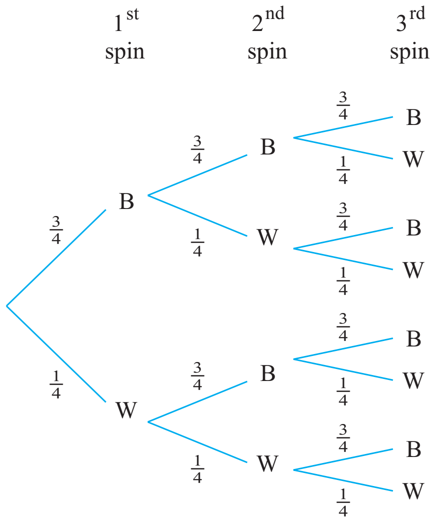
Consider twirling the spinner $n = 3$ times. Let the random variable X be the number of blue results which could occur. Then $x = 0,\ 1,\ 2$ or 3.

Now $P(0) = \text{P(all 3 are white)}$
$= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$
$= \left(\frac{1}{4}\right)^3$

$P(1) = \text{P(1 blue and 2 white)}$
$= \text{P(BWW or WBW or WWB)}$
$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \times 3$ .....(1)

$P(2) = \text{P(2 blue and 1 white)}$
$= \text{P(BBW or BWB or WBB)}$
$= \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \times 3$

$P(3) = \text{P(3 blues)}$
$= \left(\frac{3}{4}\right)^3$



The coloured factor 3 in (1) is the number of ways of getting one success in three trials, which is combination C^3_1 (sometimes written as $\binom{3}{1}$).

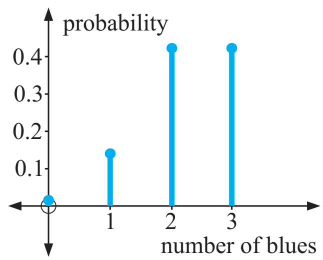
We note that

$P(0) = \left(\frac{1}{4}\right)^3 = C^3_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^3 \doteqdot 0.0156$

$P(1) = 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = C^3_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^2 \doteqdot 0.1406$

$P(2) = 3\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = C^3_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^1 \doteqdot 0.4219$

$P(3) = \left(\frac{3}{4}\right)^3 = C^3_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^0 \doteqdot 0.4219$



This suggests that: $P(x) = C^3_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{3-x}$ where $x = 0,\ 1,\ 2,\ 3$

or $= \binom{3}{x}\left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{3-x}$

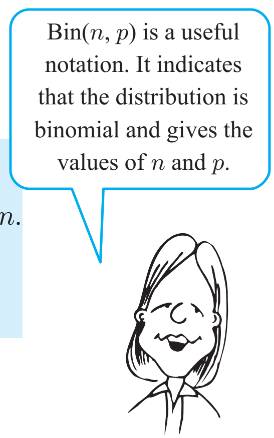
In general:

In the case of n trials where there are r *successes* and $n - r$ *failures*

$P(X = r) = C^n_r\, p^r q^{n-r}$ where $q = 1 - p$ and $r = 0, 1, 2, 3, 4, ..., n$.

p is the probability of a *success* and q is the probability of a *failure*.

$P(X = r)$ is the **binomial probability distribution function**.



Note:
- A binomial variable is often specified in the form Bin(n, p).
- $C^n_x = \binom{n}{x}$

Example 8

72% of union members are in favour of a certain change to their conditions of employment. A random sample of five members is taken. Find the probability that:

a three members are in favour of the change in conditions

b at least three members are in favour of the changed conditions.

Let X denote the number of members in favour of the changes, then as $n = 5$,

$r = 0, 1, 2, 3, 4$ or 5 and $p = 72\% = 0.72$

r is distributed as Bin(5, 0.72).

a $P(x = 3) = \text{binompdf}(5, 0.72, 3)$
$\doteqdot 0.2926$

b $\text{P}(x \geqslant 3) = 1 - \text{P}(x \leqslant 2)$
$= 1 - \text{binomcdf}(5, 0.72, 2)$
$\doteqdot 0.8623$

For binomial distributions:

- the probability distribution is discrete
- there are two outcomes, which we usually call *success* and *failure*
- the trials are independent, so the probability of success for a particular trial is not affected by the success or failure of previous trials. In other words, the probability of success is a constant for the experiment being considered.

EXERCISE 29E

1 For which of these probability experiments does the binomial distribution apply? Justify your answers, using a full sentence.

a A coin is thrown 100 times. The variable is the number of heads.

b One hundred coins are each thrown once. The variable is the number of heads.

c A box contains 5 blue and 3 red marbles. I draw out 5 marbles, replacing the marble each time. The variable is the number of red marbles drawn.

d A box contains 5 blue and 3 red marbles. I draw out 5 marbles. I do not replace the marbles that are drawn. The variable is the number of red marbles drawn.

e A large bin contains ten thousand bolts, 1% of which are faulty. I draw a sample of 10 bolts from the bin. The variable is the number of faulty bolts.

2 Assuming that the births of boys and girls are equally likely, calculate the probability that in a family of six children:

a all the children are boys

b there are exactly 2 boys

c there are more than 4 girls

d there are more boys than girls.

3 At a manufacturing plant 35% of the employees worked night-shift. If 7 employees were selected at random, find the probability that:

a exactly 3 of them worked night-shift

b less than 4 of them worked night-shift

c at least 4 of them worked night-shift.

4 There is a 5% chance that any apple in a bin of apples will have a blemish. If a sample of 25 apples is taken, find the probability that:

a exactly 2 of these have blemishes **b** at least one has a blemish.

5 Records show that 6% of the items assembled on a production line are faulty. A sample of 12 items is selected at random. Find the probability that:

a none will be faulty **b** at most one will be faulty
c at least two will be faulty **d** less than 4 will be faulty.

6 A student guesses true-false answers at random for a test of 20 questions. Find the probability that this student is correct in:

a all 20 questions **b** exactly half the questions
c less than half the questions **d** at least 15 questions.

7 The local bus service does not have a good reputation. It is known that the 8 am bus will run late on an average of two days out of every five. For any week of the year taken at random, find the probability of the 8 am bus being on time:

a all 7 days **b** only on Monday
c on any 6 days **d** on at least 4 days.

8 An infectious flu virus is spreading through a school. The probability of a randomly selected student having the flu next week is 0.3.

a Calculate the probability that out of a class of 25 students, 2 or more will have the flu next week.

b If more than 20% of the students are away with the flu next week, a class test will have to be cancelled. What is the probability that the test will be cancelled?

9 During a season, a basketball player is found to have a 94% success rate in shooting goals from the free throw line. In one match the basketballer throws from the free throw line 20 times. Find the probability that the basketballer is:

a successful on all 20 throws **b** successful on at least 18 throws.

F MEAN AND STANDARD DEVIATION OF A BINOMIAL RANDOM VARIABLE

We toss a coin $n = 20$ times; as the probability of it falling '*a head*' is $p = \frac{1}{2}$, we expect it to fall '*heads*' $np = 10$ times.

Likewise, if we roll a die $n = 30$ times, as the probability of it falling 'a 4' is $p = \frac{1}{6}$, we expect it to obtain 'a 4' on $np = 5$ occasions.

So, in the general case:

> If a binomial experiment is repeated n times and a particular variable has probability p of occurring each time, then our expectation is that the mean μ will be $\mu = np$.

Finding the standard deviation is not so simple. Consider the following theoretical approach after which we verify the generalised result by simulation.

ONE TRIAL $(n=1)$

In the case of $n = 1$ where p is the probability of success and q is the probability of failure, the number of successes x could be 0 or 1.

The table of probabilities is

x_i	0	1
p_i	q	p

Now $\mu = \sum p_i x_i$
$= q(0) + p(1)$
$= p$

and $\sigma^2 = \sum x_i^2 p_i - \mu^2$
$= [(0)^2 q + (1)^2 p] - p^2$
$= p - p^2$
$= p(1-p)$
$= pq \quad \{\text{as} \quad q = 1 - p\}$

$\therefore \quad \sigma = \sqrt{pq}$

TWO TRIALS $(n=2)$

In the case where $n = 2$,
$$\left.\begin{aligned} P(0) &= C_0^2\, p^0 q^2 = q^2 \\ P(1) &= C_1^2\, p^1 q^1 = 2pq \\ P(2) &= C_2^2\, p^2 q^0 = p^2 \end{aligned}\right\} \quad \text{as} \quad x = 0,\ 1 \text{ or } 2$$

So, the table of probabilities is

x_i	0	1	2
p_i	q^2	$2pq$	p^2

Now

$\mu = \sum p_i x_i$
$= q^2(0) + 2pq(1) + p^2(2)$
$= 2pq + 2p^2$
$= 2p(q+p)$
$= 2p \quad \{\text{as} \quad p + q = 1\}$

and $\sigma^2 = \sum x_i^2 p_i - \mu^2$
$= [0^2 \times q^2 \ + \ 1^2 \times 2pq \ + \ 2^2 \times p^2] - (2p)^2$
$= 2pq + 4p^2 - 4p^2$
$= 2pq$

$\therefore \quad \sigma = \sqrt{2pq}$

The case $n = 3$ produces $\mu = 3p$ and $\sigma = \sqrt{3pq}$ {in the following exercise}.

The case $n = 4$ produces $\mu = 4p$ and $\sigma = \sqrt{4pq}$.

These results suggest that in general:

> If x is a random variable which is binomial with parameters n and p i.e., $\text{Bin}(n,\ p)$ then the mean of x is $\mu = np$ and the standard deviation of x is $\sigma = \sqrt{npq}$.

A general proof of this statement is beyond the scope of this course. However, the following investigation should help you appreciate the truth of the statement.

INVESTIGATION 2 THE MEAN AND STANDARD DEVIATION OF A BINOMIAL RANDOM VARIABLE

In this investigation we will examine binomial distributions randomly generated by the *sorting simulation* that you may have already used.

SIMULATION STATISTICS PACKAGE

What to do:

1 Obtain experimental binomial distribution results for 1000 repetitions

a $n = 4$, $p = 0.5$ **b** $n = 5$, $p = 0.6$ **c** $n = 6$, $p = 0.75$

2 For each of the distributions obtained in **1** find the mean μ and standard deviation σ from the statistics pack.

3 Use $\mu = np$ and $\sigma = \sqrt{npq}$ to see how your experimental values for μ and σ in **2** agree with the theoretical expectation.

4 Finally, comment on the shape of a binomial distribution for changes in the value of p. Go back to the sorting simulation and for $n = 50$, say, try $p = 0.2, 0.35, 0.5, 0.68, 0.85$.

Example 9

A fair die is rolled twelve times and x is the number of sixes that could result. Find the mean and standard deviation of the x-distribution.

This is a binomial situation with $n = 12$ and $p = \frac{1}{6}$ i.e., x is Bin$(12, \frac{1}{6})$.

So, $\mu = np = 12 \times \frac{1}{6} = 2$ and $\sigma = \sqrt{npq} = \sqrt{12 \times \frac{1}{6} \times \frac{5}{6}} = \sqrt{\frac{60}{36}} \doteqdot 1.291$

This means that we expect a six to be rolled 2 times, with standard deviation 1.291 .

Example 10

5% of a batch of batteries are defective. A random sample of 80 batteries is taken with replacement. Find the mean and standard deviation of the number of defectives in the sample.

This is a binomial sampling situation with $n = 80$, $p = 5\% = \frac{1}{20}$.

If X is the random variable for the number of defectives then X is Bin$(80, \frac{1}{20})$.

So, $\mu = np = 80 \times \frac{1}{20} = 4$ and $\sigma = \sqrt{npq} = \sqrt{80 \times \frac{1}{20} \times \frac{19}{20}} \doteqdot 1.949$

This means that we expect a defective battery 4 times, with standard deviation 1.949.

EXERCISE 29F

1 Suppose x is Bin(6, p). For each of the following cases:

   i find the mean and standard deviation of the X-distribution
   ii graph the distribution using a histogram
   iii comment on the shape of the distribution

   a when $p = 0.5$ **b** when $p = 0.2$ **c** when $p = 0.8$

2 A coin is tossed 10 times and X is the number of heads which occur. Find the mean and standard deviation of the X-distribution.

3 Suppose X is Bin(3, p).

   a Find $P(0)$, $P(1)$, $P(2)$ and $P(3)$ using
   $$P(x) = C_x^3\, p^x\, q^{3-x}$$
   and display your results in a table:

x_i	0	1	2	3
p_i				

   b Show that $\mu = 3p$ by using $\mu = \sum p_i x_i$.
   c By using $\sigma^2 = \sum x_i^2 p_i - \mu^2$, show that $\sigma = \sqrt{3pq}$.

4 Bolts produced by a machine vary in quality. The probability that a given bolt is defective is 0.04. A random sample of 30 bolts is taken from the week's production. If X denotes the number of defectives in the sample, find the mean and standard deviation of the X-distribution.

5 From data over the last twenty years it is known that when a car collides with a large animal such as a cow or horse or deer, the chance of death of people in the car is approximately 0.037. There have been 243 such collisions. Find the mean and standard deviation of the number of fatalities.

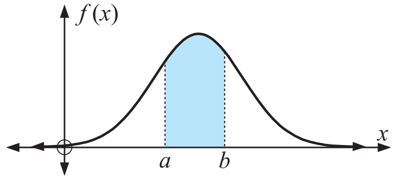
6 A city restaurant knows that 13% of reservations are not honoured, i.e., the group does not come. Suppose the restaurant receives five reservations and X is the random variable on the number of groups that do not come. Find the mean and standard deviation of the X-distribution.

G NORMAL DISTRIBUTIONS

In previous sections we have looked at discrete random variables and have examined some probability distributions where the random variable X could take non-negative integer values i.e., $x = 0, 1, 2, 3, 4, ...$

However, for a **continuous random variable** X, x can take any real value.

Consequently, a function is used to specify the probability distribution for a continuous random variable and that function is called the **probability density function**.



Probabilities are found by finding areas under the probability density function.

i.e., $\text{P}(a \leqslant x \leqslant b) = \int_a^b f(x)\,dx$ where $f(x)$ is the probability density function of random variable X.

THE NORMAL DISTRIBUTION

The normal distribution is the most important distribution for a continuous random variable.

The normal distribution lies at the heart of statistics. Many naturally occurring phenomena have a distribution that is normal, or approximately normal.

Some examples are:

- the chest sizes of English males
- the distribution of errors in many manufacturing processes
- the lengths of adult female tiger sharks
- the length of cilia on a cell
- scores on tests taken by a large population
- repeated measurements of the same quantity
- yields of corn, wheat, etc.

If X is **normally distributed** then its **probability density function** is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty.$$

This probability density function represents a **family of bell-shaped curves**. These curves are all symmetrical about the vertical line $x = \mu$. Each member of the family is specified by the **parameters** μ (the mean) and σ (the standard deviation).

Note on parameters and statistics

A **parameter** is a numerical characteristic of a *population*.

A **statistic** is a numerical characteristic of a *sample*.

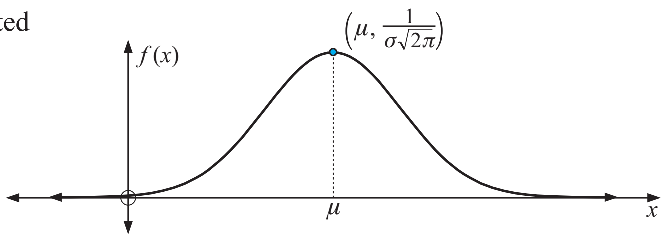
Note: **P**arameter **P**opulation, **S**ample **S**tatistic

For example, if we are examining the mean age of people in retirement villages throughout Australia the mean age found would be a *parameter*.

If we take a random sample of 300 people from the population of all retirement village persons, then the mean age would be a *statistic*.

A typical **normal curve** is illustrated alongside.

Notice that $f(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}}$.



We say that: 'x is distributed normally with mean μ and standard deviation σ'.

CHARACTERISTICS OF THE NORMAL PROBABILITY DENSITY FUNCTION

- The curve is symmetrical about the vertical line $x = \mu$.
- As $x \to \infty$ and as $x \to -\infty$ the normal curve approaches the x-axis.
- The area under the curve is one unit^2, and so $\int_{-\infty}^{\infty} f(x)dx = 1$. Why is this?

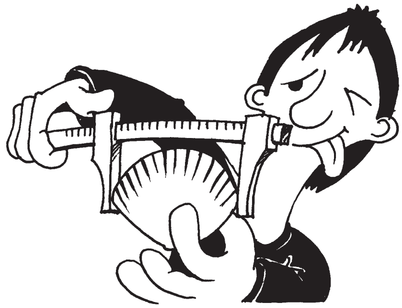
A TYPICAL NORMAL DISTRIBUTION

A large sample of cockle shells were collected and the maximum distance across each shell was measured. Click on the video clip icon to see how a histogram of the data is built up.

Now click on the demo icon to observe the effect of changing the class interval lengths for normally distributed data.

DEMO

VIDEO CLIP



THE GEOMETRICAL SIGNIFICANCE OF μ AND σ

Consider $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ and differentiate it with respect to x.

$$f'(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\times-\tfrac{1}{2}\times 2\left(\frac{x-\mu}{\sigma}\right)^1\times\frac{1}{\sigma}$$

$$=\frac{-1}{\sigma^2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\times\left(\frac{x-\mu}{\sigma}\right)$$

So $f'(x)=0$ only when $\left(\dfrac{x-\mu}{\sigma}\right)=0$ i.e., when $x=\mu$.

This result is as expected because at $x=\mu$, $f(x)$ is a maximum.

Now
$$f''(x)=-\frac{1}{\sigma^2\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\times-\left(\frac{x-\mu}{\sigma^2}\right)\times\left(\frac{x-\mu}{\sigma}\right)+e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\times\frac{1}{\sigma}\right]$$

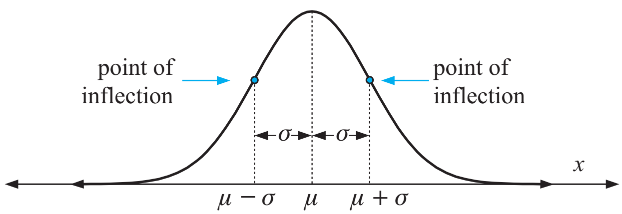
$$=\frac{-1}{\sigma^2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[\frac{1}{\sigma}-\frac{(x-\mu)^2}{\sigma^3}\right]$$

So $f''(x)=0$ when $\dfrac{(x-\mu)^2}{\sigma^3}=\dfrac{1}{\sigma}$, i.e., $(x-\mu)^2=\sigma^2$

i.e., $x-\mu=\pm\sigma$

$x=\mu\pm\sigma$

So at the points of inflection,
$x=\mu+\sigma$ and $x=\mu-\sigma$.



Consequently:

For a given normal curve the standard deviation is uniquely determined as the horizontal distance from the vertical line $x=\mu$ to a point of inflection.

HOW THE NORMAL DISTRIBUTION ARISES

Example 1:

Consider the oranges stripped from an orange tree. They do not all have the same weight. This variation may be due to several factors which could include:

- different genetic factors
- different times when the flowers were fertilised
- different amounts of sunlight reaching the leaves and fruit
- different weather conditions (some may be affected by the prevailing winds more than others), etc.

The result is that much of the fruit could have weights centred about e.g., a mean weight of 214 grams and there are far fewer oranges that are much heavier or lighter.

Invariably, a bell-shaped distribution of weights would be observed and the normal distribution model fits the data fairly closely.

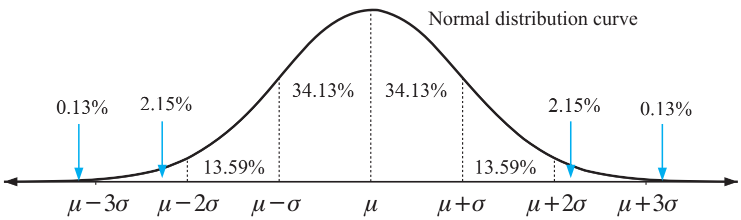
Example 2:

In manufacturing nails of a given length, say 50 mm, the machines produce nails of average length 50 mm but there is minor variation due to random errors in the manufacturing process. A small standard deviation of 0.3 mm, say, may be observed, but once again a bell-shaped distribution models the situation.

Once a normal model has been established we can use it to make predictions about the distribution and to answer other relevant questions.

THE SIGNIFICANCE OF THE STANDARD DEVIATION σ

For a normal distribution with mean μ and standard deviation σ the proportional breakdown of where the random variable could lie is given alongside.



Notice that:

- Approximately $2 \times 34.13\% = 68.26\%$ of the values lie within one standard deviation of the mean.
- Approximately $2 \times (34.13\% + 13.59\%) = 95.44\%$ of the values lie within two standard deviations of the mean.
- Approximately 99.74% of the values lie within three standard deviations of the mean.

INVESTIGATION 3 STANDARD DEVIATION SIGNIFICANCE

The purpose of this investigation is to check whether normal distributions have about

DEMO

- 68.3% of their values between $\overline{x} - s$ and $\overline{x} + s$
- 95.4% of their values between $\overline{x} - 2s$ and $\overline{x} + 2s$
- 99.7% of their values between $\overline{x} - 3s$ and $\overline{x} + 3s$.

Click on the icon to start the demonstration in Microsoft® Excel.

Choose a random sample of size $n = 1000$ from a normal distribution and follow the procedure:

- find $\overline{x}$ and s
- find $\overline{x} - s$, $\overline{x} + s$; $\overline{x} - 2s$, $\overline{x} + 2s$; $\overline{x} - 3s$, $\overline{x} + 3s$
- count all values between $\overline{x} - s$ and $\overline{x} + s$ $\overline{x} - 2s$ and $\overline{x} + 2s$ $\overline{x} - 3s$ and $\overline{x} + 3s$
- determine percentages in the intervals
- repeat several times.

Example 11

The chest measurements of 18 year old male footballers is normally distributed with a mean of 95 cm and a standard deviation of 8 cm.

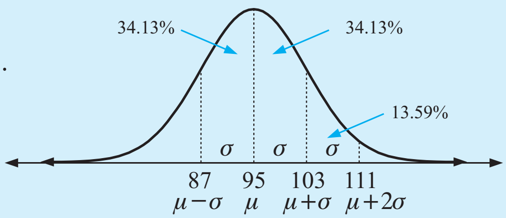
a Find the percentage of footballers with chest measurements between:
  i 87 cm and 103 cm **ii** 103 cm and 111 cm

b Find the probability that the chest measurement of a randomly chosen footballer is between 87 cm and 111 cm.

a **i** We need the percentage between $\mu - \sigma$ and $\mu + \sigma$. This is 68.26%.

ii We need the percentage between $\mu + \sigma$ and $\mu + 2\sigma$.

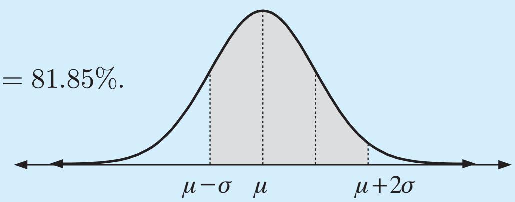
This is 13.59%.



b This is between $\mu - \sigma$ and $\mu + 2\sigma$.

The percentage is $68.26\% + 13.59\% = 81.85\%$.

So the probability is 0.8185 .



EXERCISE 29G.1

1 Draw each of the following distributions accurately on one set of axes.

Distribution	*form*	*mean (mL)*	*standard deviation (mL)*
A	normal	25	5
B	normal	30	2
C	normal	21	10

2 Explain why it is feasible that the distribution of each of the following variables is normal:

a the volume of soft drink in cans
b the life time of a given type of car tyre
c the weight of 10-cent coins when leaving the mint
d the diameter of bolts immediately after manufacture.

3 It is known that when a specific type of radish is grown in a certain manner without fertiliser the weights of the radishes produced are normally distributed with a mean of 40 g and a standard deviation of 10 g. When the same type of radish is grown in the same way except for the inclusion of fertiliser, it is known that the weights of the radishes produced are normally distributed with a mean of 140 g and a standard deviation of 40 g. Determine the proportion of radishes grown

- **a** without fertiliser with weights less than 50 grams.
- **b** with fertiliser with weights less than 60 grams.
- **c** with and without fertiliser with weights equal to or between 20 and 60 grams.
- **d** with and without fertiliser that will have weights greater than or equal to 60 grams.

4 What is the probability that a randomly selected, normally distributed value lies between:

- **a** 1σ below the mean and 1σ above the mean
- **b** the mean and the value 1σ above the mean
- **c** the mean and the value 2σ below the mean
- **d** the mean and the value 3σ above the mean?

5 The height of male students is normally distributed with a mean of 170 cm and a standard deviation of 8 cm. Find the percentage of male students whose height is:

- **a** between 162 cm and 170 cm
- **b** between 170 cm and 186 cm.

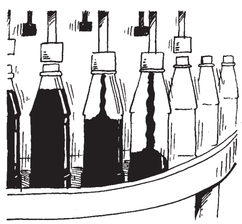
Find the probability that a randomly chosen student from this group has a height:

- **c** between 178 cm and 186 cm
- **d** less than 162 cm
- **e** less than 154 cm
- **f** greater than 162 cm.

6 A clock manufacturer investigated the accuracy of its clocks after 6 months of continuous use. They found that the mean error was 0 minutes with a standard deviation of 2 minutes. If a buyer purchases 800 of these clocks, find the expected number that will be:

- **a** on time or up to 4 minutes fast after 6 months of continuous use
- **b** on time or up to 6 minutes slow after 6 months of continuous use
- **c** between 4 minutes slow and 6 minutes fast after 6 months of continuous use.

7 A bottle filling machine fills, on average, 20 000 bottles a day with a standard deviation of 2000. If we assume that production is normally distributed and the year comprises 260 working days, calculate the approximate number of working days that:



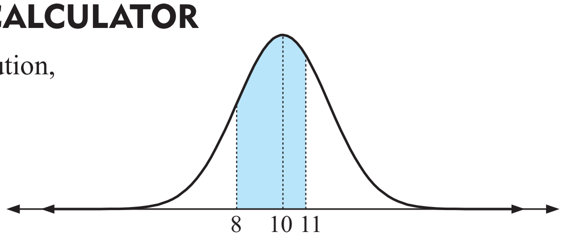
- **a** under 18 000 bottles are filled
- **b** over 16 000 bottles are filled
- **c** between 18 000 and 24 000 bottles (inclusive) are filled.

PROBABILITIES BY GRAPHICS CALCULATOR

To find probabilities for the normal distribution, the graphics calculator is a powerful tool.

Suppose X is normally distributed with mean 10 and standard deviation 2.

How do we find $P(8 \leqslant x \leqslant 11)$?



Click on the icon for your graphics calculator for instructions on how to obtain the normal distribution graph and determine probabilities.

Note: First we need to set up the **WINDOW**.

As approximately all observations lie between $\mu - 3\sigma$ and $\mu + 3\sigma$ then

$X_{\min} = 10 - 3(2) = 4 \qquad Y_{\min} = 0$

$X_{\max} = 10 + 3(2) = 16$

Now $\dfrac{1}{\sigma\sqrt{2\pi}} = \dfrac{1}{2\sqrt{2\pi}} \doteqdot 0.199, \quad \therefore \ Y_{\max} \doteqdot 0.20$, say

For a TI-83, we then use the **DISTR** function and select **DRAW**

Then **Shadenorm (**8, 11, 10, 2**)**

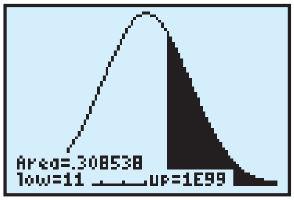
least x value greatest x value μ σ

You will need to enter **2nd** **DRAW** CLRDRAW before doing the next question

If asked to find

P($x \geqslant 11$) say, use **Shadenorm (**11, E99, 10, 2**)**.

or P($x \leqslant 9$) say, use **Shadenorm (**$-$E99, 9, 10, 2**)**.



EXERCISE 29G.2

Use a calculator to obtain a normal distribution graph and calculate probabilities:

1 X is a random variable that is distributed normally with mean 70 and standard deviation 4. Find:

a P($70 \leqslant x \leqslant 74$) **b** P($68 \leqslant x \leqslant 72$) **c** P($x \leqslant 65$)

2 Given that X is a random variable that is distributed normally with mean 60 and standard deviation 5, find:

a P($60 \leqslant x \leqslant 65$) **b** P($62 \leqslant x \leqslant 67$) **c** P($x \geqslant 64$)

d P($x \leqslant 68$) **e** P($x \leqslant 61$) **f** P($57.5 \leqslant x \leqslant 62.5$)

INVESTIGATION 4 MEAN AND STANDARD DEVIATION OF $z = \dfrac{x - \overline{x}}{s}$

Suppose a random variable X is **normally distributed** with mean $\overline{x}$ and standard deviation s.

For each value of x we can calculate a z**-value** using the algebraic transformation $z = \dfrac{x - \overline{x}}{s}$.

The question arises, "What is the mean and standard deviation of the z-distribution?"

What to do:

Question **1** could be easily done on a **graphics calculator** or **spreadsheet**.

1 Consider the following x values: 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7.

a Draw a graph of the distribution to check that it is approximately normal.

b Find the mean $\overline{x}$, and standard deviation s, of the distribution of x values.

c Use the transformation $z = \dfrac{x - \overline{x}}{s}$ to convert each of the x values into z values.

d Find the mean and standard deviation of the distribution of z values.

2 Click on the icon to bring up a large sample drawn from a normal population. By clicking appropriately we can repeat the four steps of question **1**.

DEMO

3 Write a brief report of your findings.

H THE STANDARD NORMAL DISTRIBUTION (z-DISTRIBUTION)

To obtain the **standard normal distribution** (sometimes called the z**-distribution**) the transformation $z = \dfrac{x - \mu}{\sigma}$ is applied to a normal x-distribution.

In **Investigation 4** we discovered that the mean of a z-distribution is 0 and the standard deviation is 1.

The **probability density function** for the z-distribution is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty < z < \infty$.

To get this, replace μ by 0, σ by 1 and x by z in the $f(x)$ function.

As for $f(x)$, $\displaystyle\int_{-\infty}^{\infty} f(z)\, dz$ should be 1.

Note:

The normal distribution function $f(x)$ has two parameters μ and σ whereas the **standard normal** distribution function $f(z)$ **has no parameters**. This means that a table of values of $f(z)$ can be found and is unique, whereas a vast number of tables for many values of μ and σ would need to be constructed if they are to be useful.

Before graphics calculators and computer packages the standard normal distribution was used exclusively for normal probability calculations such as those which follow. However standard normal variables are essential for the following chapter on statistical inference.

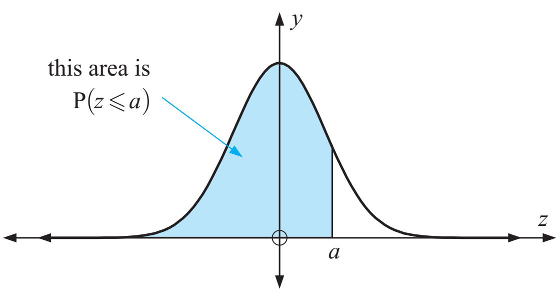
On the next page we have standard normal curve areas.

CALCULATING PROBABILITIES USING THE z-DISTRIBUTION

The table of values of $f(z)$ enables us to find $P(z \leqslant a)$.

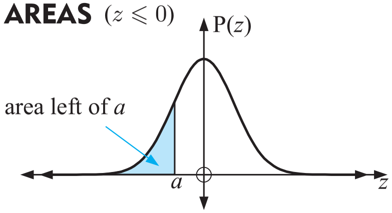
Note: $P(z \leqslant a) = Pr(z < a)$.

In fact, $P(z \leqslant a) = \displaystyle\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz.$



STANDARD NORMAL CURVE AREAS ($z \leqslant 0$)

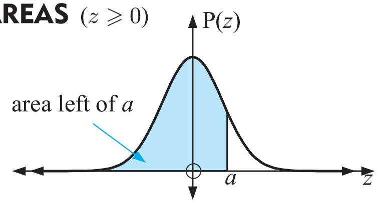
Each table value is the area to the left of the specified z-value.



	the second decimal digit of z									
z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

STANDARD NORMAL CURVE AREAS ($z \geqslant 0$)

Each table value is the area to the left of the specified z-value.



	the second decimal digit of z									
z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

USING A GRAPHICS CALCULATOR TO FIND PROBABILITIES

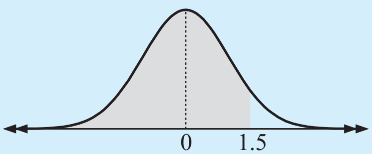
To find $P(z \leqslant a)$ or $P(z < a)$ use normalcdf($-$E99, a).
To find $P(z \geqslant a)$ or $P(z > a)$ use normalcdf(a, E99).
To find $P(a \leqslant z \leqslant b)$ or $P(a < z < b)$ use normalcdf(a, b).

Example 12

If Z is a standard normal variable, find:

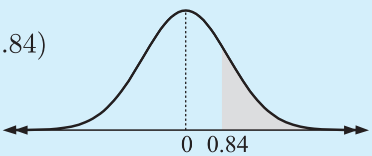
a $P(z \leqslant 1.5)$ **b** $P(z > 0.84)$ **c** $P(-0.41 \leqslant z \leqslant 0.67)$

a $P(z \leqslant 1.5)$
$= 0.9332$



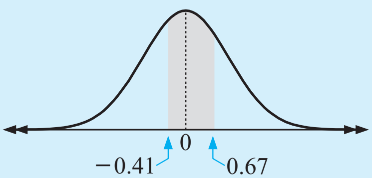
or $P(z \leqslant 1.5)$
$=$ normalcdf($-$E99, 1.5)
$= 0.9332$

b $P(z > 0.84)$
$= 1 - P(z \leqslant 0.84)$
$= 1 - 0.7995$
$= 0.2005$



or $P(z > 0.84)$
$=$ normalcdf(0.84, E99)
$= 0.2005$

c $P(-0.41 \leqslant z \leqslant 0.67)$
$= P(z \leqslant 0.67) - P(z \leqslant -0.41)$
$= 0.7486 - 0.3409$
$= 0.4077$



or $P(-0.41 \leqslant z \leqslant 0.67)$
$=$ normalcdf($-$0.41, 0.67)
$= 0.4077$

EXERCISE 29H.1

1 If Z has standard normal distribution, find *using tables* and a sketch:

a $P(z \leqslant 1.2)$ **b** $P(z \geqslant 0.86)$ **c** $P(z \leqslant -0.52)$
d $P(z \geqslant -1.62)$ **e** $P(-0.86 \leqslant z \leqslant 0.32)$

2 If Z has standard normal distribution, find *using technology*:

a $P(z \geqslant 0.837)$ **b** $P(z \leqslant 0.0614)$ **c** $P(z \geqslant -0.876)$
d $P(-0.3862 \leqslant z \leqslant 0.2506)$ **e** $P(-2.367 \leqslant z \leqslant -0.6503)$

3 If Z is standard normal distributed, find:

a $P(-0.5 < z < 0.5)$ **b** $P(-1.960 < z < 1.960)$

4 Find a if Z has standard normal distribution and:

a $P(z \leqslant a) = 0.95$ **b** $P(z \geqslant a) = 0.90$

STANDARDISING ANY NORMAL DISTRIBUTION

To find probabilities for a random variable x which is normally distributed we could follow these steps:

Step 1: Convert x values to z using $z = \dfrac{x-\mu}{\sigma}$.

Step 2: Sketch a standard normal curve and shade the required region.

Step 3: Use the standard normal tables or a graphics calculator to find the probability.

Example 13

Given that X is a normal variable with mean 62 and standard deviation 7, find:

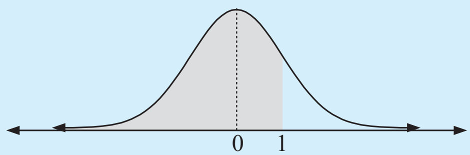
a $P(x \leqslant 69)$ **b** $P(x \geqslant 54.3)$ **c** $P(58.5 \leqslant x \leqslant 71.8)$

a $P(x \leqslant 69)$

$= P\left(\dfrac{x-62}{7} \leqslant \dfrac{69-62}{7}\right)$

$= P(z \leqslant 1)$

$= 0.8413$ i.e., 84.13% chance that a randomly selected x-value is 69 or less.

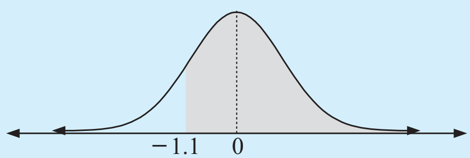


b $P(x \geqslant 54.3)$

$= P\left(\dfrac{x-62}{7} \geqslant \dfrac{54.3-62}{7}\right)$

$= P(z \geqslant -1.1)$

$= 0.8643$ i.e., 86.43% chance that a randomly selected x-value is 69 or more.



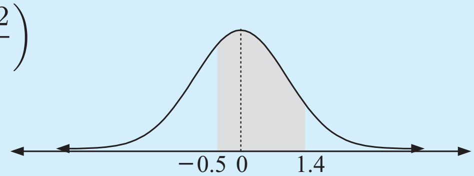
c $P(58.5 \leqslant x \leqslant 71.8)$

$= P\left(\dfrac{58.5-62}{7} \leqslant \dfrac{x-62}{7} \leqslant \dfrac{71.8-62}{7}\right)$

$= P(-0.5 \leqslant z \leqslant 1.4)$

$= 0.9192 - 0.3085$

$= 0.6107$



This means that there is an 61.07% chance that a randomly selected x-value is between 58.5 and 71.8, inclusive.

These probabilities can be easily found using a graphics calculator without actually converting to standard normal z-scores. Click on the icon for your calculator for instructions.

Note: If a normal distribution has mean μ and standard deviation σ

- to find $P(x \leqslant a)$ or $P(x < a)$ use normalcdf($-$E99, a, μ, σ)
- to find $P(x \geqslant a)$ or $P(x > a)$ use normalcdf(a, E99, μ, σ)
- to find $P(a \leqslant x \leqslant b)$ or $P(a < x < b)$ use normalcdf(a, b, μ, σ)

EXERCISE 29H.2

1 Given that a random variable X is normally distributed with a mean 70 and standard deviation 4, find the following probabilities by *first converting to the standard variable* z, and then using the *tabled* probability values for z:

a $\mathrm{P}(x \geqslant 74)$ **b** $\mathrm{P}(x \leqslant 68)$ **c** $\mathrm{P}(60.6 \leqslant x \leqslant 68.4)$

2 Given that the random variable X is normally distributed with mean 58.3 and standard deviation 8.96, find the following probabilities by *first converting to the standard variable* z and then using your graphics calculator:

a $\mathrm{P}(x \geqslant 61.8)$ **b** $\mathrm{P}(x \leqslant 54.2)$ **c** $\mathrm{P}(50.67 \leqslant x \leqslant 68.92)$

3 The length of a nail L, is normally distributed with mean 50.2 mm and standard deviation 0.93 m. Find by first converting to z values:

a $\mathrm{P}(l \geqslant 50)$ **b** $\mathrm{P}(l \leqslant 51)$ **c** $\mathrm{P}(49 \leqslant l \leqslant 50.5)$

FINDING QUANTILES (k values)

Suppose we have a population of adult snails where the length of a snail shell, X mm, is normally distributed with mean 23.6 mm and standard deviation 3.1 mm.

Consider the question: "What is, with 95% probability, the longest snail shell length?".

To answer this question we need to find the value of k such that $\mathrm{P}(x \leqslant k) = 0.95$.

We could either:
- read the standard normal table in reverse *or*
- use the **invNorm(** option of a graphics calculator.

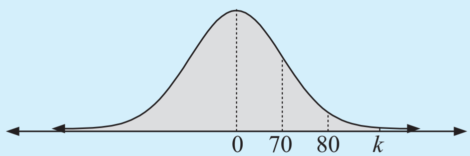
Example 14

Find k for which $\mathrm{P}(x \leqslant k) = 0.95$ given that x is normally distributed with mean 70 and standard deviation 10.

$$\mathrm{P}(x \leqslant k) = 0.95$$

$$\therefore \quad \mathrm{P}\left(\frac{x-70}{10} \leqslant \frac{k-70}{10}\right) = 0.95$$

$$\therefore \quad \mathrm{P}\left(z \leqslant \frac{k-70}{10}\right) = 0.95$$



Searching amongst the standard normal tables or from your graphics calculator:

For $a = 1.64$, $P = 0.9495$ and for $a = 1.65$, $P = 0.9505$

Now, $a \doteqdot 1.645 \quad \therefore \quad \dfrac{k-70}{10} \doteqdot 1.645$ and so $k \doteqdot 86.45$

This means that approximately 95% of the values are expected to be 86.45 or less.

Your **graphics calculator** can be used effectively to find k-values to greater accuracy if required.

For example, in the previous example, when we get to $\mathrm{P}\left(z \leqslant \dfrac{k-70}{10}\right) = 0.95$

then $\dfrac{k-70}{10} = \text{invNorm}(0.95)$

$\therefore\quad k = 70 + 10 \times \text{invNorm}(0.95)$

and so, $k \doteqdot 86.45$

Note: We do not have to convert to the standard normal variable to find k.

For example, to find k when $\mathrm{P}(x \leqslant k) = 0.95$ where X is normally distributed with mean 70 and standard deviation 10 we can use

$$\begin{aligned} k &= \text{invNorm}(0.95, 70, 10) \\ &\doteqdot 86.45 \end{aligned}$$

EXERCISE 29H.3

1 Z has a standard normal distribution. Find k using the *tabled values* if:

a $\mathrm{P}(z \leqslant k) = 0.81$ **b** $\mathrm{P}(z \leqslant k) = 0.58$ **c** $\mathrm{P}(z \leqslant k) = 0.17$

2 If Z is standard normal distributed, find k using technology if:

a $\mathrm{P}(z \leqslant k) = 0.384$ **b** $\mathrm{P}(z \leqslant k) = 0.878$ **c** $\mathrm{P}(z \leqslant k) = 0.1384$

3 **a** Show that $\mathrm{P}(-k \leqslant z \leqslant k) = 2\,\mathrm{P}(z \leqslant k) - 1$.

b If Z is standard normally distributed, find k if:

i $\mathrm{P}(-k \leqslant z \leqslant k) = 0.238$ **ii** $\mathrm{P}(-k \leqslant z \leqslant k) = 0.7004$

I APPLICATIONS OF THE NORMAL DISTRIBUTION

Example 15

In 1972 the heights of rugby players was found to be normally distributed with mean 179 cm and standard deviation 7 cm. Find the probability that in 1972 a randomly selected player was: **a** at least 175 cm tall **b** between 170 cm and 190 cm.

If X is the height of a player then X is normally distributed with mean $\mu = 179$ and standard deviation $\sigma = 7$.

a We need to find

$$\begin{aligned} &\mathrm{P}(x \geqslant 175) \\ &= \text{normalcdf}\,(175, \text{E99}, 179, 7) \\ &= 0.7161 \quad \{\text{graphics calculator}\} \end{aligned}$$

b We need to find

$$\begin{aligned} &\mathrm{P}(170 \leqslant x \leqslant 190) \\ &= \text{normalcdf}\,(170, 190, 179, 7) \\ &= 0.8427 \quad \{\text{graphics calculator}\} \end{aligned}$$

Note: It is good practice to **always** verify your results using an alternative method (if possible).

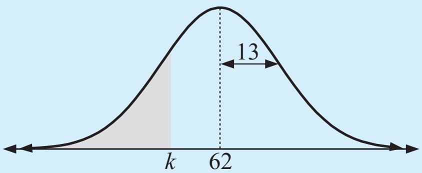
Example 16

A University professor determines that 80% of this year's History candidates should pass the final examination. The examination results are expected to be normally distributed with mean 62 and standard deviation 13. Find the lowest score necessary to pass the examination.

Let the random variable X denote the final examination result, then X is normally distributed with $\mu = 62$ and $\sigma = 13$.

We need to find k such that $P(x \geqslant k) = 0.8$ or $P(x \leqslant k) = 0.2$

Now, for $P(x \leqslant k) = 0.2$, $\therefore$ $P\left(z \leqslant \dfrac{k-62}{13}\right) = 0.2$



$\therefore$ $\dfrac{k-62}{13} = \text{invNorm}(0.2)$

$\therefore$ $k = 13 \times \text{invNorm}(0.2) + 62$

$\therefore$ $k \doteqdot 51.059$

So, the minimum pass mark is 51.

EXERCISE 29I

1 A machine produces metal bolts. The lengths of these bolts have a normal distribution with a mean of 19.8 cm and a standard deviation of 0.3 cm. A bolt is selected at random from the machine. Find the probability that it will have a length between 19.7 cm and 20 cm.

2 Max's customers put money for charity in a collection box on the front counter of his shop. The average weekly collection is approximately normally distributed with a mean of \$40 and a standard deviation of \$6. What proportion of weeks would he expect to collect: **a** between \$30.00 and \$50.00 **b** at least \$50.00?

3 The students of Class X sat a Physics test. From the results, the average score was 46 with a standard deviation of 25. The teacher decided to award an A to the top 7% of the students in the class. Assuming that the scores were normally distributed, find the lowest score that a student must obtain in order to achieve an A.

4 Eels are washed onto a beach after a storm. Their lengths are found to have a normal distribution with a mean of 41 cm and a variance of 11 cm^2 (standard deviation $= \sqrt{\text{variance}}$).

a If an eel is randomly selected, find the probability that it is at least 50 cm.

b Find the proportion of eels measuring between 40 cm and 50 cm.

c How many eels from a sample of 200 would you expect to measure at least 45 cm?

5 Rewi grows a species of palm tree suitable for patios. The heights of these palms, when mature, are found to be normally distributed with a mean of 181 cm and a standard deviation of 4 cm. A mature palm of this species is selected at random. Find the probability that this palm is: **a** at least 175 cm tall **b** between 177 cm and 180 cm tall.

Example 17

Find the mean and the standard deviation of a normally distributed random variable X, if $P(x \geqslant 50) = 0.2$ and $P(x \leqslant 20) = 0.3$.

$$\begin{aligned} P(x \leqslant 20) &= 0.3 \\ \therefore\ P\left(z \leqslant \frac{20-\mu}{\sigma}\right) &= 0.3 \\ \therefore\ \frac{20-\mu}{\sigma} &= \text{invNorm}(0.3) \\ \therefore\ \frac{20-\mu}{\sigma} &\doteqdot -0.5244 \\ \therefore\ 20-\mu &\doteqdot -0.5244\sigma \ \ldots\ldots (1) \end{aligned}$$

$$\begin{aligned} P(x \geqslant 50) &= 0.2 \\ \therefore\ P(x \leqslant 50) &= 0.8 \\ \therefore\ P\left(z \leqslant \frac{50-\mu}{\sigma}\right) &= 0.8 \\ \therefore\ \frac{50-\mu}{\sigma} &= 0.8416 \\ \therefore\ 50-\mu &\doteqdot 0.8416\sigma \ \ldots\ldots (2) \end{aligned}$$

Solving (1) and (2) simultaneously we get $\mu \doteqdot 31.5$, $\sigma \doteqdot 22.0$ Check with a GC!

6 **a** A random variable, X, is normally distributed. Find the mean and the standard deviation of X, given that $P(X \geqslant 80) = 0.1$ and $P(X \leqslant 30) = 0.15$.

b It was found that 10% of the students scored at least 80 marks and no more than 15% scored less than 30 marks in the Mathematics examination at the end of the year. What proportion of students scored more than 50 marks?

7 Circular metal tokens are used to operate a washing machine in a laundromat. The diameters of the tokens are known to be normally distributed. Only tokens with diameters between 1.94 and 2.06 cm will operate the machine.

a Find the mean and the standard deviation of the distribution given that 2% of the tokens are too small, and 3% are too large.

b Find the probability that less than two tokens out of a batch of 20 will not operate the machine.

8 A student scored 70 for a Science exam and 66 for a Geography exam. If the class scores are normally distributed with a mean and a standard deviation for Science of 60 and 10 and for Geography 50 and 12, in which subject did the student achieve a higher standard, and what percentage of others achieved lower marks in each subject?

REVIEW SET 29A

1 $P(x) = \dfrac{a}{x^2+1}$ for $x = 0, 1, 2, 3$ is a probability distribution function.

a Find a. **b** Hence, find $P(x \geqslant 1)$.

2 A random variable X has probability distribution function $P(x) = C_x^4 \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$ for $x = 0, 1, 2, 3, 4$.

a Find $P(x)$ for $x = 0, 1, 2, 3, 4$. **b** Find μ and σ for this distribution.

3 A manufacturer finds that 18% of the items produced from one of the assembly lines are defective. During a floor inspection, the manufacturer randomly selects ten items. Find the probability that the manufacturer finds:

a one defective **b** two defective **c** at least two defectives.

4 A random sample of 120 toothbrushes is made (with replacement) from a very large batch where 4% are known to be defective. Find:

a the mean
b the standard deviation of the number of defectives in the sample.

5 At a social club function, a dice game is played where on a single roll of a six-sided die the following payouts are made:
\$2 for an odd number, \$3 for a 2, \$6 for a 4 and \$9 for a 6.

a What is the expected return for a single roll of the die?
b If the club charges \$5 for each roll, how much money would the club expect to make if 75 people played the game once each?

6 Lakshmi rolls a normal six-sided die and wins twice the number of dollars as the number shown on the face.

a How much does Lakshmi expect to win from one roll of the die?
b If it costs \$8 to play the game, would you advise Lakshmi to play several games? Explain your answer.

7 The arm lengths of 18 year old females are normally distributed with mean 64 cm and standard deviation 4 cm.

a Find the percentage of 18 year old females whose arm lengths are:
  i between 60 cm and 72 cm **ii** greater than 60 cm.
b Find the probability that a randomly chosen 18 year old female has an arm length in the range 56 cm to 68 cm.

8 The length of steel rods produced by a machine is normally distributed with a standard deviation of 3 mm. It is found that 2% of all rods are less than 25 mm long. Find the mean length of rods produced by the machine.

REVIEW SET 29B

1 A discrete random variable X has probability distribution function $P(x)$ where
$P(x) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$ and k is a constant.

a Find k. **b** Find $P(x \geqslant 1)$.

2 An insurance company covers a \$45 000 painting against fire, theft and accidental damage. If the painting is destroyed by fire the policy is paid out in full. If it is stolen the company will pay \$30 000 and if accidentally damaged \$10 000. From past experience the company knows that the probabilities of fire, theft and accidental damage are 0.000 72, 0.0023 and 0.0088 respectively. How much should the company charge to cover the painting if they want a \$250 expected return?

3 A pistol shooter has a probability of 0.96 of hitting a target with each shot. If she fires four times at a target, find the probability that:

a all four shots hit the target **b** she does not hit the target
c at least three shots hit the target **d** only one shot hits the target.

4 Suppose X is Bin(4, p).

a Find $P(0)$, $P(1)$, $P(2)$, $P(3)$ and $P(4)$ using $P(x) = C^4_x p^x q^{4-x}$ and tabulate the results in:

x_i	0	1	2	3	4
p_i					

b Use $\mu = \sum p_i x_i$ to show that $\mu = 4p$.

c Use $\sigma^2 = \sum x_i^2 p_i - \mu^2$ to show that $\sigma = \sqrt{4pq}$. [**Reminder**: $q = 1 - p$]

5 The contents of a certain brand of soft drink can is normally distributed with mean 377 mL and standard deviation 4.2 mL.

a Find the percentage of cans with contents:

i less than 368.6 mL **ii** between 372.8 mL and 389.6 mL

b Find the probability that a randomly selected can has contents between 364.4 mL and 381.2 mL.

6 The life of a Xenon battery has a normal distribution. The mean is 33.2 weeks and the standard deviation is 2.8 weeks. If a battery is selected at random,

a find the probability that it will last at least 35 weeks.

b Find the maximum number of weeks for which the manufacturer can expect that not more than 8% of batteries will fail.

REVIEW SET 29C

1 A random variable X has probability distribution function given by:

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

a Find k.

b Find the mean μ, and standard deviation σ, for the distribution of X.

2 Only 40% of young trees that are planted will survive the first year. The Botanical Gardens buys five young trees. Assuming independence, calculate the probability that during the first year:

a exactly one tree will survive

b at most one tree will survive

c at least one tree will survive.

3 From data over the last fifteen years it is known that the chance of a netballer needing major knee surgery in any one season is 0.0132. In 1998 there were 487 cases in which this happened. Find the mean and standard deviation of the number of cases.

4 The edible part of a batch of Coffin Bay oysters is normally distributed with mean 38.6 grams and standard deviation 6.3 grams. Given that the random variable X is the mass of a Coffin Bay oyster:

a find a if $P(38.6 - a \leqslant x \leqslant 38.6 + a) = 0.6826$

b find b if $P(x \geqslant b) = 0.8413$.

5 Staplers are manufactured for \$5.00 each and are sold for \$20.00 each. The staplers have a money-back guarantee to last three years. The mean life is actually 3.42 years and the standard deviation is 0.4 years. If the life of these staplers is normally distributed, how much profit would we expect from selling all of a batch of 2000?

6 A random variable X has probability distribution function $f(x) = ax^2(2-x)$ for $0 < x < 2$.

- **a** Show that $a = \frac{3}{4}$. (**Hint**: What is $\int_0^2 f(x)\,dx$ equal to?)
- **b** Find the mode of X.
- **c** Find the median of X.
- **d** Find $P(0.6 < x < 1.2)$.

ANSWERS

EXERCISE 1A

1 **a**, **d**, **e** **2** **a**, **b**, **c**, **e**, **h** **3** No, e.g., $x = 1$ **4** $y = \pm\sqrt{9 - x^2}$

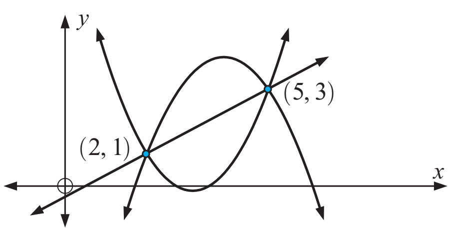
EXERCISE 1B

1 **a** Domain $\{x : x \geqslant -1\}$, Range $\{y : y \leqslant 3\}$
b Domain $\{x : -1 < x \leqslant 5\}$, Range $\{y : 1 < y \leqslant 3\}$
c Domain $\{x : x \neq 2\}$, Range $\{y : y \neq -1\}$
d Domain $\{x : x \text{ is in } \mathcal{R}\}$, Range $\{y : 0 < y \leqslant 2\}$
e Domain $\{x : x \text{ is in } \mathcal{R}\}$, Range $\{y : y \geqslant -1\}$
f Domain $\{x : x \text{ is in } \mathcal{R}\}$, Range $\{y : y \leqslant \frac{25}{4}\}$
g Domain $\{x : x \geqslant -4\}$, Range $\{y : y \geqslant -3\}$
h Domain $\{x : x \text{ is in } \mathcal{R}\}$, Range $\{y : y > -2\}$
i Domain $\{x : x \neq \pm 2\}$, Range $\{y : y \leqslant -1 \text{ or } y > 0\}$

2 **a** Domain $\{x : x \geqslant 0\}$, Range $\{y : y \geqslant 0\}$
b Domain $\{x : x \neq 0\}$, Range $\{y : y > 0\}$
c Domain $\{x : x \leqslant 4\}$, Range $\{y : y \geqslant 0\}$
d Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \geqslant -2\frac{1}{4}\}$
e Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \leqslant 2\frac{1}{12}\}$
f Domain $\{x : x \neq 0\}$, Range $\{y : y \leqslant -2 \text{ or } y \geqslant 2\}$
g Domain $\{x : x \neq 2\}$, Range $\{y : y \neq 1\}$
h Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \in \mathcal{R}\}$
i Domain $\{x : x \neq -1 \text{ or } 2\}$, Range $\{y : y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$
j Domain $\{x : x \neq 0\}$, Range $\{y : y \geqslant 2\}$
k Domain $\{x : x \neq 0\}$, Range $\{y : y \leqslant -2 \text{ or } y \geqslant 2\}$
l Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \geqslant -8\}$

EXERCISE 1C

1 **a** 2 **b** 8 **c** -1 **d** -13 **e** 1
2 **a** -3 **b** 3 **c** 3 **d** -3 **e** $7\frac{1}{2}$
3 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$
4 **a** $7-3a$ **b** $7+3a$ **c** $-3a-2$ **d** $10-3b$ **e** $1-3x$
5 **a** $2x^2+19x+43$ **b** $2x^2-11x+13$ **c** $2x^2-3x-1$
d $2x^4+3x^2-1$ **e** $2x^4-x^2-2$
6 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$ **b** $x = 4$ **c** $\dfrac{2x+7}{x-2}$ **d** $x = \frac{9}{5}$
7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.
8 $f(a)f(b) = 2^a 2^b = 2^{a+b}$ {index law} and $f(a+b) = 2^{a+b}$
9 **a** $x + 3$ **b** $4 + h$
10 **a** \$6210, value after 4 years **b** $t = 4.5$, the time for the photocopier to reach a value of \$5780. **c** \$9650
11



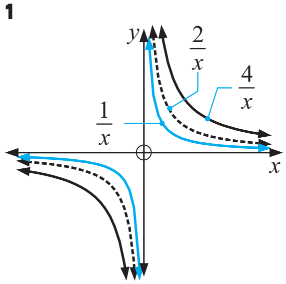
12 $f(x) = -2x + 5$ **13** $a = 3$, $b = -2$
14 $a = 3$, $b = -1$, $c = -4$, $T(x) = 3x^2 - x - 4$

EXERCISE 1D

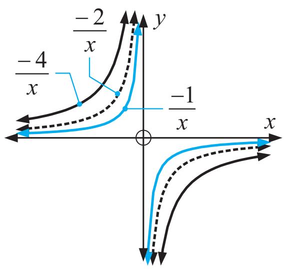
1 **a** $5 - 2x$ **b** $-2x - 2$ **c** 11
2 $f(g(x)) = (2-x)^2$ $\quad g(f(x)) = 2 - x^2$
3 **a** $x^2 - 6x + 10$ **b** $2 - x^2$ **c** $x = \pm\frac{1}{\sqrt{2}}$
4 **a** Let $x = 0$, $\therefore$ $b = d$ and so
$ax + b = cx + b$
$\therefore$ $ax = cx$ for all x
Let $x = 1$, $\therefore$ $a = c$
b $(f \circ g)(x) = [2a]x + [2b+3] = 1x + 0$ for all x
$\therefore$ $2a = 1$ and $2b + 3 = 0$
$\therefore$ $a = \frac{1}{2}$ and $b = -\frac{3}{2}$
c Yes, $\{(g \circ f)(x) = [2a]x + [3a+b]\}$

EXERCISE 1E

1

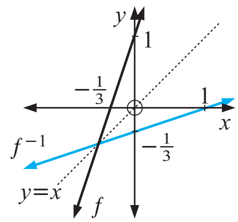


2



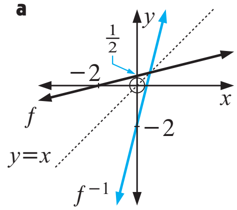
EXERCISE 1F

1 **a**



b, c $f^{-1}(x) = \dfrac{x-1}{3}$

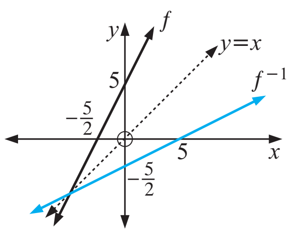
2 **a**



b, c $f^{-1}(x) = 4x - 2$

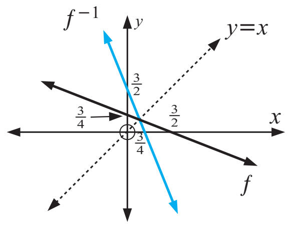
3 **a** **i** $f^{-1}(x) = \dfrac{x-5}{2}$

ii



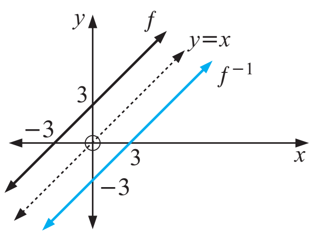
b **i** $f^{-1}(x) = -2x + \frac{3}{2}$

ii

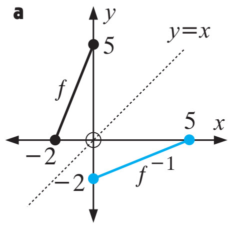


c **i** $f^{-1}(x)$
$= x - 3$

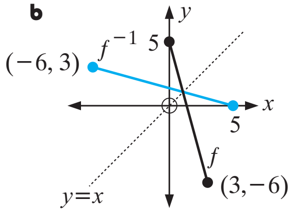
ii



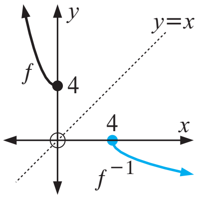
4 **a**



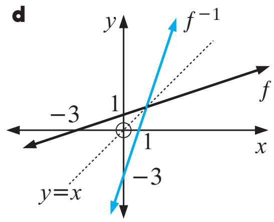
b



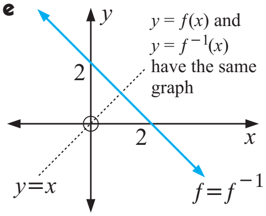
c



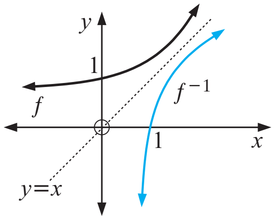
d



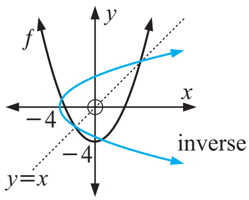
e



f



5 **a**

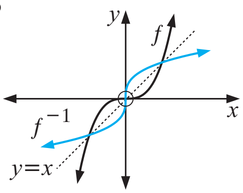


b No

c Yes, it is

$y = \sqrt{x+4}$

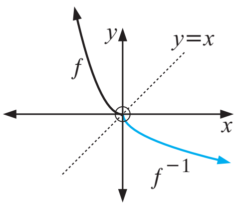
6



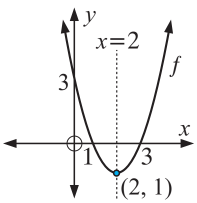
7 **b** **i** is the only one

8 **a** $f^{-1}(x) = -\sqrt{x}$

b



9 **a**



A horizontal line above the vertex cuts the graph **twice**. So, it does not have an inverse.

b For $x \geqslant 2$, all horizontal lines cut 0 or once only. ∴ has an inverse.

c **Hint:** Inverse is $x = y^2 - 4y + 3$ for $y \geqslant 2$

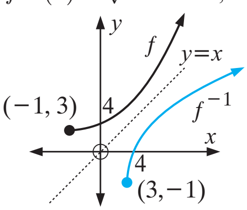
d **i** Domain is $\{x : x \geqslant 2\}$, Range is $\{y : y \geqslant -1\}$

ii Domain is $\{x : x \geqslant -1\}$, Range is $\{y : y \geqslant 2\}$

10 **a** $f^{-1}(x) = 2x + 2$ **b** **i** x **ii** x

11 **a** $f^{-1}(x) = \sqrt{x-3} - 1,\ x \geqslant 3$

b



c **i** Domain $\{x : x \geqslant -1\}$
Range $\{y : y \geqslant 3\}$

ii Domain $\{x : x \geqslant 3\}$
Range $\{y : y \geqslant -1\}$

12 **a** 10 **b** $x = 3$ **13** **a** **i** 25 **ii** 16 **b** $x = 1$

14 $(f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

15 **a** Is not **b** Is **c** Is **d** Is **e** Is

EXERCISE 1G

1 $f^{-1}(x) = \dfrac{x-1}{3}$ **2** $f^{-1}(x) = 4x - 3$

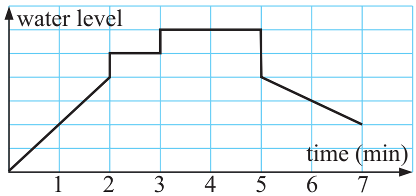
3 $f^{-1}(x) = x^2$ for $x \geqslant 0$

4 **a** B is $(f(x), x)$ **b** $x = f^{-1}(f(x)) = (f^{-1} \circ f)(x)$

c Start with B first and repeat the process used in **a** and **b**.

REVIEW SET 1A

1



2 **a** 0 **b** −15 **c** $-\frac{5}{4}$

3 **a** **i** Range $= \{y : y \geqslant -5\}$, Domain $= \{x : x$ is in $\mathcal{R}\}$

ii x-intercepts −1, 5; y-intercept $-\frac{25}{9}$ **iii** is a function

b **i** Range $= \{y : y = 1$ or $-3\}$ Domain $= \{x : x$ is in $\mathcal{R}\}$

ii no x-intercepts; y-intercept 1 **iii** is a function

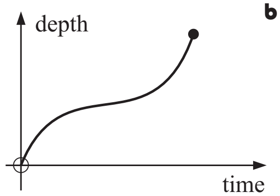
4 **a** Domain $= \{x : x \geqslant -2\}$, Range $= \{y : 1 \leqslant y < 3\}$

b Domain $= \{x$ is in $\mathcal{R}\}$, Range $= \{y : y = -1, 1$ or $2\}$

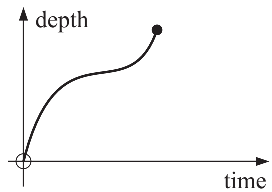
5 **a** $10 - 6x$ **b** $x = 2$ **6** **a** $a = -6,\ b = 13$

7 $a = 1,\ b = -6,\ c = 5$

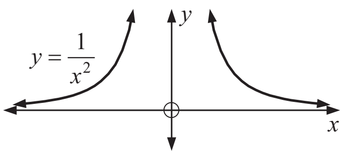
8 **a**



b



9 **a** $x = 0$ **b**



c Domain $= \{x : x \neq 0\}$, Range $= \{y : y > 0\}$

10 **a** $2x^2 + 1$ **b** $4x^2 - 12x + 11$

11 **a** $1 - 2\sqrt{x}$ **b** $\sqrt{1-2x}$

12 **a** $f(x) = \sqrt{x},\ g(x) = 1 - x^2$

b $g(x) = x^2,\ f(x) = \dfrac{x-2}{x+1}$

REVIEW SET 1B

1 **a** 5 **b** −5 **c** 11 **d** 4

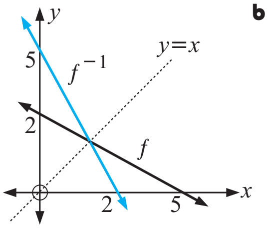
2 **a** $x^2 - x - 2$ **b** $x^4 - 7x^2 + 10$

3 **a** $f^{-1}(x) = \dfrac{7-x}{4}$ **b** $f^{-1}(x) = \dfrac{5x-3}{2}$

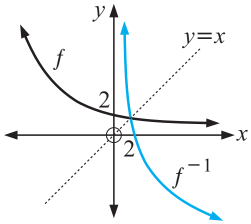
4 **a** Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \geqslant -4\}$

b Domain $\{x : x \neq 0, 2\}$, Range $\{y : y \leqslant -1$ or $y > 0\}$

5 **a**

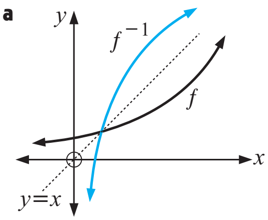


b

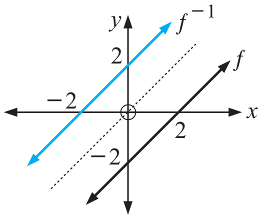


6 **a** $f^{-1}(x) = \dfrac{x-2}{4}$ **b** $f^{-1}(x) = \dfrac{3-4x}{5}$

7 **a**

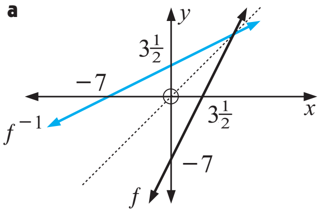


b



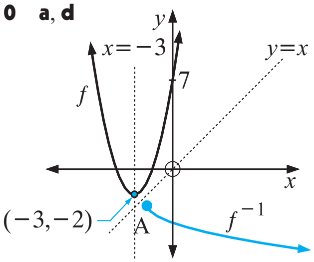
8 16

9 **a**



b, c $f^{-1}(x) = \dfrac{x+7}{2}$

10 **a, d**



b If $x \leqslant -3$, we have the graph to the left of $x = -3$ and any horizontal line cuts it at most once.

c $y = -3 - \sqrt{x+2}$

11 $h^{-1}(x) = 4 + \sqrt{x-3}$

12 $(f^{-1} \circ h^{-1})(x) = x - 2$ and $(h \circ f)^{-1}(x) = x - 2$

EXERCISE 2A

1 **a** 4, 13, 22, 31, ... **b** 45, 39, 33, 27, ...
c 2, 6, 18, 54, ... **d** 96, 48, 24, 12, ...

2 **a** Starts at 8 and each term is 8 more than the previous term. Next two terms 40, 48.
b Starts at 2, each term is 3 more than the previous term; 14, 17.
c Starts at 36, each term is 5 less than the previous term; 16, 11.
d Starts at 96, each term is 7 less than the previous term; 68, 61.
e Starts at 1, each term is 4 times the previous term; 256, 1024.
f Starts at 2, each term is 3 times the previous term; 162, 486.
g Starts at 480, each term is half the previous term; 30, 15.
h Starts at 243, each term is $\frac{1}{3}$ of the previous term; 3, 1.
i Starts at 50 000, each term is $\frac{1}{5}$ of the previous term; 80, 16.

3 **a** 79, 75 **b** 1280, 5120 **c** 81, 90

4 **a** Each term is the square of the number of the term; 25, 36, 49.
b Each term is the cube of the number of the term; 125, 216, 343.
c Each term is $n \times (n+1)$ where n is the number of the term; 30, 42, 56.

5 **a** 625, 1296 **b** 13, 21 **c** 9, 11 **d** 13, 17 (primes)
e 16, 22 **f** 14, 18

EXERCISE 2B

1 **a** 2, 4, 6, 8, 10 **b** 4, 6, 8, 10, 12 **c** 1, 3, 5, 7, 9
d -1, 1, 3, 5, 7 **e** 5, 7, 9, 11, 13 **f** 13, 15, 17, 19, 21
g 4, 7, 10, 13, 16 **h** 1, 5, 9, 13, 17
i 9, 14, 19, 24, 29

2 **a** 2, 4, 8, 16, 32 **b** 6, 12, 24, 48, 96
c 3, $1\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$ **d** -2, 4, -8, 16, -32

3 17, 11, 23, -1, 47

EXERCISE 2C

1 **a** $u_1 = 6,\ d = 11$ **b** $u_n = 11n - 5$ **c** 545
d yes, u_{30} **e** no

2 **a** $u_1 = 87,\ d = -4,$ **b** $u_n = 91 - 4n$ **c** -69 **d** no

3 **b** $u_1 = 1,\ d = 3$ **c** 169 **d** $u_{151} = 451$

4 **b** $u_1 = 32,\ d = -\frac{7}{2}$ **c** -227 **d** $n \geqslant 68$

5 **a** $k = 17\frac{1}{2}$ **b** $k = 4$ **c** $k = 4$ **d** $k = 0$
e $k = 3,\ k = -2$ **f** $k = 3,\ k = -1$

6 **a** $u_n = 6n - 1$ **b** $u_n = -\frac{3}{2}n + \frac{11}{2}$ **c** $u_n = -5n + 36$
d $u_n = -\frac{3}{2}n + \frac{1}{2}$

7 **a** $6\frac{1}{4}$, $7\frac{1}{2}$, $8\frac{3}{4}$ **b** $3\frac{5}{7}$, $8\frac{3}{7}$, $13\frac{1}{7}$, $17\frac{6}{7}$, $22\frac{4}{7}$, $27\frac{2}{7}$

8 **a** $u_1 = 36,\ d = -\frac{2}{3}$ **b** 100 **9** 100 006

EXERCISE 2D

1 **a** $b = 18,\ c = 54$ **b** $b = 2\frac{1}{2},\ c = 1\frac{1}{4}$ **c** $b = 3,\ c = -1\frac{1}{2}$

2 **a** $u_1 = 5,\ r = 2$ **b** $u_n = 5 \times 2^{n-1},\ u_{15} = 81\,920$

3 **a** $u_1 = 12,\ r = -\frac{1}{2}$ **b** $u_n = 12 \times (-\frac{1}{2})^{n-1},\ u_{13} = \frac{3}{1024}$

4 **a** $u_1 = 8,\ r = -\frac{3}{4},\ u_{10} = -0.600\,677\,49$

5 **a** $u_1 = 8,\ r = \frac{1}{\sqrt{2}},\ u_n = 2^{\frac{7}{2} - \frac{n}{2}}$

6 **a** $k = \pm 14$ **b** $k = 2$ **c** $k = -2$ or 4

7 **a** $u_n = 3 \times 2^{n-1}$ **b** $u_n = 32 \times (-\frac{1}{2})^{n-1}$
c $u_n = 3 \times (\sqrt{2})^{n-1}$ **d** $u_n = 10 \times (\sqrt{2})^{1-n}$

8 **a** $u_9 = 13\,122$ **b** $u_{14} = 2916\sqrt{3} \doteqdot 5050.66$
c $u_{18} \doteqdot 0.000\,091\,55$

9 **a** \$3993.00 **b** \$993.00

10 11 470.39 Yen **11** **a** 43 923 Yen **b** 13 923 Yen

12 \$23 602.32 **13** 148 024.43 Yen **14** £51 249.06

15 \$14 976.01 **16** £11 477.02 **17** 19 712.33 Euro

18 19 522.47 Yen

19 **a** **i** 1550 ants **ii** 4820 ants **b** 12.2 weeks

20 **a** 278 animals **b** Year 2037

EXERCISE 2E.1

1 **a** **i** $S_n = 3 + 11 + 19 + 27 + ... + (8n - 5)$ **ii** 95
b **i** $S_n = 42 + 37 + 32 + ... + (47 - 5n)$ **ii** 160
c **i** $S_n = 12 + 6 + 3 + 1\frac{1}{2} + ... + 12(\frac{1}{2})^{n-1}$ **ii** $23\frac{1}{4}$
d **i** $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + ... + 2(\frac{3}{2})^{n-1}$ **ii** $26\frac{3}{8}$

e **i** $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{n-1}}$ **ii** $1\frac{15}{16}$

f **i** $S_n = 1 + 8 + 27 + 64 + ... + n^3$ **ii** 225

EXERCISE 2E.2

1 **a** 820 **b** 3087.5 **c** -1460 **d** -740
2 **a** 1749 **b** 2115 **c** $1410\frac{1}{2}$ **3** 203
4 -115.5 **5** 18 **6** **a** 65 **b** 1914 **c** 47 850
7 **a** 14 025 **b** 71 071 **c** 3367
9 **a** $u_n = 2n - 1$ **c** $S_1 = 1,\ S_2 = 4,\ S_3 = 9,\ S_4 = 16$
10 56, 49 **11** 10, 4, -2 or -2, 4, 10
12 2, 5, 8, 11, 14 or 14, 11, 8, 5, 2

EXERCISE 2E.3

1 **a** $23.9766 \doteqdot 24.0$ **b** $\doteqdot 189\,134$ **c** $\doteqdot 4.000$ **d** $\doteqdot 0.5852$
2 **a** $S_n = \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)$ **b** $S_n = 24(1 - (\frac{1}{2})^n)$
c $S_n = 1 - (0.1)^n$ **d** $S_n = \frac{40}{3}(1 - (-\frac{1}{2})^n)$
3 **c** \$26 361.59
4 **a** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$ **b** $S_n = \frac{2^n - 1}{2^n}$
c $1 - (\frac{1}{2})^n = \frac{2^n - 1}{2^n}$ **d** as $n \to \infty$, $S_n \to 1$
5 **b** $S_n = 1 + 18(1 - (0.9)^{n-1})$ **c** 19 seconds
6 **a** **i** $u_1 = \frac{3}{10}$ **ii** $r = 0.1$ **b** $S_\infty = \frac{1}{3}$

EXERCISE 2F

1 **a** 10 **b** 25 **c** 168 **d** 310
2 $2 + 5 + 8 + 11 + ... + 59 = 610$
3 **a** 160 **b** -630 **c** 135
4 **a** 3069 **b** $\frac{4095}{1024} \doteqdot 3.999$ **c** $-134\,217\,732$
5 **a** 420 **b** 2231.868 211

REVIEW SET 2A

1 **a** $\frac{1}{3}$, 1, 3, 9 **b** $\frac{5}{4}, \frac{8}{5}, \frac{11}{6}, 2$ **c** 5, -5, 35, -65
2 **b** $u_1 = 63,\ d = -5$ **c** -117 **d** $u_{54} = -202$
3 **a** $u_1 = 3,\ r = 4$ **b** $u_n = 3 \times 4^{n-1},\ u_9 = 196\,608$
4 $k = -\frac{11}{2}$ **5** $u_n = 73 - 6n,\ u_{34} = -131$
6 **b** $u_1 = 6,\ r = \frac{1}{2}$ **c** 0.000 183
7 $u_n = 33 - 5n,\ S_n = \frac{n}{2}(61 - 5n)$ **8** $k = \pm\frac{2\sqrt{3}}{3}$
9 $u_n = \pm\frac{1}{6} \times 2^{n-1}$

REVIEW SET 2B

1 **a** 81 **b** $-1\frac{1}{2}$ **c** -486 **2** 21, 19, 17, 15, 13, 11
3 **a** $u_n = 89 - 3n$ **b** $u_n = \frac{2n+1}{n+3}$ **c** $u_n = 100(0.9)^{n-1}$
4 **a** $1 + 4 + 9 + 16 + 25 + 36 + 49$
b $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$
5 **a** $\sum_{r=1}^{n}(7r-3)$ **b** $\sum_{r=1}^{n}(\frac{1}{2})^{r+1}$ **6** **a** 1587 **b** $47\frac{253}{256} \doteqdot 47.99$
7 **a** 70 **b** 241.2 **8** $u_{12} = 10\,240$
9 **a** 8415.31 Euro **b** 8488.67 Euro **c** 8505.75 Euro

REVIEW SET 2C

1 $u_n = (\frac{3}{4})2^{n-1}$ **a** 49 152 **b** 24 575.25 **2** 12
3 $u_{11} = \frac{8}{19\,683} \doteqdot 0.000\,406$ **4** **a** 17 **b** $255\frac{511}{512} \doteqdot 256.0$
5 **a** \$18 726.65 **b** \$18 885.74
6 \$13 972.28 **7** **a** 3470 **b** Year 2008

EXERCISE 3A

1 **a** $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64$
b $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81$
c $5^1 = 5,\ 5^2 = 25,\ 5^3 = 125,\ 5^4 = 625$
d $7^1 = 7,\ 7^2 = 49,\ 7^3 = 343$

EXERCISE 3B

1 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1 **f** -1 **g** -1
h -8 **i** -8 **j** 8 **k** -25 **l** 125
2 **a** 512 **b** -3125 **c** -243 **d** 16 807 **e** 512
f 6561 **g** -6561 **h** 5.117 264 691
i $-0.764\,479\,956$ **j** $-20.361\,584\,96$
3 **a** $0.\overline{142\,857}$ **b** $0.\overline{142\,857}$ **c** $0.\overline{1}$ **d** $0.\overline{1}$ **e** 0.015 625
f 0.015 625 **g** 1 **h** 1 **4** 3 **5** 7

EXERCISE 3C

1 **a** 7^5 **b** 5^7 **c** a^9 **d** a^5 **e** b^{13} **f** a^{3+n}
g b^{7+m} **h** m^9
2 **a** 5^7 **b** 11^4 **c** 7^3 **d** a^4 **e** b^3 **f** p^{5-m}
g y^{a-5} **h** b^{2x-1}
3 **a** 3^8 **b** 5^{15} **c** 2^{28} **d** a^{10} **e** p^{20} **f** b^{5n}
g x^{3y} **h** a^{10x}
4 **a** 2^3 **b** 5^2 **c** 3^3 **d** 2^6 **e** 3^4 **f** 3^{a+2} **g** 5^{t-1}
h 3^{3n} **i** 2^{4-x} **j** 3^2 **k** 5^{4x-4} **l** 2^2 **m** 2^{y-2x}
n 2^{2y-3x} **o** 3^{2x} **p** 2^3
5 **a** a^3b^3 **b** a^4c^4 **c** b^5c^5 **d** $a^3b^3c^3$ **e** $16a^4$ **f** $25b^2$
g $81n^4$ **h** $8b^3c^3$ **i** $64a^3b^3$ **j** $\frac{a^3}{b^3}$ **k** $\frac{m^4}{n^4}$ **l** $\frac{32c^5}{d^5}$
6 **a** $8b^{12}$ **b** $\frac{9}{x^4y^2}$ **c** $25a^8b^2$ **d** $\frac{m^{12}}{16n^8}$ **e** $\frac{27a^9}{b^{15}}$
f $32m^{15}n^{10}$ **g** $\frac{16a^8}{b^4}$ **h** $125x^6y^9$
7 **a** $4a^2$ **b** $36b^4$ **c** $-8a^3$ **d** $-27m^6n^6$ **e** $16a^4b^{16}$
f $\frac{-8a^6}{b^6}$ **g** $\frac{16a^6}{b^2}$ **h** $\frac{9p^4}{q^6}$
8 **a** a^2 **b** $8b^5$ **c** m^3n **d** $7a^5$ **e** $4ab^2$ **f** $\frac{9m^3}{2}$
g $40h^5k^3$ **h** $\frac{1}{m^5}$ **i** p^3
9 **a** 1 **b** $\frac{1}{3}$ **c** $\frac{1}{6}$ **d** 1 **e** 4 **f** $\frac{1}{4}$ **g** 8 **h** $\frac{1}{8}$
i 25 **j** $\frac{1}{25}$ **k** 100 **l** $\frac{1}{100}$
10 **a** 1 **b** 1 **c** 3 **d** 1 **e** 2 **f** 1 **g** $\frac{1}{25}$ **h** $\frac{1}{32}$
i 3 **j** $\frac{5}{2}$ **k** $\frac{3}{4}$ **l** 12 **m** $2\frac{1}{4}$ **n** $\frac{4}{5}$ **o** $\frac{8}{7}$ **p** $\frac{7}{2}$
11 **a** $\frac{1}{2a}$ **b** $\frac{2}{a}$ **c** $\frac{3}{b}$ **d** $\frac{1}{3b}$ **e** $\frac{b^2}{4}$ **f** $\frac{1}{4b^2}$ **g** $\frac{1}{9n^2}$
h $\frac{n^2}{3}$ **i** $\frac{a}{b}$ **j** $\frac{1}{ab}$ **k** $\frac{a}{b^2}$ **l** $\frac{1}{a^2b^2}$ **m** $\frac{1}{2ab}$
n $\frac{2}{ab}$ **o** $\frac{2a}{b}$ **p** a^2b^3
12 **a** 3^{-1} **b** 2^{-1} **c** 5^{-1} **d** 2^{-2} **e** 3^{-3} **f** 5^{-2}
g 2^{-3x} **h** 2^{-4y} **i** 3^{-4a} **j** 3^{-2} **k** 5^{-2} **l** 5^{-3}
m 2^4 **n** $2^0 = 3^0 = 5^0$ **o** $2^{-3} \times 3^{-3}$ **p** $2^4 \times 5^2$
13 25 days **14** 63 sums
15 **a** $5^3 = 21 + 23 + 25 + 27 + 29$

b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$

c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

16 5^{75}

EXERCISE 3D

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$ **f** $2^{\frac{4}{3}}$
g $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$ **f** $7^{-\frac{1}{3}}$
g $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** 2.280 **b** 1.834 **c** 0.794 **d** 0.435

5 **a** 3 **b** 1.682 **c** 1.933 **d** 0.523

6 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4 **f** $\frac{1}{2}$ **g** $\frac{1}{27}$
h $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 3E

1 **a** $x^5+2x^4+x^2$ **b** $2^{2x}+2^x$ **c** $x+1$ **d** $e^{2x}+2e^x$
e $2\times3^x-1$ **f** x^2+2x+3 **g** $1+5\times2^{-x}$ **h** 5^x+1
i $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$

2 **a** $4^x+2^{2+x}+3$ **b** $9^x+7\times3^x+10$ **c** $25^x-6\times5^x+8$
d $4^x+6\times2^x+9$ **e** $9^x-2\times3^x+1$ **f** $16^x+14\times4^x+49$
g $x-4$ **h** 4^x-9 **i** $x-x^{-1}$ **j** $x^2+4+\dfrac{4}{x^2}$
k $e^{2x}-2+e^{-2x}$ **l** $25-10\times2^{-x}+4^{-x}$

EXERCISE 3F

1 **a** $x=1$ **b** $x=2$ **c** $x=3$ **d** $x=0$ **e** $x=-1$
f $x=-1$ **g** $x=-3$ **h** $x=2$ **i** $x=0$
j $x=-4$ **k** $x=5$ **l** $x=1$

2 **a** $x=2\frac{1}{2}$ **b** $x=-\frac{2}{3}$ **c** $x=-\frac{1}{2}$ **d** $x=-\frac{1}{2}$
e $x=-1\frac{1}{2}$ **f** $x=-\frac{1}{2}$ **g** $x=-\frac{1}{3}$ **h** $x=\frac{5}{3}$
i $x=\frac{1}{4}$ **j** $x=\frac{7}{2}$ **k** $x=-2$ **l** $x=-4$
m $x=0$ **n** $x=\frac{5}{2}$ **o** $x=-2$ **p** $x=-6$

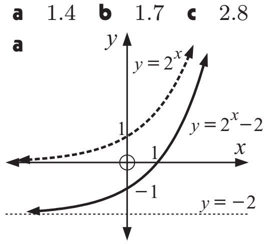
3 **a** $x=\frac{1}{7}$ **b** has no solutions **c** $x=2\frac{1}{2}$

4 **a** $x=3$ **b** $x=3$ **c** $x=2$ **d** $x=2$ **e** $x=-2$
f $x=-2$

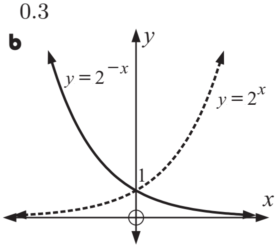
EXERCISE 3G

1 **a** 1.4 **b** 1.7 **c** 2.8 **d** 0.3

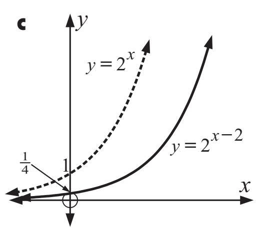
2 **a**



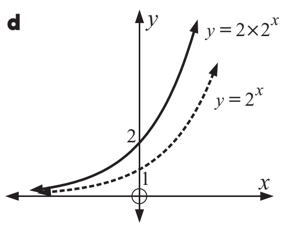
b



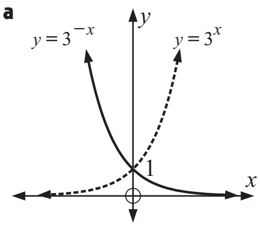
c



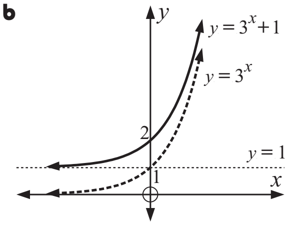
d



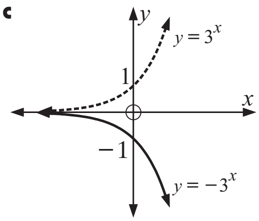
4 **a**



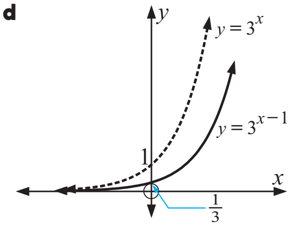
b



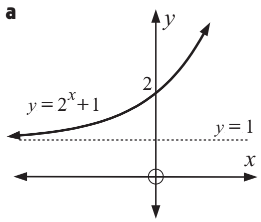
c



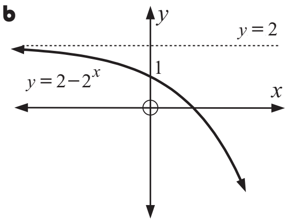
d



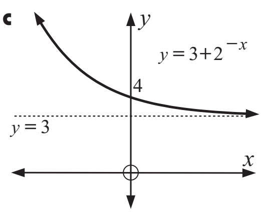
5 **a**



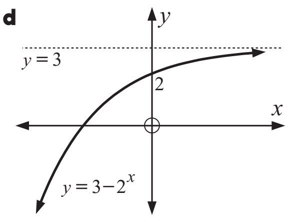
b



c

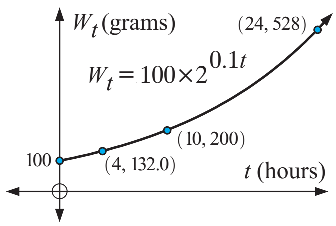


d

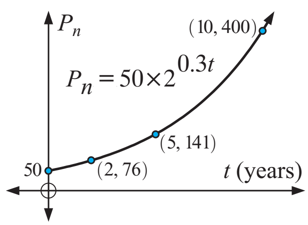


EXERCISE 3H

1 **a** 100 grams
b **i** 132 g
ii 200 g
iii 528 g
c



2 **a** 50
b **i** 76
ii 141
iii 400
c



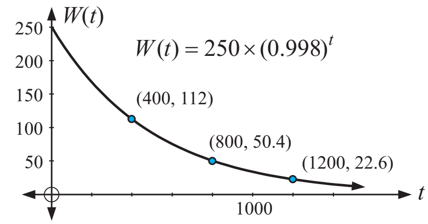
3 **a** V_0 **b** $2V_0$ **c** 100% **d** 183% increase, percentage increase at 50°C compared with 20°C

4 **a** 12 bears **b** 146 bears **c** 248% increase

EXERCISE 3I

1 **a** 250 g **b** **i** 112 g **ii** 50.4 g **iii** 22.6 g

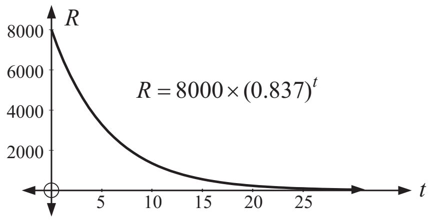
c



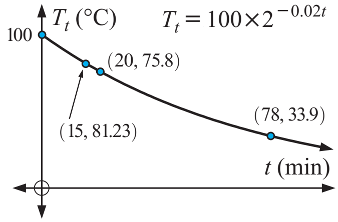
d $\doteqdot$ 346 years

2 **1** 8000 rabbits **3** $\doteqdot$ 26 weeks **4** no

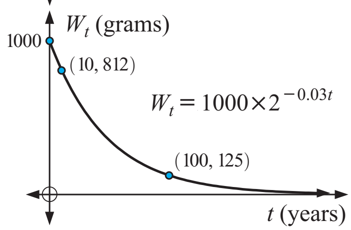
5



3 **a** 100^oC
b **i** 81.2^oC
ii 75.8^oC
iii 33.9^oC
c



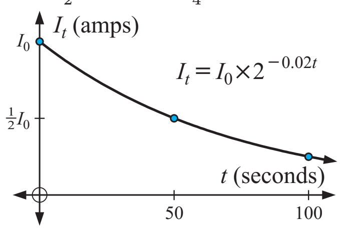
4 **a** 1000 g
b **i** 812 g
ii 125 g
iii 9.31×10^{-7} g
c



5 **a** W_0 **b** 12.9%

6 **a** I_0 **b** $0.986 I_0$ **c** 1.38% decrease

d $I_{50} = \frac{1}{2} I_0$ $I_{100} = \frac{1}{4} I_0$



REVIEW SET 3A

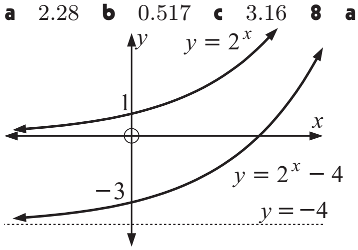
1 **a** -1 **b** 27 **c** $\frac{2}{3}$ **2** **a** $a^6 b^7$ **b** $\dfrac{2}{3x}$ **c** $\dfrac{y^2}{5}$

3 **a** 2^{-3} **b** 2^7 **c** 2^{12} **4** **a** $\dfrac{1}{b^3}$ **b** $\dfrac{1}{ab}$ **c** $\dfrac{a}{b}$

5 **a** $x = -2$ **b** $x = \frac{3}{4}$ **6** **a** 4 **b** $\frac{1}{9}$

7 **a** 2.28 **b** 0.517 **c** 3.16 **8** **a** 3 **b** 24 **c** $\frac{3}{4}$

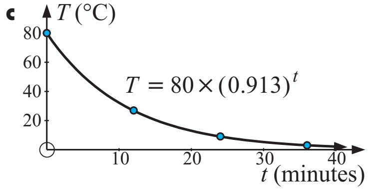
9



a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$

b $y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$

10 **a** 80^oC
b **i** 26.8^oC
ii 9.00^oC
iii 3.02^oC
d $\doteqdot$ 12.8 min
c



REVIEW SET 3B

1 **a** 8 **b** $-\frac{4}{5}$ **2** **a** a^{21} **b** $p^4 q^6$ **c** $\dfrac{4b}{a^3}$

3 **a** 2^{-4} **b** 2^{x+2} **c** 2^{2x-3}

4 **a** $\dfrac{1}{x^5}$ **b** $\dfrac{2}{a^2 b^2}$ **c** $\dfrac{2a}{b^2}$ **5** **a** $x = 4$ **b** $x = -\frac{2}{5}$

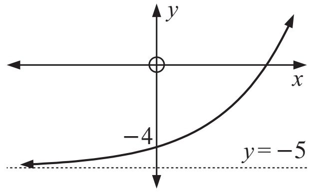
6 **a** 3^4 **b** 3^0 **c** 3^{-3} **d** 3^{-5} **7** **a** 3^{3-2a} **b** $3^{\frac{5}{2} - \frac{9}{2}x}$

8 **a**

x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

b as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to -5$ (above)

c



d $y = -5$

9 **a** $x = \frac{1}{3}$ **b** $x = -\frac{4}{5}$ **10** $x = \frac{1}{2}$, $y = 3$

REVIEW SET 3C

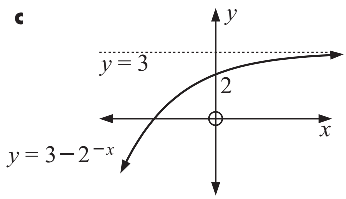
1 **a** 2^{-2} **b** 2^5 **c** $2^{-\frac{1}{2}}$ **d** $2^{\frac{3}{2}}$ **2** **a** 2^{2a+3b} **b** 2^{-3x-3}

3 **a**

x	-2	1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

b as $x \to \infty$, $y \to 3$ (below); as $x \to -\infty$, $y \to -\infty$

c



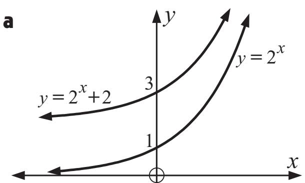
d $y = 3$

4 **a** $x = -\frac{1}{6}$ **b** $x = -\frac{4}{3}$

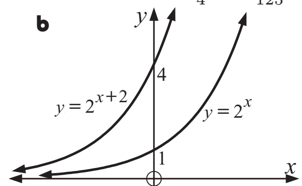
5 **a** $\dfrac{m}{n^2}$ **b** $\dfrac{1}{m^3 n^3}$ **c** $\dfrac{m^2 p^2}{n}$ **d** $\dfrac{16 n^2}{m^2}$

6 **a** $5^{\frac{1}{4}}$ **b** $3^{\frac{3}{2}}$ **c** $2^{-\frac{4}{3}}$ **d** $5^{-\frac{3}{2}}$ **7** **a** 16 **b** 81 **c** $\frac{1}{4}$ **d** $\frac{1}{125}$

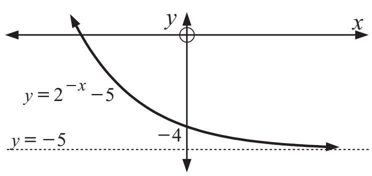
8 **a**



b



9



10 $x = 0$, $y = 2$

REVIEW SET 3D

1 **a** 2^{n+2} **b** $-\frac{6}{7}$ **c** $3\frac{3}{8}$ **d** $\frac{4}{a^2b^4}$ **2** **a** $288 = 2^5 \times 3^2$ **b** 2^{2x}

3 **a** 5^0 **b** $5^{\frac{3}{2}}$ **c** $5^{-\frac{1}{4}}$ **d** 5^{2a+6}

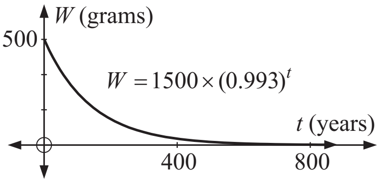
4 **a** -4 **b** $\frac{1}{4a^6}$ **c** $-\frac{b^3}{27}$

5 **a** $1+e^{2x}$ **b** $2^{2x}+10\times 2^x+25$ **c** $x-49$

6 **a** $9-6\times 2^a+2^{2a}$ **b** $x-4$ **c** 2^x+1

7 **a** $x=5$ **b** $x=-4$

8 **a** 1500 g **c**



b **i** 90.3 g **ii** 5.4 g

d 386 years

EXERCISE 4A

1 **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$ **d** $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$ **f** $3^{1.5} = \sqrt{27}$

2 **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$ **c** $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$ **f** $\log_3(\frac{1}{27}) = -3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 3 **e** 6 **f** 7 **g** 2 **h** 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$ **m** 5 **n** $\frac{1}{3}$ **o** n **p** $\frac{1}{3}$ **q** -1 **r** $\frac{3}{2}$ **s** 0 **t** 1

4 **a** $\doteq 2.18$ **b** $\doteq 1.40$ **c** $\doteq 1.87$ **d** $\doteq -0.0969$

5 **a** $x=8$ **b** $x=2$ **c** $x=3$ **d** $x=14$

6 **a** 2 **b** -1 **c** -3 **d** $-\frac{1}{2}$ **e** $\frac{2}{3}$ **f** $\frac{3}{2}$ **g** $\frac{5}{2}$ **h** $-\frac{1}{3}$

EXERCISE 4B

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$ **g** $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$ **m** n **n** $a+2$ **o** $1-m$ **p** $a-b$

2 **a** log 10 000 ENTER **b** log 0.001 ENTER

c log 2nd √ 10)) ENTER

d log 10 ^ (1 ÷ 3)) ENTER

e log 100 ^ (1 ÷ 3)) ENTER

f log 10 × 2nd √ 10)) ENTER

g log 1 ÷ 2nd √ 10)) ENTER

h log 1 ÷ 10 ^ 0.25) ENTER

3 **a** $10^{0.7782}$ **b** $10^{1.7782}$ **c** $10^{3.7782}$ **d** $10^{-0.2218}$ **e** $10^{-2.2218}$ **f** $10^{1.1761}$ **g** $10^{3.1761}$ **h** $10^{0.1761}$ **i** $10^{-0.8239}$ **j** $10^{-3.8239}$

4 **a** **i** 0.477 **ii** 2.477 **b** $\log 300 = \log(3\times 10^2)$

5 **a** **i** 0.699 **ii** -1.301 **b** $\log 0.05 = \log(5\times 10^{-2})$

6 **a** 100 **b** 10 **c** 1 **d** $\frac{1}{10}$ **e** $10^{\frac{1}{2}}$ **f** $10^{-\frac{1}{2}}$ **g** 10 000 **h** 0.000 01 **i** 6.84 **j** 140 **k** 0.0419 **l** 0.000 631

EXERCISE 4C

1 **a** $\log 16$ **b** $\log 4$ **c** $\log 8$ **d** $\log 20$ **e** $\log 2$ **f** $\log 24$ **g** $\log 30$ **h** $\log 0.4$ **i** $\log 10$ **j** $\log 200$ **k** $\log 0.4$ **l** $\log 1$ **m** $\log 0.005$ **n** $\log 20$ **o** $\log 28$

2 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log\left(\frac{25}{8}\right)$ **e** $\log 6$ **f** $\log\frac{1}{2}$ **g** $\log 20$ **h** $\log 25$ **i** 1

3 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

5 **a** $\log y = x\log 2$ **b** $\log y \doteq 1.301 + 3\log b$ **c** $\log M = \log a + 4\log d$ **d** $\log T \doteq 0.6990 + \frac{1}{2}\log d$ **e** $\log R = \log b + \frac{1}{2}\log l$ **f** $\log Q = \log a - n\log b$ **g** $\log y = \log a + x\log b$ **h** $\log F \doteq 1.301 - \frac{1}{2}\log n$ **i** $\log L = \log a + \log b - \log c$ **j** $\log N = \frac{1}{2}\log a - \frac{1}{2}\log b$ **k** $\log S \doteq 2.301 + 0.301t$ **l** $\log y = m\log a - n\log b$

6 **a** $D = 2e$ **b** $F = \frac{5}{t}$ **c** $P = \sqrt{x}$ **d** $M = b^2c$ **e** $B = \frac{m^3}{n^2}$ **f** $N = \frac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $Q = \frac{100}{x}$

7 **a** $p+q$ **b** $2p+3q$ **c** $2q+r$ **d** $r+\frac{1}{2}q-p$ **e** $r-5p$ **f** $p-2q$

8 **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$ **e** $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{2}y-3x$

9 **a** 0.86 **b** 2.15 **c** 1.075

10 **a** $x=9$ **b** $x=2$ or 4 **c** $x=25\sqrt{5}$ **d** $x=200$ **e** $x=5$ **f** $x=3$

EXERCISE 4D

1 **a** $x \doteq 3.32$ **b** $x \doteq 2.73$ **c** $x \doteq 3.32$ **d** $x \doteq 37.9$ **e** $x \doteq -3.64$ **f** $x \doteq -7.55$ **g** $x \doteq 7.64$ **h** $x \doteq 32.0$ **i** $x \doteq 1150$

2 **a** $t \doteq 6.34$ **b** $t \doteq 74.9$ **c** $t \doteq 8.38$ **d** $t \doteq 133$ **e** $t \doteq 122$ **f** $t \doteq 347$

EXERCISE 4E

1 **a** 3.90 h **b** 15.5 h **2** **a** 66.7 min **b** 222 min

3 **a** 25 years **b** 141 years **c** 166 years

4 **a** 10 000 years **b** 49 800 years **5** 15.9°C

6 166 seconds **7** 11.6 seconds

EXERCISE 4F

1 6.17 years, i.e., 6 years 63 days

2 8.65 years, i.e., 8 years 237 days

3 **a** $\frac{8.4\%}{12} = 0.7\% = 0.007$ $r = 1 + 0.007 = 1.007$

b $\therefore$ after 74 months

EXERCISE 4G

1 **a** $\doteq 2.26$ **b** $\doteq -10.3$ **c** $\doteq -2.46$ **d** $\doteq 5.42$

2 **a** $x \doteq -4.29$ **b** $x \doteq 3.87$ **c** $x \doteq 0.139$

3 **a** $x \doteq 0.683$ **b** $x \doteq -1.89$

4 **a** $x = 16$ **b** $x \doteq 1.71$ **5** $x = \frac{\log 8}{\log 25}$ or $\log_{25} 8$

REVIEW SET 4A

1 **a** 3 **b** 8 **c** -2 **d** $\frac{1}{2}$ **e** 0 **f** 1 **g** $\frac{1}{4}$ **h** -1 **i** $\frac{1}{3}$ **j** $\frac{1}{2}$ **k** $\frac{5}{2}$ **l** -2

2 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a+b+1$

3 **a** $10^{1.51}$ **b** $10^{-2.89}$ **c** $10^{-4.05}$

4 **a** $x = 0.001$ **b** $x \doteq 553$ **c** $x \doteq 0.000\,716$

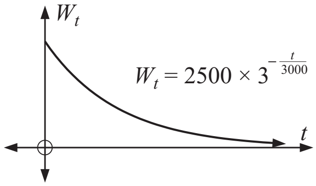
5 **a** 2 **b** $\log 6$ **c** $\log 4$

6 **a** $\log P = \log 3 + x\log b$ **b** $\log m = 3\log n - 2\log p$

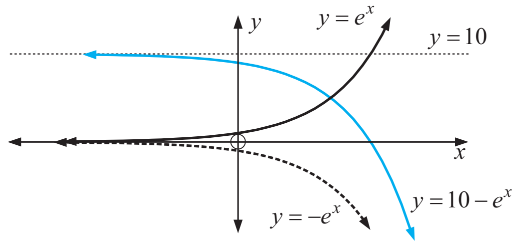
7 **a** $k = 50\times 10^x$ **b** $Q = P^3R$ **c** $A = \frac{B^5}{400}$

8 **a** $x \doteq 1.21$ **b** $x \doteq 1.82$ **c** $x \doteq 0.980$

9 **a** 2500 g **b** 3290 years **c** 42.3% **d**

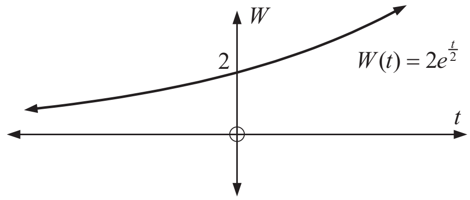


11

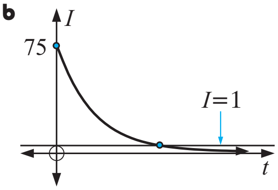


Domain of f, g and h is $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$ Range of g is $\{y : y < 0\}$
Range of h is $\{y : y < 10\}$

12 **a** **i** 2 g **ii** 2.57 g **iii** 4.23 g **iv** 40.2 g
b

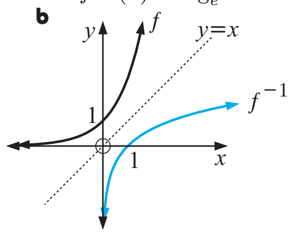


13 **a** **i** 64.5 amps **ii** 16.7 amps
b



c 28.8 sec

14 **a** $f^{-1}(x) = \log_e x$
b



REVIEW SET 4B

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a+b$ **2** **a** $10^{2.30}$ **b** $10^{5.57}$ **c** $10^{-3.43}$

3 **a** $x = 1000$ **b** $x \doteq 52.0$ **c** $x \doteq 2.13$

4 **a** $\log 144$ **b** $\log \frac{16}{9}$ **c** $\log 500$

5 **a** $\log M = \log a + n \log b$ **b** $\log T \doteq 0.6990 - \frac{1}{2}\log l$
c $\log G = 2\log a + \log b - \log c$

6 **a** $T = \dfrac{x^2}{y}$ **b** $K = n\sqrt{t}$

7 **a** $x \doteq 5.19$ **b** $x \doteq 4.29$ **c** $x \doteq -0.839$

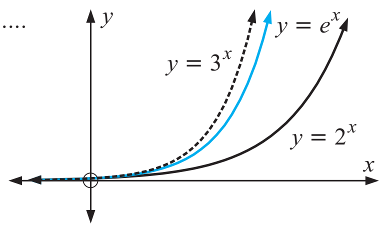
8 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$ **d** $4B - 2A$
e $3A - 2B$

9 **a** 3 years **b** 152% **c** 4.75 years

EXERCISE 5A

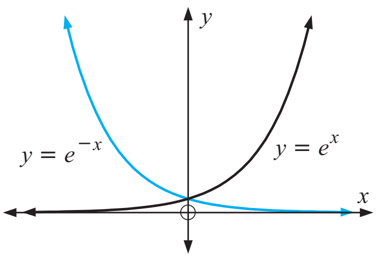
1 $e^1 \doteq 2.718\,281\,828\ldots$

2



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

3 One is the other reflected in the y-axis.



4 *a*

5 **a** $e^x > 0$ for all x **b** **i** $0.000\,000\,004\,12$ **ii** $970\,000\,000$

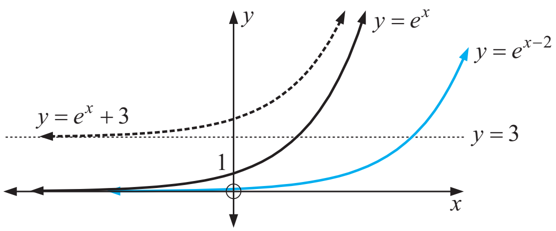
6 **a** $\doteq 7.39$ **b** $\doteq 20.1$ **c** $\doteq 2.01$ **d** $\doteq 1.65$ **e** $\doteq 0.368$

7 **a** $e^{\frac{1}{2}}$ **b** $e^{\frac{3}{2}}$ **c** $e^{-\frac{1}{2}}$ **d** e^{-2}

8 **a** $e^{0.18t}$ **b** $e^{0.004t}$ **c** $e^{-0.005t}$ **d** $\doteq e^{-0.167t}$

9 **a** 10.074 **b** 0.099 261 **c** 125.09 **d** 0.007 994 5
e 41.914 **f** 42.429 **g** 3540.3 **h** 0.006 342 4

10



Domain of f, g and h is $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$ Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > 3\}$

EXERCISE 5B

1 **a** 3 **b** 0 **c** $\frac{1}{3}$ **d** -2

3 x does not exist such that $e^x = -2$ or 0

4 **a** 0.693 **b** 2.30 **c** 7.68 **d** -6.39

5 **a** a **b** $a+1$ **c** $n-1$ **d** $a+b$ **e** ab **f** $a-b$

6 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$ **e** $e^{-5.1160}$
f $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$ **i** $e^{-1.8971}$ **j** $e^{-8.8049}$

7 **a** $x \doteq 20.1$ **b** $x \doteq 2.72$ **c** $x = 1$ **d** $x \doteq 0.368$
e $x \doteq 0.006\,74$ **f** $x \doteq 2.30$ **g** $x \doteq 8.54$ **h** $x \doteq 0.0370$

EXERCISE 5C

1 **a** $\ln 16$ **b** $\ln 4$ **c** $\ln 8$ **d** $\ln 20$ **e** $\ln 2$ **f** $\ln 24$
g $\ln 3e$ **h** $\ln \frac{4}{e}$ **i** $\ln 10$ **j** $\ln 2e^2$ **k** $\ln \frac{40}{e^2}$ **l** $\ln 1$

2 **a** $\ln 96$ **b** $\ln 72$ **c** $\ln 8$ **d** $\ln \frac{25}{8}$ **e** $\ln 6$ **f** $\ln \frac{1}{2}$
g $\ln \frac{1}{2}$ **h** $\ln 3$ **i** $\ln 16$

3 **a** e.g., $\ln 9 = \ln 3^2 = 2\ln 3$

5 **a** $D = ex$ **b** $F = \dfrac{e^2}{p}$ **c** $P = \sqrt{ex}$ **d** $M = e^3y^2$
e $B = \dfrac{t^3}{e}$ **f** $N = \dfrac{1}{3\sqrt{g}}$ **g** $Q \doteq 8.66x^3$ **h** $D \doteq 0.518n^{0.4}$

EXERCISE 5D

1 **a** $x \doteq 2.303$ **b** $x \doteq 6.908$ **c** $x \doteq -4.754$
d $x \doteq 3.219$ **e** $x \doteq 15.18$ **f** $x \doteq -40.85$
g $x \doteq -14.63$ **h** $x \doteq 137.2$ **i** $x \doteq 4.868$

EXERCISE 5E

1 **a** 1.488 h (1 h 29 min) **b** 10.730 h (10 h 44 min)

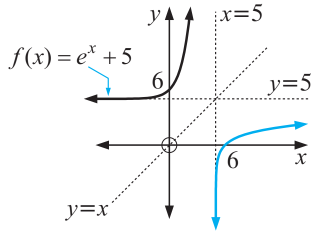
2 **a** 40.13 minutes (40 minutes 8 seconds)
b 153.51 minutes (2 hours 33 mins 30 seconds)

3 **a** 17.329 years **b** 92.222 years **c** 115.129 years

4 8.047 sec **5** 21.320 min

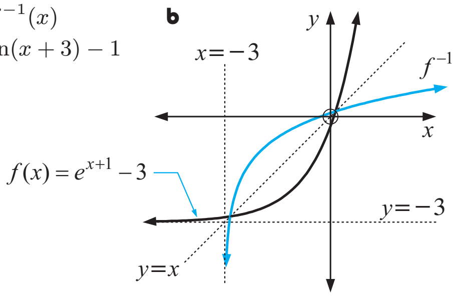
EXERCISE 5F

1 **a** $f^{-1}(x) = \ln(x-5)$ **b**



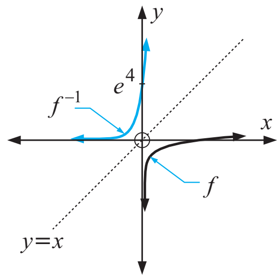
c domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 5\}$
domain of f^{-1} is $\{x : x > 5\}$, range is $\{y : y \in \mathcal{R}\}$

2 **a** $f^{-1}(x) = \ln(x+3) - 1$ **b**



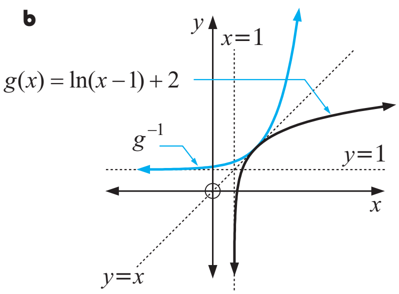
c domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > -3\}$
domain of f^{-1} is $\{x : x > -3\}$, range is $\{y : y \in \mathcal{R}\}$

3 **a** $f^{-1}(x) = e^{x+4}$ **b**



c domain of f is $\{x : x > 0\}$, range of f is $\{y : y \in \mathcal{R}\}$
domain of f^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 0\}$

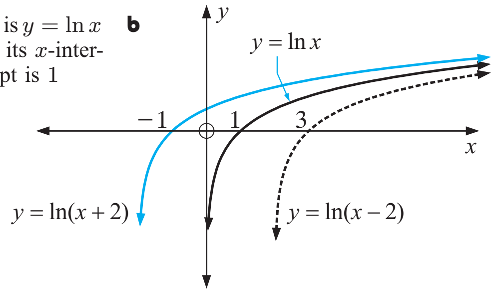
4 **a** $g^{-1}(x) = 1 + e^{x-2}$ **b**



c domain of g is $\{x : x > 1\}$, range is $\{y : y \in \mathcal{R}\}$
domain of g^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 1\}$

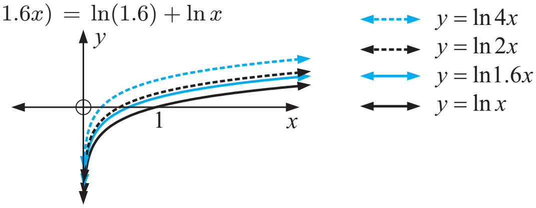
5 $f^{-1}(x) = \frac{1}{2}\ln x$ **a** $\frac{1}{2}\ln(2x-1)$ **b** $\frac{1}{2}\ln\left(\frac{x+1}{2}\right)$

6 **a** A is $y = \ln x$ **b**
as its x-intercept is 1



7 $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

8
$$\ln(2x) = \ln 2 + \ln x$$
$$\ln(4x) = \ln 4 + \ln x$$
$$\ln(1.6x) = \ln(1.6) + \ln x$$

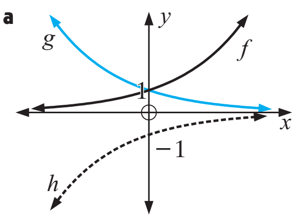


$\therefore$ $y = \ln(2x)$ is $\ln 2$ units above $y = \ln x$
$y = \ln(4x)$ is $\ln 4$ units above $y = \ln x$
$y = \ln(1.6x)$ is $\ln(1.6)$ units above $y = \ln x$

REVIEW SET 5A

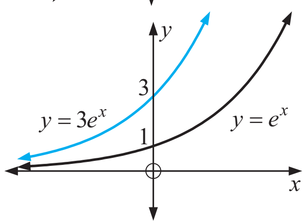
1 **a** $\doteq 54.6$ **b** $\doteq 22.2$ **c** $\doteq 0.0613$ **d** $\doteq 6.07$

2 **a**

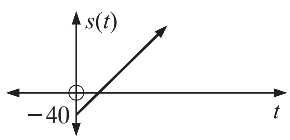


b **i** g is the reflection of f in the y-axis
ii h is the reflection of g in the x-axis

3



4 **a** **i** -40 m **ii** 585 m **iii** 2400 m **b**



5 **a** 5 **b** $\frac{1}{2}$ **c** -1 **6** **a** $2x$ **b** $2+x$ **c** $1-x$

7 **a** $x \doteq 148$ **b** $x \doteq 0.513$

8 **a** $\ln 24$ **b** $\ln 3$ **c** $\ln 4$ **d** $\ln 125$

9 **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

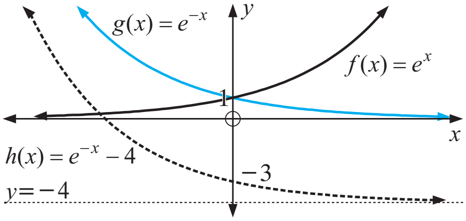
10 **a** $x \doteq 5.99$ **b** $x \doteq 0.699$ **c** $x \doteq 6.80$
d $x \doteq 1.10$ or 1.39

REVIEW SET 5B

1 **a** e^{-3} **b** $e^{-1.5}$ **c** $e^{4.5}$ **d** $\doteq e^{1.65}$

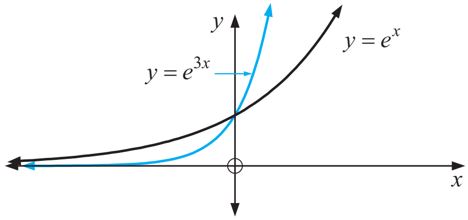
2 **a** $\doteq 26.9401$ **b** $\doteq 0.109\,447$

3



Each function has domain $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$, Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > -4\}$

4



5 **a** $\frac{3}{2}$ **b** -3 **c** $-\frac{3}{2}$

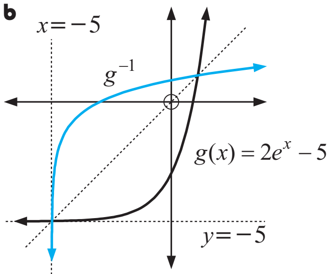
6 **a** $\doteq e^{3.00}$ **b** $\doteq e^{8.01}$ **c** $\doteq e^{-2.59}$

7 **a** $\ln 144$ **b** $\ln(\frac{3}{2})$ **c** $\ln(\frac{25}{e})$ **d** $\ln 3$

8 **a** $P = TQ^{1.5}$ **b** $M = \dfrac{e^{1.2}}{\sqrt{N}}$

9 **a** $g^{-1}(x)$
$= \ln\left(\dfrac{x+5}{2}\right)$

b



c domain of g is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > -5\}$
domain of g^{-1} is $\{x : x > -5\}$, range is $\{y : y \in \mathcal{R}\}$

10 **a** 13.9 weeks **b** 41.6 weeks **c** 138 weeks

EXERCISE 6A

1 **a** $2x$ **b** $x + 2$ **c** $\dfrac{x}{2}$ **d** $2x + 3$

2 **a** $9x^2$ **b** $\dfrac{x^2}{4}$ **c** $3x^2$ **d** $2x^2 - 4x + 7$

3 **a** $64x^3$ **b** $4x^3$ **c** $x^3 + 3x^2 + 3x + 1$
d $2x^3 + 6x^2 + 6x - 1$

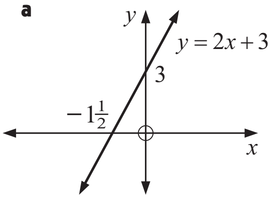
4 **a** $|2x| + 3$ **b** $|x|$ **c** $|x - 2|$ **d** $|x + 1| + 2$

5 **a** 4^x **b** $2^{-x} + 1$ **c** $2^{x-2} + 3$ **d** $2^{x+1} + 3$

6 **a** $-\dfrac{1}{x}$ **b** $\dfrac{2}{x}$ **c** $\dfrac{2 + 3x}{x}$ **d** $\dfrac{2x + 1}{x - 1}$

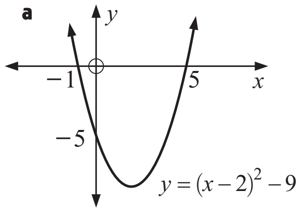
EXERCISE 6B

1 **a**



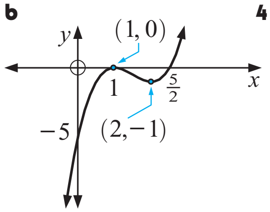
b **i** $-1\frac{1}{2}$ **ii** 3
iii 2

2 **a**

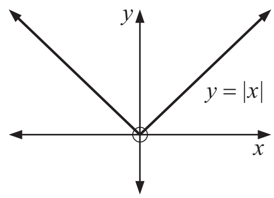


b **i** -1 and 5
ii -5

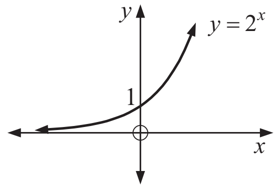
3 **a, b**



4

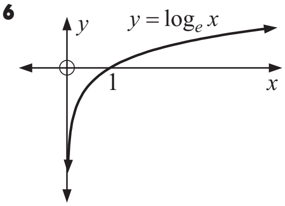


5

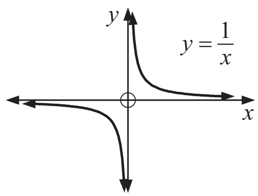


When $x = 0$,
$y = 2^0 = 1$ ✓
$2^x > 0$ for all x as the graph is always above the y-axis. ✓

6

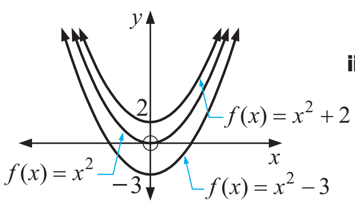


7



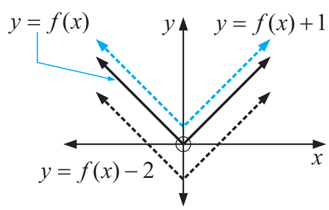
EXERCISE 6C.1

1 **a, b**

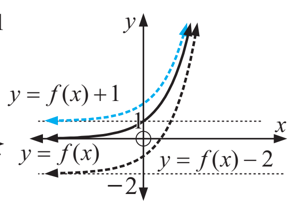


c **i** If $b > 0$, the function is translated vertically upwards through b units.
ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

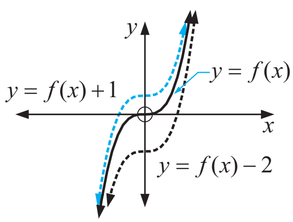
2 **a**



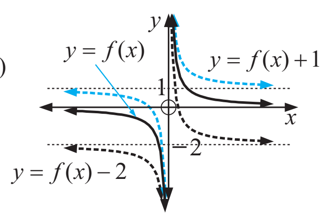
b



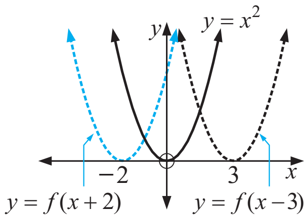
c



d



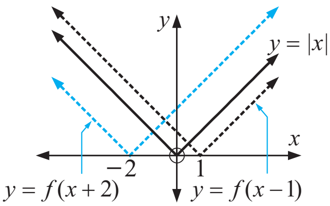
3 **a**



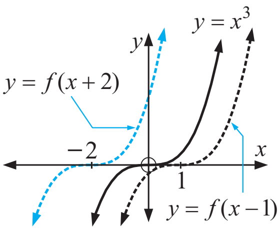
b **i** If $a > 0$, the graph is translated a units right.

ii If $a < 0$, the graph is translated $|a|$ units left.

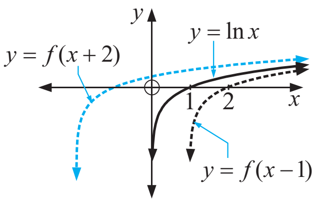
4 **a**



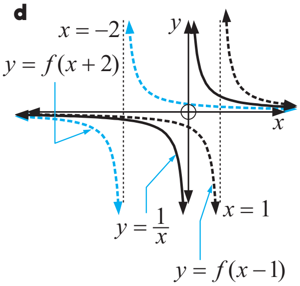
b



c

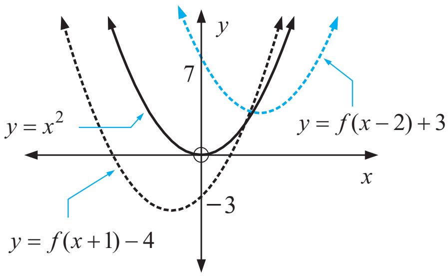


d

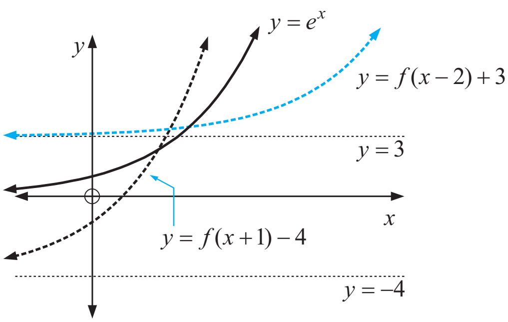


$y = f(x-a)$ is a horizontal translation of $y = f(x)$ through $\begin{bmatrix} a \\ 0 \end{bmatrix}$.

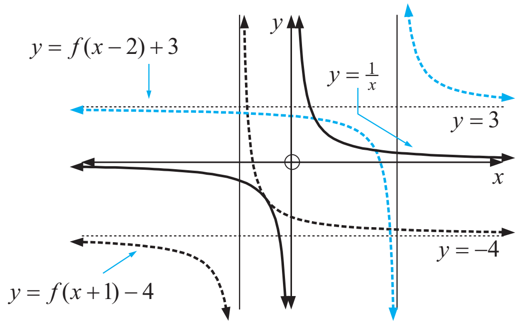
5 **a**



b

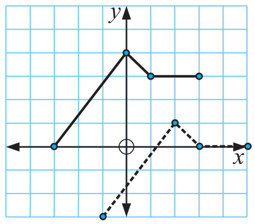


c

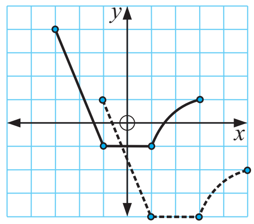


6 A translation of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

a

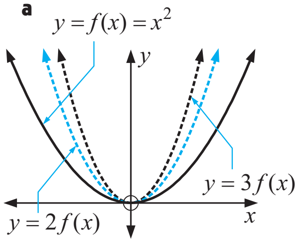


b

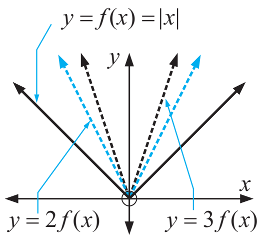


EXERCISE 6C.2

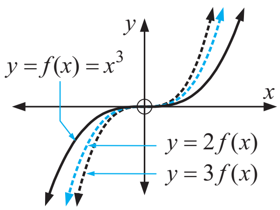
1 **a**



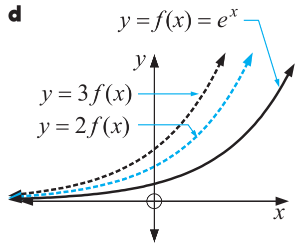
b



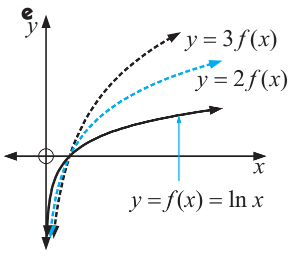
c



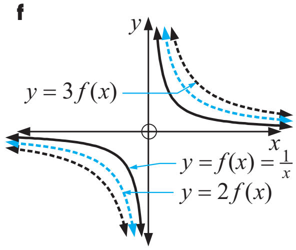
d



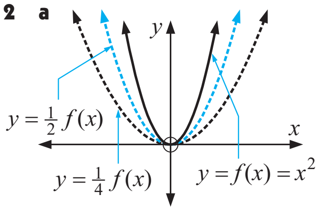
e



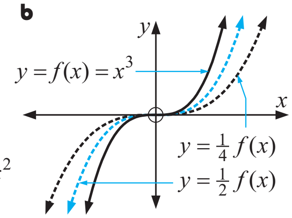
f



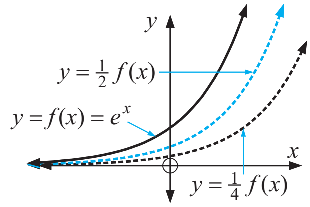
2 **a**



b

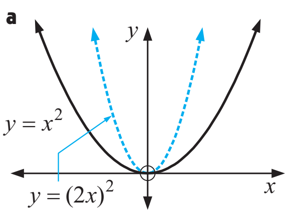


c

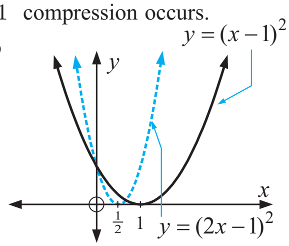


3 p affects the vertical stretching or compressing of the graph of $y = f(x)$ by a factor of p. If $p > 1$ stretching occurs. If $0 < p < 1$ compression occurs.

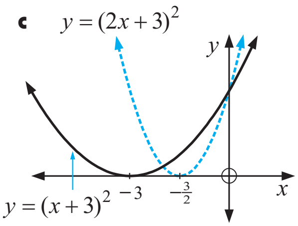
4 **a**



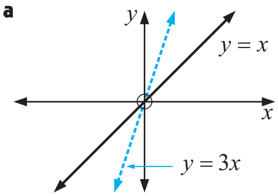
b



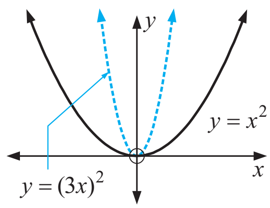
c



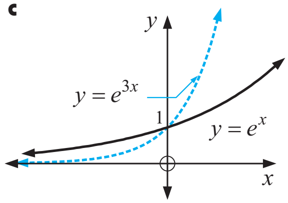
5 **a**



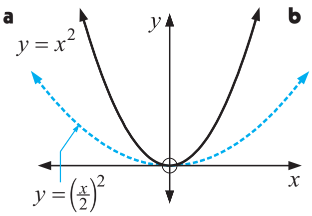
b



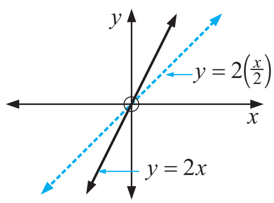
c



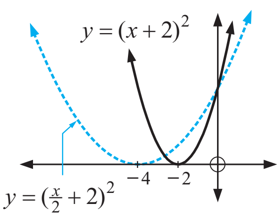
6 **a**



b



c

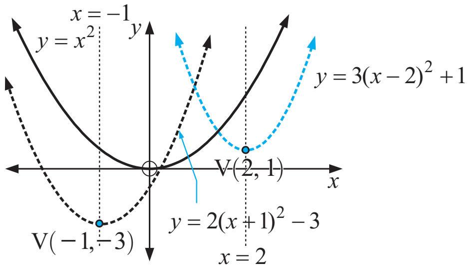


7 k affects the horizontal stretching or compressing of $y = f(x)$ by a factor of $\frac{1}{k}$.

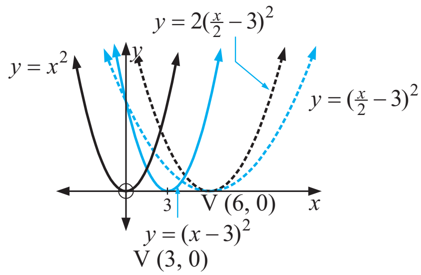
If $k > 1$ it moves closer to the y-axis.

If $0 < k < 1$ it moves further from the y-axis.

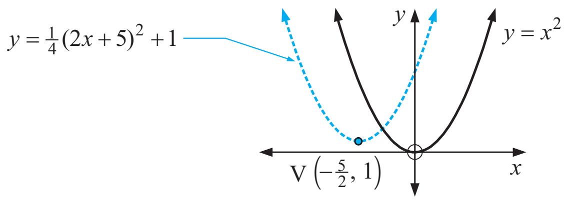
8 **a**



b

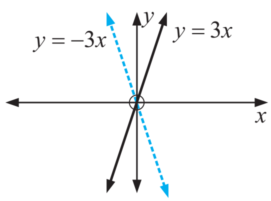


c

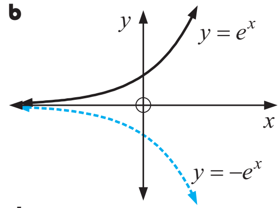


EXERCISE 6C.3

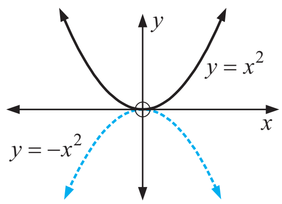
1 **a**



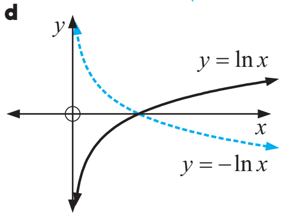
b



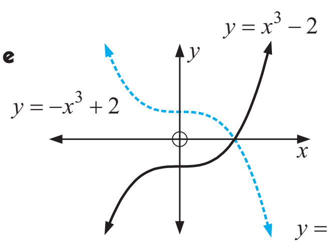
c



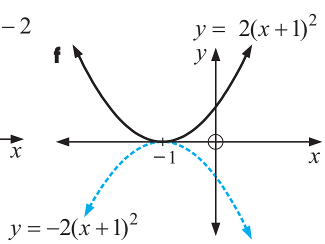
d



e



f

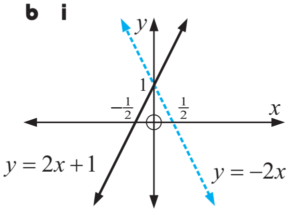


2 $y=-f(x)$ is the reflection of $y=f(x)$ in the x-axis.

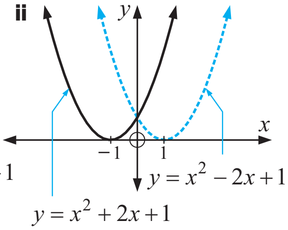
3 **a** **i** $f(-x)=-2x+1$ **ii** $f(-x)=x^2-2x+1$

iii $f(-x)=|-x-3|$

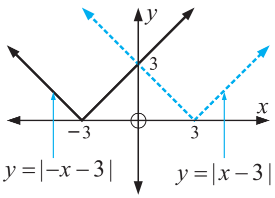
b **i**



ii



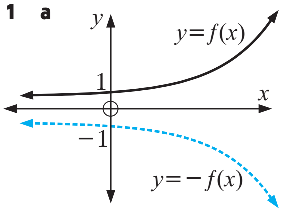
iii



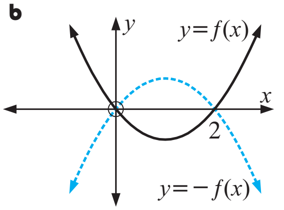
4 $y=f(-x)$ is the reflection of $y=f(x)$ in the y-axis.

EXERCISE 6D

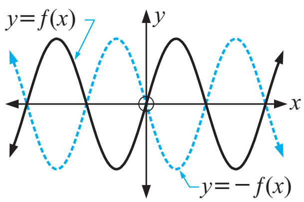
1 **a**



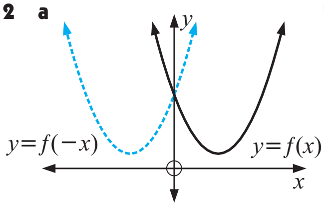
b



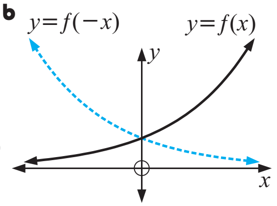
c



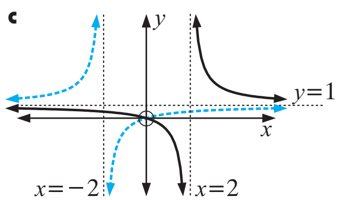
2 **a**



b

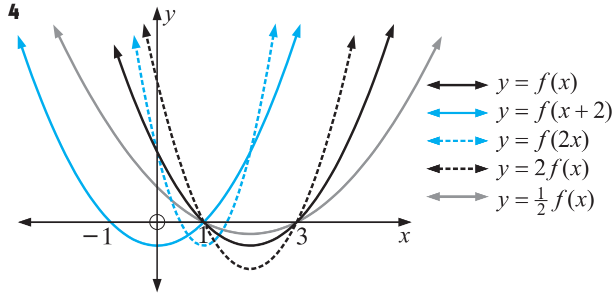


c

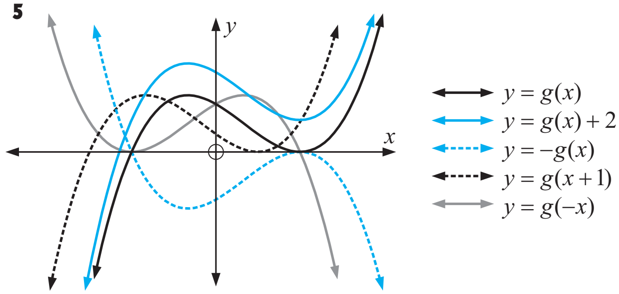


3 **a** A **b** B **c** D **d** C

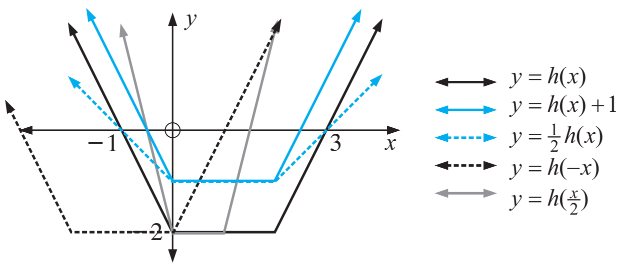
4



5



6



REVIEW SET 6

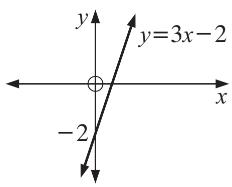
1 **a** linear **b** exponential **c** reciprocal **d** modulus
e quadratic **f** logarithmic

2 **a** 3 **b** 8 **c** $4x^2-4x$ **d** x^2+2x **e** $3x^2-6x-2$

3 **a** -15 **b** 5 **c** $-x^2+x+5$ **d** $5-\frac{1}{2}x-\frac{1}{4}x^2$
e $-x^2-3x+5$

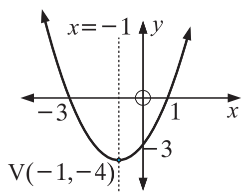
4 **a** -1 **b** $\dfrac{2}{x}$ **c** $\dfrac{8}{x}$ **d** $\dfrac{10-3x}{x+2}$

5 **a**



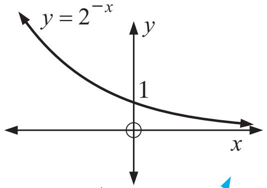
b **i** $\frac{2}{3}$ **ii** -2 **iii** 3
c **i** -1.1 **ii** 0.9

6 **a**



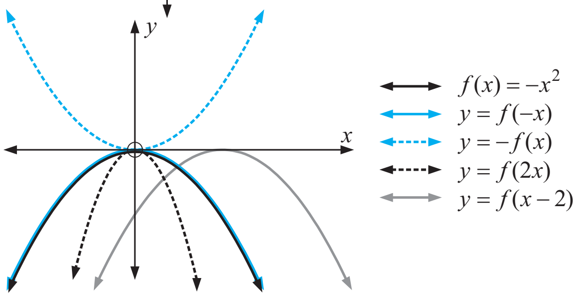
b **i** 1 and -3 **ii** -3
c V(-1, -4)

7 **a**

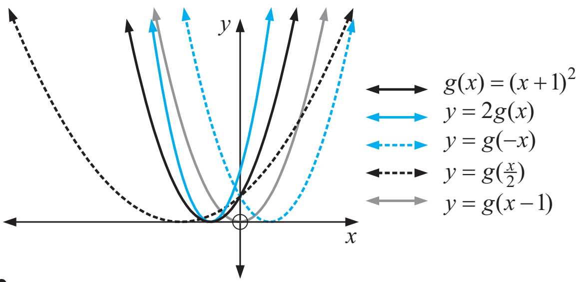


b **i** true **ii** false **iii** false **iv** true

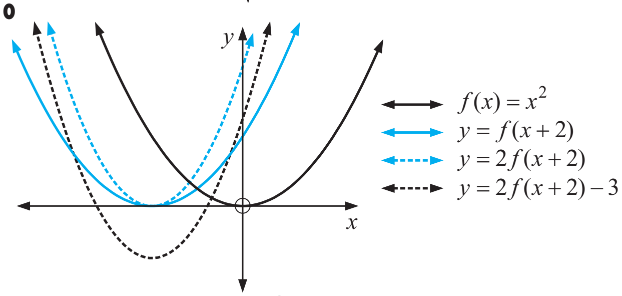
8



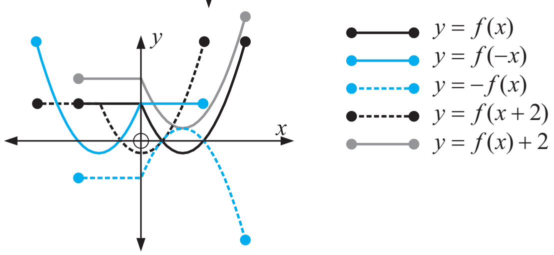
9



10



11



EXERCISE 7A

1 **a** $\sqrt{13}$ units **b** $\sqrt{34}$ units **c** $\sqrt{20}$ units **d** $\sqrt{10}$ units

2 **a** $(2, 3)$ **b** $(2, -1)$ **c** $(3\frac{1}{2}, 1\frac{1}{2})$ **d** $(2, -2\frac{1}{2})$

3 **a** $\frac{2}{3}$ **b** $-\frac{2}{5}$ **c** 1 **d** 0 **e** -4 **f** undefined

4 **a** 2 **b** 3 **c** undefined **d** 0 **e** -4 **f** $\frac{1}{4}$

5 **a** parallel, slopes $\frac{2}{5}$ **b** parallel, slopes $-\frac{1}{7}$
c perpendicular, slopes $\frac{1}{2}, -2$ **d** neither, slopes $-4, \frac{1}{3}$
e neither, slopes $\frac{2}{7}, \frac{1}{5}$ **f** perpendicular, slopes $2, -\frac{1}{2}$

6 **a** $-\frac{4}{3}$ **b** $-\frac{3}{11}$ **c** $-\frac{1}{4}$ **d** 3 **e** $\frac{1}{5}$ **f** undefined

7 **a** $y=2x-7$ **b** $y=-2x+4$ **c** $y=3x-15$
d $y=-3x+4$ **e** $y=-4x+9$ **f** $y=x+5$

8 **a** $3x-2y=4$ **b** $3x+2y=11$ **c** $x-3y=4$
d $4x+y=6$ **e** $3x-y=0$ **f** $4x+9y=-2$

9 **a** $x-3y=-3$ **b** $5x-y=1$ **c** $x-y=3$
d $4x-5y=10$ **e** $x-2y=-1$ **f** $2x+3y=-5$

10 **a** $y=-2$ **b** $x=6$ **c** $x=-3$

11

	Equation of line	Slope	x-int.	y-int.
a	$2x-3y=6$	$\frac{2}{3}$	3	-2
b	$4x+5y=20$	$-\frac{4}{5}$	5	4
c	$y=-2x+5$	-2	$\frac{5}{2}$	5
d	$x=8$	undef.	8	no y-int.
e	$y=5$	0	no x-int.	5
f	$x+y=11$	-1	11	11
g	$4x+y=8$	-4	2	8
h	$x-3y=12$	$\frac{1}{3}$	12	-4

12 **a** yes **b** no **c** yes

13 **a** $(4, 2)$ **b** $(-2, 3)$ **c** $(-3, 6)$ **d** $(4, 0)$
e parallel lines do not meet **f** coincident lines

EXERCISE 7B.1

1 **a** $y=-4$ **b** $x=5$ **c** $x=-1$ **d** $y=2$ **e** $y=0$ **f** $x=0$

2 **a** $3x-4y=-19$ **b** $x-y=7$ **c** $y=3x$
d $2x+3y=19$ **e** $x-2y=5$ **f** $x+3y=13$

3 **a** $x-8y=-83$ **b** $8x+y=41$
c $9x-2y=23$ for $5 \leqslant x \leqslant 7$ **d** $(8, 11\frac{3}{8})$

4 **a** $x-3y=-16$ **b** $2x-y=-3$ **c** $x=5$

EXERCISE 7B.2

1 **a** $x-2y=2$ **b** $2x-3y=-19$ **c** $3x-4y=15$
d $3x-y=11$ **e** $x+3y=13$ **f** $3x+y=4$

2 **a** $-\frac{2}{3}$ **b** $\frac{3}{7}$ **c** $\frac{6}{11}$ **d** $-\frac{5}{6}$ **e** $-\frac{1}{2}$ **f** 3

3 **a** parallel lines have the same slope
b $-\frac{3}{5} \times \frac{5}{3} = -1$ for perpendicular lines

4 **a** $3x+4y=10$ **b** $2x-5y=3$ **c** $3x+y=-12$
d $x-3y=0$

5 **a** $\frac{2}{3}, -\frac{6}{k}$ **b** $k=-9$ **c** $k=4$

EXERCISE 7C

1 **a** AB = AC = $\sqrt{221}$ units **b** all sides are 2 units long
c KL = $\sqrt{68}$ units, KM = $\sqrt{85}$ units, LM = $\sqrt{17}$ units
and $(\sqrt{85})^2 = (\sqrt{68})^2 + (\sqrt{17})^2$, so right angled at L.
d AB = $2\sqrt{10}$ units, BC = $7\sqrt{10}$ units,
AC = $5\sqrt{10}$ units and AB + AC = BC
e AB = $\sqrt{29}$ units, BC = $\sqrt{116}$ units, AC = $\sqrt{145}$ units
and $AC^2 = AB^2 + BC^2$, so right angled at B.

2 $(4\frac{3}{4}, 8)$ **3** **a** $(2\frac{1}{3}, 7)$ **b** 8.03 km **c** yes (6.16 km)

4 **a** $a=-2$ or 14 **b** $a=-6$ or 4 **c** $a=-\frac{3}{5}$ or 1

5 $(13.41, 8)$ or $(-5.41, 8)$

6 **a** $(\frac{13}{16}, 0)$ **b** $(0, \frac{5}{2})$ **c** $(2, 6)$ or $(-1, -3)$

EXERCISE 7D.1

1 A$(5, -3)$, B$(7, -4)$, C$(9, -5)$

2 C$(3, -2)$, D$(5, -5)$, E$(7, -8)$

3 **a** midpoint of AC is $(-2, 0)$, midpoint of BD is $(-2, 0)$
b slope AB = slope DC = 3,
slope AD = slope BC = $-\frac{1}{3}$

4 **a** slope AB $= 2$, slope AC $= -\frac{1}{2}$ **b** $(3, 1)$ **c** $y = 6$

5 **a** D$(9, -1)$ **b** R$(3, 1)$ **c** X$(2, -1)$

EXERCISE 7D.2

1 **a** $x - y = 4$ **b** $2x - y = -6$ **c** $12x - 10y = -35$
d $y = 1$

2 $2x - 3y = -5$

3 **a** $x + 2y = 5$, $3x + y = 10$, $x - 3y = 0$ **b** $(3, 1)$

4 $(3, 3)$ **5** $(149.1, 127.1)$

6 **a** $ax + cy = a^2 + c^2$, $(b - a)x - cy = b^2 - a^2 - c^2$
b $x = b$ **c** RS is the vertical line $x = b$
d ...... intersect at a point.

REVIEW SET 7A

1 5 units **2** $(5, 1\frac{1}{2})$

3 **a** $y = -3x + 5$ **b** $y = -\frac{3}{2}x + \frac{5}{2}$

4 **a** $2x - 3y = 17$ **b** $x - 3y = 11$

5 **a** $(0, 7)$, $(\frac{14}{3}, 0)$ **b** $(0, -4)$, $(\frac{12}{5}, 0)$ **6** yes

7 **a** $4x - 3y = 19$ **b** $4y - x = 6$

8 **a** **i** $y = 2$ **ii** $x = 0$ **iii** $4x - 3y = -18$
b **i** $(-3, 2)$ **ii** $(0, 6)$

9 **a** $k = \frac{3}{4}$ **b** $k = -12$ **10** $x^2 + y^2 - 9x + y + 16 = 0$

REVIEW SET 7B

1 **a** $3x + 5y = 12$ **b** $7x + 2y = 32$

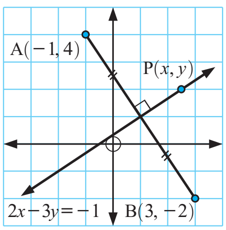
2 **a** $k = -\frac{21}{5}$ **b** $k = \frac{15}{7}$ **3** $(-1, 4\frac{1}{2})$

4 isosceles (KL = LM) $\measuredangle$KLM $= 90^o$ **5** $(\frac{11}{4}, 5)$

6 $k = 4 \pm 2\sqrt{6}$

7 **a** $T_1(0, 2 + 2\sqrt{2})$, $T_2(0, 2 - 2\sqrt{2})$
b $2\sqrt{2}x - y = -2\sqrt{2} - 2$

8 **a**



b $2x - 3y = -1$

9 C$(3, 0)$

10 $(2\frac{1}{2}, 1\frac{1}{2})$

REVIEW SET 7C

1 K$(10, 15)$ **2** $5x - 3y = 4$ **3** $k = 5\frac{1}{3}$

4 A is at $(-12, 7)$ **5** **a** $2x + 3y = -6$ **b** D is at $(-6, 2)$

6 A is at $(2, 1)$ **7** $x - 2y = -9$ **8** **a** T$(5, 1)$ **b** 50 km

9 **a** $\sqrt{41}$ units **b** $-\frac{4}{5}$ **c** $5x - 4y = -13$ **d** $5x - 4y = 69$

10 yes (distance between ships $\doteqdot$ 84.85 km)

EXERCISE 8A

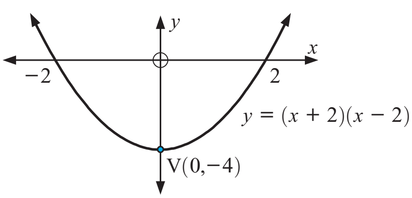
1 **a**, **c**, **d**, **e** **2** **a** $y = 20$ **b** $y = 27$ **c** $y = -4$ **d** $y = 37$

3 **a** 3 **b** -5 **c** -4 **d** 8

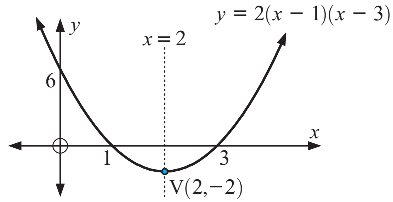
4 **a** no **b** no **c** no **d** no **e** yes **f** yes

EXERCISE 8B.1

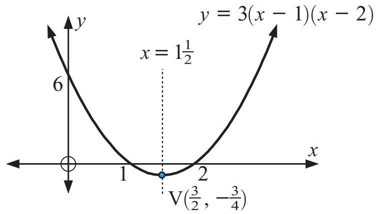
1 **a** **i** ± 2
ii $x = 0$
iii $(0, -4)$
iv -4



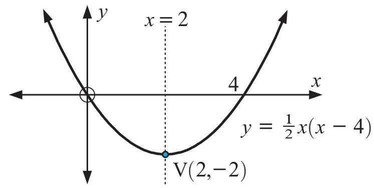
b **i** 1, 3
ii $x = 2$
iii $(2, -2)$
iv 6



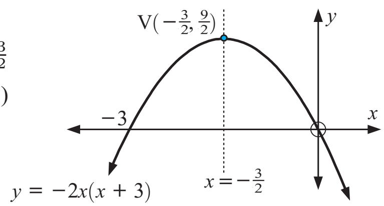
c **i** 1, 2
ii $x = \frac{3}{2}$
iii $(\frac{3}{2}, -\frac{1}{4})$
iv 6



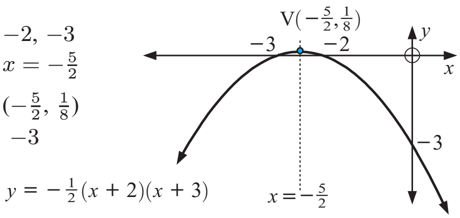
d **i** 0, 4
ii $x = 2$
iii $(2, -2)$
iv 0



e **i** $0, -3$
ii $x = -\frac{3}{2}$
iii $(-\frac{3}{2}, \frac{9}{2})$
iv 0



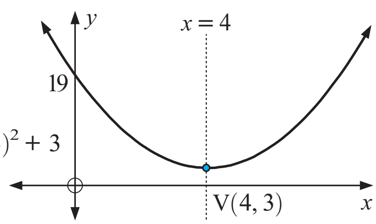
f **i** $-2, -3$
ii $x = -\frac{5}{2}$
iii $(-\frac{5}{2}, \frac{1}{8})$
iv -3



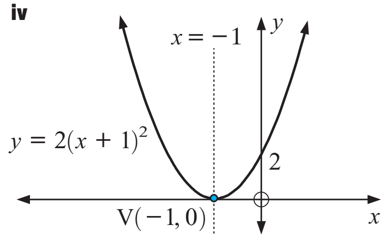
2 **a** B **b** A **c** F **d** D **e** E **f** C

EXERCISE 8B.2

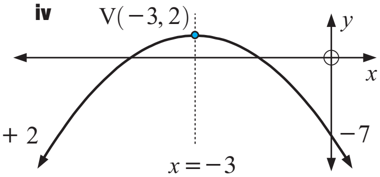
1 **a** **i** $x = 4$ **iv**
ii $(4, 3)$
iii 19



b **i** $x = -1$ **iv**
ii $(-1, 0)$
iii 2

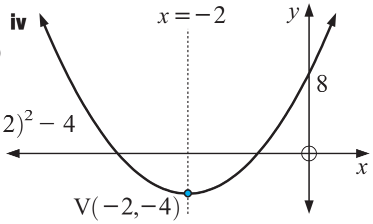


c **i** $x=-3$ **ii** $(-3,\ 2)$ **iii** -7 **iv**



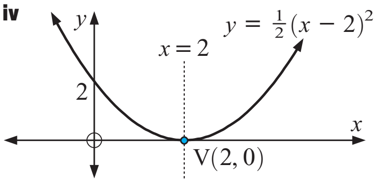
$y=-(x+3)^2+2$

d **i** $x=-2$ **ii** $(-2,\ -4)$ **iii** 8 **iv**



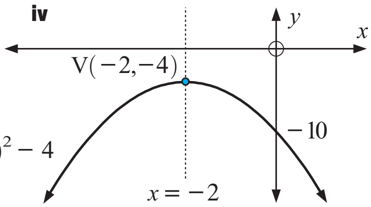
$y=3(x+2)^2-4$

e **i** $x=2$ **ii** $(2,\ 0)$ **iii** 2 **iv**



$y=\frac{1}{2}(x-2)^2$

f **i** $x=-2$ **ii** $(-2,\ -4)$ **iii** -10 **iv**

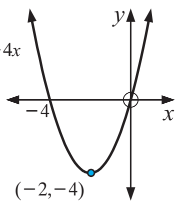


$y=-\frac{3}{2}(x+2)^2-4$

2 **a** G **b** A **c** E **d** B **e** I **f** C **g** D **h** F **i** H

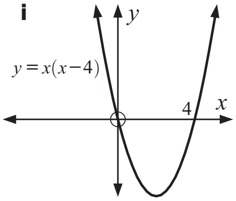
3 **a** $x=2$ **b** $x=-\frac{5}{2}$ **c** $x=1$ **d** $x=3$ **e** $x=-4$ **f** $x=-4$

4 **a** **i**



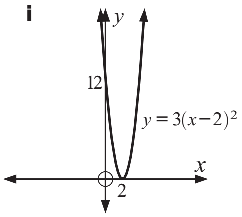
ii $x=-2$ **iii** $(-2,\ -4)$

b **i**



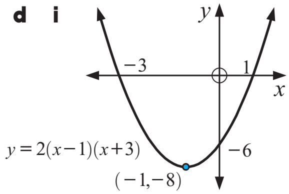
ii $x=2$ **iii** $(2,\ -4)$

c **i**



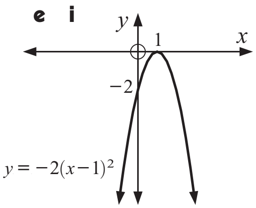
ii $x=2$ **iii** $(2,\ 0)$

d **i**



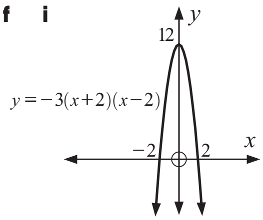
ii $x=-1$ **iii** $(-1,\ -8)$

e **i**



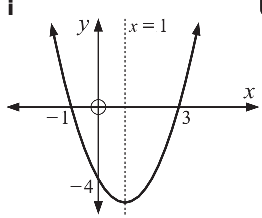
ii $x=1$ **iii** $(1,\ 0)$

f **i**



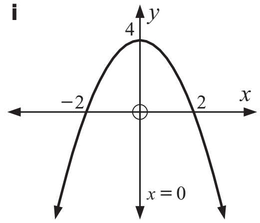
ii $x=0$ **iii** $(0,\ 12)$

5 **a** **i**



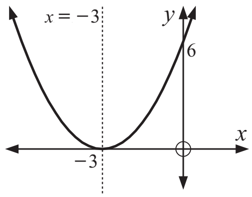
ii axis of sym. $x=1$

b **i**



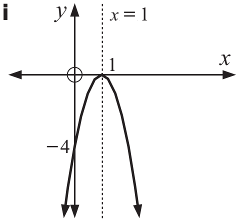
ii axis of sym. $x=0$

c **i**



ii axis of sym. $x=-3$

d **i**

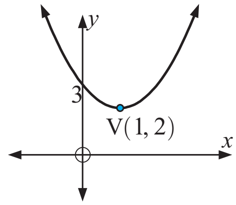


ii axis of sym. $x=1$

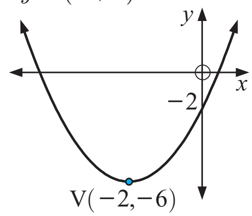
6 **a** 2 and 6 **b** -1 and -5 **c** 3 (touching)

EXERCISE 8C

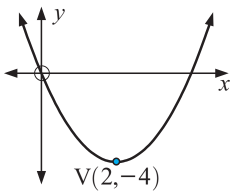
1 **a** $y=(x-1)^2+2$



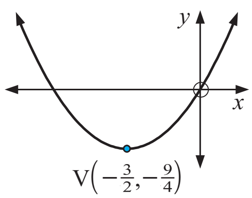
b $y=(x+2)^2-6$



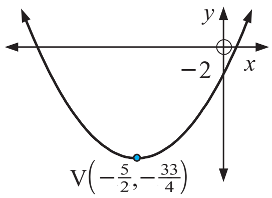
c $y=(x-2)^2-4$



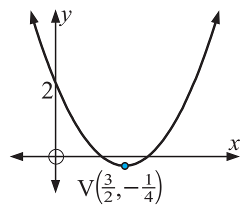
d $y=\left(x+\frac{3}{2}\right)^2-\frac{9}{4}$



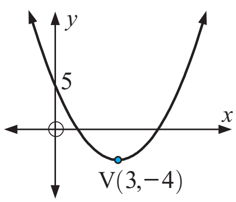
e $y=\left(x+\frac{5}{2}\right)^2-\frac{33}{4}$



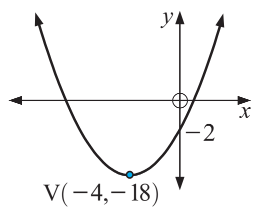
f $y=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$



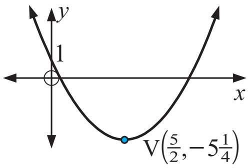
g $y=(x-3)^2-4$



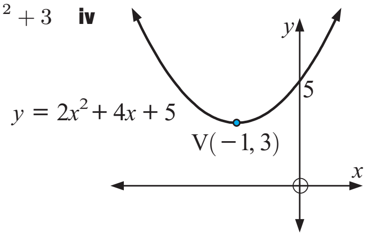
h $y=(x+4)^2-18$



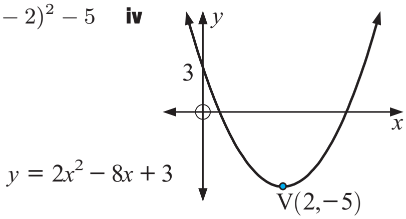
i $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



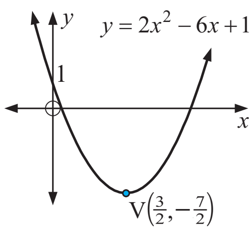
2 **a** **i** $y = 2(x+1)^2 + 3$ **iv**
ii $(-1, 3)$
iii 5



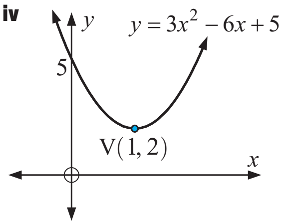
b **i** $y = 2(x-2)^2 - 5$ **iv**
ii $(2, -5)$
iii 3



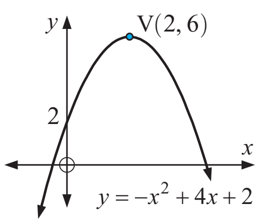
c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$ **iv**
ii $(\frac{3}{2}, -\frac{7}{2})$
iii 1



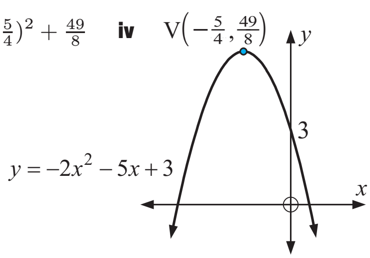
d **i** $y = 3(x-1)^2 + 2$ **iv**
ii $(1, 2)$
iii 5



e **i** $y = -(x-2)^2 + 6$ **iv**
ii $(2, 6)$
iii 2



f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$ **iv**
ii $(-\frac{5}{4}, \frac{49}{8})$
iii 3



3 **a** $y = (x-2)^2 + 3$ **b** $y = (x+3)^2 - 6$
c $y = -(x-2)^2 + 9$ **d** $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{17}{2}$
e $y = -2\left(x + \frac{5}{2}\right)^2 + \frac{27}{2}$ **f** $y = 3\left(x - \frac{3}{2}\right)^2 - \frac{47}{4}$

EXERCISE 8D.1

1 **a** $x = 0, -\frac{7}{4}$ **b** $x = 0, -\frac{1}{3}$ **c** $x = 0, \frac{7}{3}$ **d** $x = 0, \frac{11}{2}$
e $x = 0, \frac{8}{3}$ **f** $x = 0, \frac{3}{2}$ **g** $x = 3, 2$ **h** $x = 4, -2$
i $x = 3, 7$ **j** $x = 3$ **k** $x = -4, 3$ **l** $x = -11, 3$

2 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}, 7$ **c** $x = -\frac{2}{3}, 6$ **d** $x = \frac{1}{3}, -2$
e $x = \frac{3}{2}, 1$ **f** $x = -\frac{2}{3}, 2$ **g** $x = -\frac{2}{3}, 4$ **h** $x = \frac{1}{2}, -\frac{3}{2}$
i $x = -\frac{1}{4}, 3$ **j** $x = -\frac{3}{4}, \frac{5}{3}$ **k** $x = \frac{1}{7}, -1$ **l** $x = -2, \frac{28}{15}$

3 **a** $x = 2, 5$ **b** $x = -3, 2$ **c** $x = 0, -\frac{3}{2}$ **d** $x = 1, 2$
e $x = \frac{1}{2}, -1$ **f** $x = 3$

4 **a** $x = -3$ **b** $x = -3$ or -2 **c** $x = 1$ or 4 **d** no solution

5 **a** $x = 0$ or $\frac{2}{3}$ **b** $x = 3$ or -2 **c** $x = \frac{1}{2}$ or -7 **d** $x = 3$

6 **a** **i** 25 m **ii** 25 m **iii** 45 m
b **i** 2 secs and 4 secs **ii** 0 secs and 6 secs
c once going up and once coming down

7 **a** **i** $-\$30$ **ii** $\$105$ **b** 6 or 58 cakes

EXERCISE 8D.2

1 **a** $x = -5 \pm \sqrt{2}$ **b** $x = -6 \pm \sqrt{11}$ **c** $x = 4 \pm 2\sqrt{2}$
d $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$
g $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$
d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$
g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solutions

3 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$
d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

EXERCISE 8E

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$
d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$
g $x = \sqrt{2}$ **h** $x = -\frac{4}{9} \pm \frac{\sqrt{7}}{9}$ **i** $x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}$ **c** $x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$
d $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$ **e** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **f** $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$

EXERCISE 8F

1 **a** $x = -3.414$ or -0.586 **b** $x = 0.317$ or -6.317
c $x = 2.77$ or -1.27 **d** $x = -1.08$ or 3.41
e $x = -0.892$ or 3.64 **f** $x = 1.34$ or -2.54

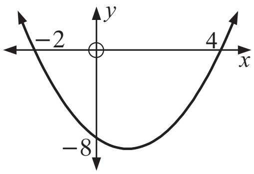
2 **a** $x = -4.83$ or 0.828 **b** $x = -1.57$ or 0.319
c $x = 0.697$ or 4.30 **d** $x = -0.823$ or 1.82
e $x = -0.618$ or 1.62 **f** $x = -0.281$ or 1.78

EXERCISE 8G

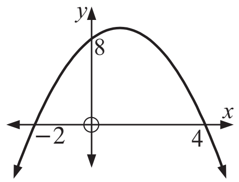
1 7 and -5 or -7 and 5 **2** 5 or $\frac{1}{5}$
3 14 **4** 18 and 20 or -18 and -20
5 15 and 17 or -15 and -17 **6** 15 sides
7 6 cm by 6 cm by 7 cm **8** 3.48 cm **9** 11.2 cm square
10 no **12** 221 ha **13** 2.03 m **14** 52.1 kmph
15 553 kmph **16** 51.1 kmph **17** 32

EXERCISE 8H

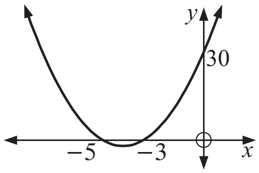
1 **a** $y=(x-4)(x+2)$



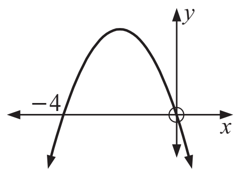
b $y=-(x-4)(x+2)$



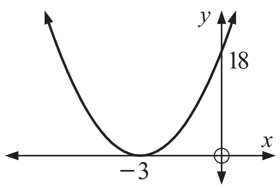
c $y=2(x+3)(x+5)$



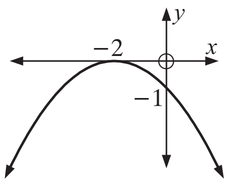
d $y=-3x(x+4)$



e $y=2(x+3)^2$

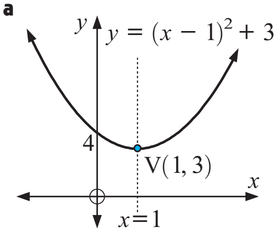


f $y=-\frac{1}{4}(x+2)^2$

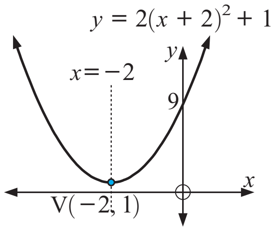


2 **a** $x=1$ **b** $x=1$ **c** $x=-4$ **d** $x=-2$ **e** $x=-3$
f $x=-2$

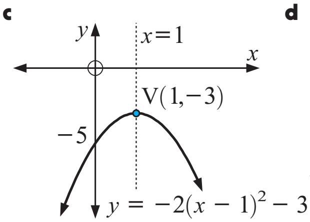
3 **a**



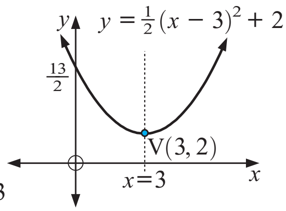
b



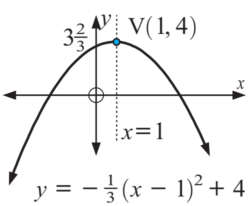
c



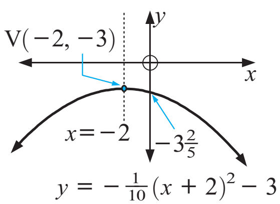
d



e



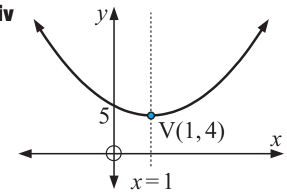
f



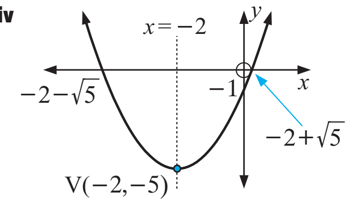
4 **a** $(2,-2)$ **b** $(-1,-4)$ **c** $(0,4)$ **d** $(0,1)$ **e** $(-2,-15)$
f $(-2,-5)$ **g** $(-\frac{3}{2},-\frac{11}{2})$ **h** $(\frac{5}{2},-\frac{19}{2})$ **i** $(1,-\frac{9}{2})$ **j** $(2,6)$

5 **a** ± 3 **b** $\pm\sqrt{3}$ **c** -5 and -2 **d** 3 and -4 **e** 0 and 4
f -4 and -2 **g** -1 (touching) **h** 3 (touching)
i $2\pm\sqrt{3}$ **j** $-2\pm\sqrt{7}$ **k** $3\pm\sqrt{11}$ **l** $-4\pm\sqrt{5}$

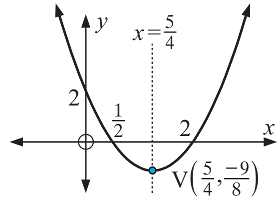
6 **a** **i** $x=1$
ii $(1,4)$
iii no x-intercept, y-intercept 5
iv



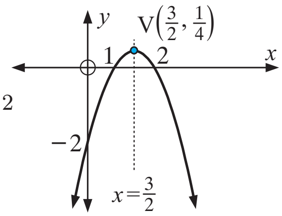
b **i** $x=-2$
ii $(-2,-5)$
iii x-int. $-2\pm\sqrt{5}$, y-intercept -1
iv



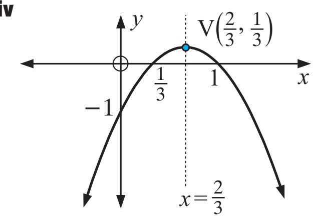
c **i** $x=\frac{5}{4}$
ii $(\frac{5}{4},-\frac{9}{8})$
iii x-intercepts $\frac{1}{2}$, 2, y-intercept 2
iv



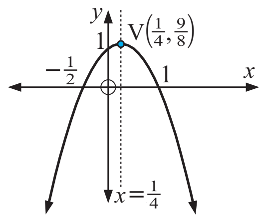
d **i** $x=\frac{3}{2}$
ii $(\frac{3}{2},\frac{1}{4})$
iii x-intercepts $x=1,2$ y-intercept -2
iv



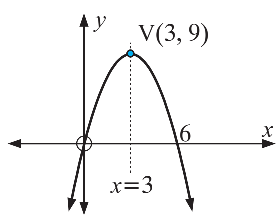
e **i** $x=\frac{2}{3}$
ii $(\frac{2}{3},\frac{1}{3})$
iii x-intercepts $\frac{1}{3}$, 1, y-intercept -1
iv



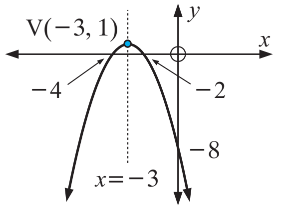
f **i** $x=\frac{1}{4}$
ii $(\frac{1}{4},\frac{9}{8})$
iii x-intercepts $-\frac{1}{2}$, 1, y-intercept 1
iv



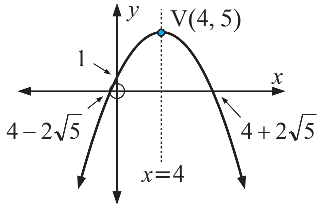
g **i** $x=3$
ii $(3,9)$
iii x-intercepts 0, 6, y-intercept 0
iv



h **i** $x=-3$ **ii** $(-3,\ 1)$ **iii** x-int. -2, -4, y-intercept -8 **iv**



i **i** $x=4$ **ii** $(4,\ 5)$ **iii** x-int. $4\pm 2\sqrt{5}$, y-intercept 1 **iv**



EXERCISE 8I.1

1 **a** 2 real distinct roots **b** a repeated root **c** 2 real distinct roots **d** 2 real distinct roots **e** no real roots **f** a repeated root

2 **a**, **b**, **d**, **f**

3 **a** $\Delta=9-4m$ **i** $m=\frac{9}{4}$ **ii** $m<\frac{9}{4}$ **iii** $m>\frac{9}{4}$
b $\Delta=25-4m$ **i** $m=\frac{25}{4}$ **ii** $m<\frac{25}{4}$ **iii** $m>\frac{25}{4}$
c $\Delta=1-4m$ **i** $m=\frac{1}{4}$ **ii** $m<\frac{1}{4}$ **iii** $m>\frac{1}{4}$
d $\Delta=4-12m$ **i** $m=\frac{1}{3}$ **ii** $m<\frac{1}{3}$ **iii** $m>\frac{1}{3}$
e $\Delta=49-8m$ **i** $m=\frac{49}{8}$ **ii** $m<\frac{49}{8}$ **iii** $m>\frac{49}{8}$
f $\Delta=25-16m$ **i** $m=\frac{25}{16}$ **ii** $m<\frac{25}{16}$ **iii** $m>\frac{25}{16}$

EXERCISE 8I.2

1 **a** cuts x-axis twice **b** touches x-axis **c** cuts x-axis twice **d** cuts x-axis twice **e** cuts x-axis twice **f** touches x-axis

2 **a** $a=1$ which is >0 and $\Delta=-15$ which is <0
b $a=-1$ which is <0 and $\Delta=-8$ which is <0
c $a=2$ which is >0 and $\Delta=-40$ which is <0
d $a=-2$ which is <0 and $\Delta=-23$ which is <0

3 $a=3$ which is >0 and $\Delta=k^2+12$ which is always >0 {as $k^2>0$ for all k}

4 $a=2$ which is >0 and $\Delta=k^2-16$ $\therefore$ positive definite when $k^2-16<0$ i.e., $k^2<16$ i.e., $-4<k<4$

EXERCISE 8J

1 **a** $y=2(x-1)(x-2)$ **b** $y=2(x-2)^2$ **c** $y=(x-1)(x-3)$ **d** $y=-(x-3)(x+1)$ **e** $y=-3(x-1)^2$ **f** $y=-2(x+2)(x-3)$

2 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

3 **a** $y=\frac{3}{2}(x-2)(x-4)$ **b** $y=-\frac{1}{2}(x+4)(x-2)$ **c** $y=-\frac{4}{3}(x+3)^2$

4 **a** $y=3x^2-18x+15$ **b** $y=-4x^2+6x+4$ **c** $y=-x^2+6x-9$ **d** $y=4x^2+16x+16$ **e** $y=\frac{3}{2}x^2-6x+\frac{9}{2}$ **f** $y=-\frac{1}{3}x^2+\frac{2}{3}x+5$

5 **a** $y=-(x-2)^2+4$ **b** $y=2(x-2)^2-1$ **c** $y=-2(x-3)^2+8$ **d** $y=\frac{2}{3}(x-4)^2-6$ **e** $y=-2(x-2)^2+3$ **f** $y=2(x-\frac{1}{2})^2-\frac{3}{2}$

EXERCISE 8K

1 **a** (1, 7) and (2, 8) **b** (4, 5) and (−3, −9) **c** (3, 0) (touching) **d** graphs do not meet

2 **a** (0.59, 5.59) and (3.41, 8.41) **b** (3, −4) touching **c** graphs do not meet **d** (−2.56, −18.81) and (1.56, 1.81)

3 **a** (2, 4), (−1, 1) **b** (1, 0), (−2, −3) **c** (1, 4) **d** (1, 4), (−4, −1)

EXERCISE 8L

1 **a** 9 seconds **b** 162 m **c** 18 seconds

2 **a** 12 **b** \$100 **c** \$244

3 **a** 15 m/s **b** $\frac{1}{2}$ sec; since the car was travelling downhill, it was accelerating. $\therefore$ when the brake was applied, the speed of the vehicle still increased for a short time.
c $15\frac{1}{8}$ m/s **d** 6 seconds

4 **a** 21 **b** \$837 **c** \$45 **5** **a** 30^oC **b** 5.00 am **c** 5^oC

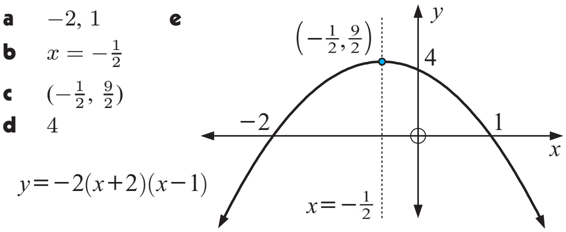
6 **b** $x=10$ **c** 200 m^2

7 **a** $y=-\frac{1}{100}x^2+70$ **b** supports are 21 m, 34 m, 45 m, 54 m, 61 m, 66 m, 69 m

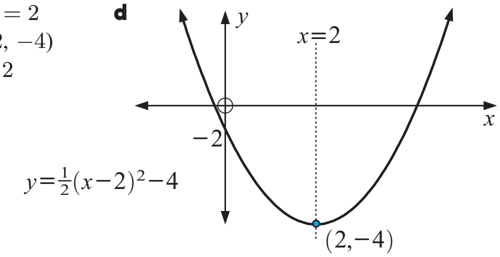
8 **a** vertex (30, 30) **b** $y=\frac{1}{45}(x-30)^2+30$ **c** 38.89 m

REVIEW SET 8A

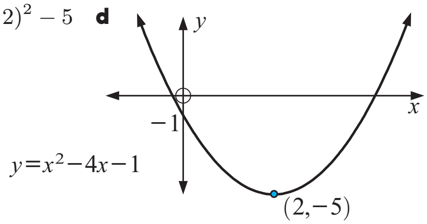
1 **a** -2, 1 **b** $x=-\frac{1}{2}$ **c** $(-\frac{1}{2},\ \frac{9}{2})$ **d** 4 **e**



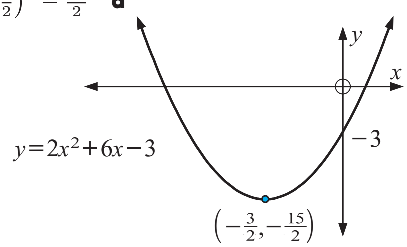
2 **a** $x=2$ **b** $(2,\ -4)$ **c** -2 **d**



3 **a** $y=(x-2)^2-5$ **b** $(2,\ -5)$ **c** -1 **d**



4 **a** $y=2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}$ **b** $(-\frac{3}{2},\ -\frac{15}{2})$ **c** -3 **d**



5 **a** $x=15$ or -4 **b** $x=-\frac{5}{3}$ or 2 **c** $x=0$ or 4

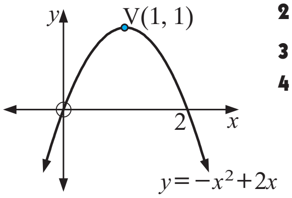
6 **a** $x=5$ or 2 **b** $x=3$ or 4 **c** $x=\frac{1}{2}$ or 3

7 $x=-\frac{7}{2}\pm\frac{\sqrt{65}}{2}$ **8** $x=-2\pm\sqrt{3}$

9 **a** $x=\frac{7}{2}\pm\frac{\sqrt{37}}{2}$ **b** no real roots

REVIEW SET 8B

1



2 $x = \frac{4}{3}$, $V(\frac{4}{3}, 12\frac{1}{3})$

3 $x = -1$, $V(-1, -5)$

4 **a** no real solutions **b** two real distinct solutions

5 $a = 3$ which is > 0 and $\Delta = -11$ which is < 0

6 $a = -2$ which is < 0 $\therefore$  a max.
max. $= 5$ when $x = 1$

7 $(4, 4)$ and $(-3, 18)$ **8** $k < -3\frac{1}{8}$ **9** **b** 15 m by 30 m

REVIEW SET 8C

1 **a** $x = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$ **b** $x = \frac{-11 \pm \sqrt{145}}{6}$

2 **a** $x = \frac{5}{2} \pm \frac{\sqrt{37}}{2}$ **b** $x = \frac{7}{4} \pm \frac{\sqrt{73}}{4}$

3 **a** $x = -5.828$ or -0.1716 **b** $x = -1.135$ or 1.468

4 **a** $x = 0.5858$ or 3.414 **b** $x = -0.1861$ or 2.686

5 **a** two distinct real rational roots **b** a repeated root

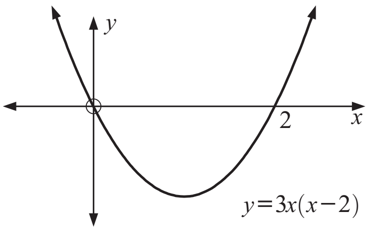
6 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$

7 **a** $t = \frac{4}{3}$ **b** $t < \frac{4}{3}$ **c** $t > \frac{4}{3}$

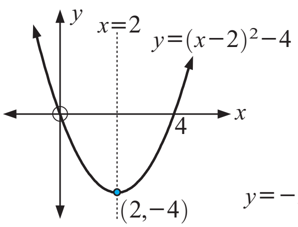
8 12.92 cm **9** 17 cm

REVIEW SET 8D

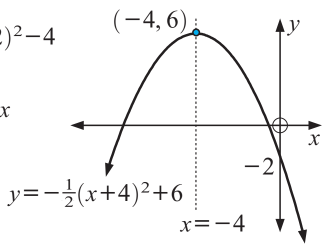
1



2 **a**

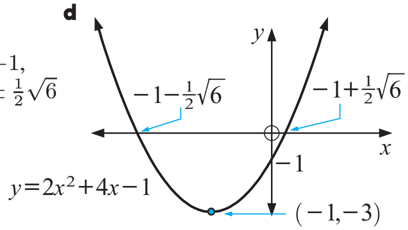


b

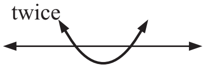


3 **a** $x = -1$
b $(-1, -3)$
c y-intercept -1, x-ints. $-1 \pm \frac{1}{2}\sqrt{6}$

d



4 **a** graph cuts x-axis twice



b graph cuts x-axis twice

5 **a** neither **b** positive definite

6 **a** $y = 3(x-3)(x+3)$ **b** $y = -6(x-2)^2 + 25$

7 $\frac{6}{5}$ or $\frac{5}{6}$ **8** 13.48 cm by 13.48 cm **9** touch at $(-2, 9)$

REVIEW SET 8E

1 **a** $y = -6(x+3)(x-1)$ **b** $y = \frac{20}{9}(x-2)^2 - 20$

2 **a** $y = -\frac{2}{7}(x-1)(x-7)$ **b** $y = \frac{2}{9}(x+3)^2$

3 $y = -4x^2 + 4x + 24$ **4** $y = 3x^2 - 24x + 48$

5 $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

6 **a** min. $= 5\frac{2}{3}$ when $x = -\frac{2}{3}$ **b** max. $= 5\frac{1}{8}$ when $x = -\frac{5}{4}$

7 $k < 1$

8 **b** $A = x\left(\frac{600 - 8x}{9}\right)$ **c** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **d** 1250 m^2

9 **b** $x = 6$

EXERCISE 9A

1 **a** $x^3 + 3x^2 + 3x + 1$ **b** $x^3 + 6x^2 + 12x + 8$
c $x^3 - 12x^2 + 48x - 64$ **d** $8x^3 + 12x^2 + 6x + 1$
e $8x^3 - 12x^2 + 6x - 1$ **f** $27x^3 - 27x^2 + 9x - 1$
g $8x^3 + 60x^2 + 150x + 125$ **h** $8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

2 **a** $x^4 + 8x^3 + 24x^2 + 32x + 16$ **b** $x^4 - 8x^3 + 24x^2 - 32x + 16$
c $16x^4 + 96x^3 + 216x^2 + 216x + 81$
d $81x^4 - 108x^3 + 54x^2 - 12x + 1$
e $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$ **f** $16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

3 **a** $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
b $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
c $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$
d $32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5}$

4 **a** 1 6 15 20 15 6 1
b **i** $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
ii $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$
iii $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5 **a** $7 + 5\sqrt{2}$ **b** $56 + 24\sqrt{5}$ **c** $232 - 164\sqrt{2}$

6 **a** $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$
b 65.944 160 601 201

7 $2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$ **8** **a** 270 **b** 4320

EXERCISE 9B

1 **a** $1^{11} + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + ... + \binom{11}{10}(2x)^{10} + (2x)^{11}$

b $(3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right) + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + ...$
$.... + \binom{15}{14}(3x)\left(\frac{2}{x}\right)^{14} + \left(\frac{2}{x}\right)^{15}$

c $(2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right) + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + ...$
$.... + \binom{20}{19}(2x)\left(-\frac{3}{x}\right)^{19} + \left(-\frac{3}{x}\right)^{20}$

2 **a** $T_6 = \binom{15}{5}(2x)^{10}5^5$ **b** $T_4 = \binom{9}{3}(x^2)^6\left(\frac{5}{x}\right)^3$

c $T_{10} = \binom{17}{9}x^8\left(-\frac{2}{x}\right)^9$ **d** $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\frac{1}{x}\right)^8$

3 **a** $\binom{10}{5}3^5 2^5$ **b** $\binom{6}{3}2^3(-3)^3$ **c** $\binom{12}{4}2^8(-1)^4$

4 **a** $\binom{15}{5}2^5$ **b** $\binom{9}{3}(-3)^3$

5 **a**

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

b sum: 2, 4, 8, 16, 32

c It seems that the sum of the numbers in row n of Pascal's triangle is 2^n.
d After the first part let $x = 1$.

6 **a** $\binom{8}{6} = 28$ **b** $2\binom{9}{3}3^6 - \binom{9}{4}3^5 = 91\,854$

7 **b** $84x^3$ **c** $n = 6$ and $k = -2$

REVIEW SET 9

1 $(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
a $x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$
b $1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$

2 $\binom{6}{3}5^3 2^3 = 20\,000$ **3** $362 + 209\sqrt{3}$

4 $64.964\,808$ **5** It does not have one.

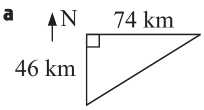
6 $243a^{10} - \frac{810a^8}{b} + \frac{1080a^6}{b^2} - \frac{720a^4}{b^3} + \frac{240a^2}{b^4} - \frac{32}{b^5}$

7 $\binom{12}{6}2^6(-3)^6$ **8** $\binom{15}{5}5^5$ **9** $8\binom{6}{2} - 6\binom{6}{1} = 84$

EXERCISE 10A

1 **a** $x = 0.663$ **b** $x = 4.34$ **c** $x = 2.23$

2 **a** 4.54 m **b** 4.17 m **3** 237 m

4 **a**


b 87.1 km

EXERCISE 10B

1 50.3 m **2** 17.3 cm **3** 8.60 m **4** 53.4 m

5 **a** both are 90^o **b** $a^2 + b^2$

EXERCISE 10C

1 **a** 3, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$
b 12, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$
c $\sqrt{11}$, $\sin\theta = \frac{5}{6}$, $\cos\theta = \frac{\sqrt{11}}{6}$, $\tan\theta = \frac{5}{\sqrt{11}}$
d $\sqrt{5}$, $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\tan\theta = 2$
e $\sqrt{52}$, $\sin\theta = \frac{4}{\sqrt{52}}$, $\cos\theta = \frac{6}{\sqrt{52}}$, $\tan\theta = \frac{2}{3}$
f $\sqrt{15}$, $\sin\theta = \frac{7}{8}$, $\cos\theta = \frac{\sqrt{15}}{8}$, $\tan\theta = \frac{7}{\sqrt{15}}$

2 **a** $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$ **b** $\cos\alpha = \frac{\sqrt{5}}{3}$, $\tan\alpha = \frac{2}{\sqrt{5}}$
c $\sin\beta = \frac{4}{5}$, $\cos\beta = \frac{3}{5}$

3 **a** $\sin\theta = \frac{b}{c}$, $\cos\theta = \frac{a}{c}$, $\tan\theta = \frac{b}{a}$

4 **a** **i** $\frac{a}{b}$ **ii** $\frac{c}{b}$ **iii** $\frac{c}{b}$ **iv** $\frac{a}{b}$ **b** **i** complement **ii** complement

5 **a** $\sqrt{2}$ **b** $\sin 45^o = \frac{1}{\sqrt{2}}$, $\cos 45^o = \frac{1}{\sqrt{2}}$, $\tan 45^o = 1$

6 **a** $\measuredangle ABN = 60^o$, $\measuredangle BAN = 30^o$ **b** $BN = 1$, $AN = \sqrt{3}$
c **i** $\sin 60^o = \frac{\sqrt{3}}{2}$, $\cos 60^o = \frac{1}{2}$, $\tan 60^o = \sqrt{3}$
ii $\sin 30^o = \frac{1}{2}$, $\cos 30^o = \frac{\sqrt{3}}{2}$, $\tan 30^o = \frac{1}{\sqrt{3}}$

EXERCISE 10D

1 **a** $x = 17.2$ **b** $x = 257$ **c** $x = 15.1$
d $x = 7.10$ **e** $x = 554$ **f** $x = 457$

2 **a** $\theta = 69.5$ **b** $\theta = 76.2$ **c** $\theta = 60.0$ **d** $\theta = 73.4$
e $\theta = 19.5$ **f** $\theta = 77.9$ **g** $\theta = 9.06$ **h** $\theta = 34.7$

3 **a** $\theta = 56.4$ **b** $\alpha = 4.8$ **c** $\beta = 48.2$

4 **a** AC = 6.40 m, $\measuredangle A = 38.7^o$, $\measuredangle C = 51.3^o$
b $\measuredangle R = 39^o$, PQ = 8.10 m, PR = 12.9 m

5 **a** $x = 2.65$, $\theta = 37.1$
b $x = 6.16$, $\theta = 50.3$, $y = 13.0$

6 **a** $x = 4.13$ **b** $\alpha = 75.5$ **c** $\beta = 41.0$

7 **a** $\theta = 36.9$ **b** $r = 11.3$ **c** $\alpha = 61.9$

8 7.99 cm **9** 89.2^o **10** 47.2^o **11** 22.4^o

12 11.8 cm **13** 119^o **14** 36.5 cm

15 **a** $x = 3.44$ **b** $\alpha = 51.5$ **16** 129^o

EXERCISE 10E

1 18.3 m **2** **a** 371 m **b** 1.62 km

3 159 m **4** 1.575^o

5 angle of elevn. $= 26.4^o$, angle of depn. $= 26.4^o$

6 418.5 m **7** 111 m **8** 72.0 m **9** 9.91 m

10 **a** 16.2 m/s **b** 11.5^o **11** $\theta = 12.6$ **12** 9.56 m

13 77.7 litres **14** 2.22 m **15** 10.95 m **16** 786 m

17 962 m **18** 2.10 km **19** 3.17 km **20** 33.8 m

EXERCISE 10F

1 **a** $\doteqdot 18.4^o$ **b** $\doteqdot 26.6^o$ **c** $\doteqdot 116.6^o$ **d** 135^o

2 **a** $\tan 60^o = \sqrt{3}$ **b** $\tan 165^o \doteqdot -0.268$

3 **a** $y = \sqrt{3}x + 2$ **b** $y = -\sqrt{3}x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

REVIEW SET 10A

1 $\frac{7}{11}$ **2** $\sin\theta = \frac{7}{\sqrt{74}}$ $\cos\theta = \frac{5}{\sqrt{74}}$ **3** $\frac{3}{\sqrt{55}}$

4 unknown side $= \sqrt{45} \doteqdot 6.71$, $\sin\theta = \frac{\sqrt{45}}{9}$,
$\cos\theta = \frac{2}{3}$, $\tan\theta = \frac{\sqrt{45}}{6}$

5 unknown side $= \sqrt{48} \doteqdot 6.93$ cm, $\sin\theta = \frac{\sqrt{48}}{13}$,
$\cos\theta = \frac{11}{13}$, $\tan\theta = \frac{\sqrt{48}}{11}$

6 **a** 36.7 cm **b** 60.6^o **7** **a** 6.03 m **b** 6.13^o

8 $\measuredangle K = 66^o$, KL = 6.91 cm, LM = 15.5 cm

9 $\measuredangle R = 52^o$, PQ = 8.96 cm, PR = 11.4 cm **10** 7.28 m

REVIEW SET 10B

1 15.8 km on a bearing of 147^o **2** 22.3^o

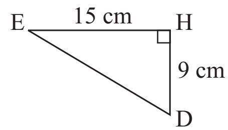
3 369 kmph **4** **a** $\theta = 54.6$ **b** $\theta = 84.7$ **c** $\theta = 79.2$

5 **a** $\theta = 41.6$ **b** $\theta = 44.4$ **c** $\theta = 36.7$ **6** 6.2 cm

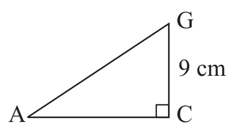
7 25.2^o **8** **a** 5 cm **b** 36.9^o **9** 9.78 cm **10** 70.9 m

REVIEW SET 10C

1 **a** 3.61 m **b** 33.7^o **2** $\doteqdot 121^o$ **3** $y = \frac{1}{\sqrt{3}}x - 3$

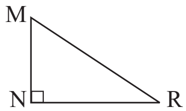
4 **a**


b 59.0^o

c


d 68.35^o

5 **a** 50.2^o **b** 22.6^o

6 **a** 

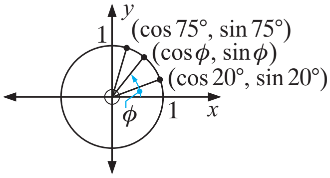
b 11.2 cm **c** 57.95^o

7 **a** 141 m **b** 45^o **8** $\doteq 51.3^o$ **10** $\doteq 314$ m

EXERCISE 11A

1 **a** 0 **b** 0.26 **c** 0.42 **d** 0.5 **e** 0.71 **f** 0.87 **g** 0.97 **h** 1

3 **a** 1 **b** 0.97 **c** 0.91 **d** 0.87 **e** 0.71 **f** 0.5 **g** 0.26 **h** 0

5 (0.57, 0.82) **6** 

7 **a** $\sin\theta = 0.6$ **b** $\cos\theta \doteq 0.714$

EXERCISE 11B

1 **a** 0.98 **b** 0.98 **c** 0.87 **d** 0.87 **e** 0.5 **f** 0.5 **g** 0 **h** 0

3 **a** $\sin(180-\theta)^o = \sin\theta^o$

4 **a** -0.34 **b** 0.34 **c** -0.64 **d** 0.64 **e** -0.77 **f** 0.77 **g** -1 **h** 1

6 $\cos(180-\theta)^o = -\cos\theta^o$

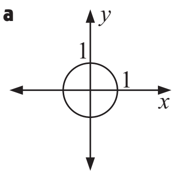
7 **a** 135^o **b** 129^o **c** 106^o **d** 98^o

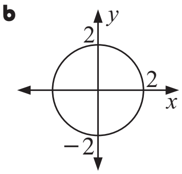
8 **a** 50^o **b** 34^o **c** 18^o **d** 9^o

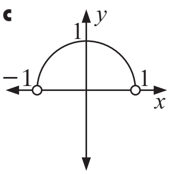
9 **a** 0.6820 **b** 0.8572 **c** -0.7986 **d** 0.9135 **e** 0.9063 **f** -0.6691

10 **a** $(180-\theta)^o$
b OQ is a reflection of OP in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$
c $\cos(180-\theta) = -\cos\theta$ and $\sin(180-\theta) = \sin\theta$

EXERCISE 11C

1 **a** 

b 

c 

2 **a** **i** A$(\cos 26^o, \sin 26^o)$ B$(\cos 146^o, \sin 146^o)$ C$(\cos 199^o, \sin 199^o)$
ii A(0.899, 0.438) B(-0.829, 0.559) C(-0.946, -0.326)
b **i** A$(\cos 123^o, \sin 123^o)$ B$(\cos 251^o, \sin 251^o)$ C$(\cos(-35^o), \sin(-35^o))$
ii A(-0.545, 0.839) B(-0.326, -0.946) C(0.819, -0.574)

3 **a** $\cos 0^o = 1, \sin 0^o = 0$ **b** $\cos 90^o = 0, \sin 90^o = 1$
c $\cos 180^o = -1, \sin 180^o = 0$
d $\cos 270^o = 0, \sin 270^o = -1$
e $\cos(-90^o) = 0, \sin(-90^o) = -1$
f $\cos 450^o = 0, \sin 450^o = 1$

REVIEW SET 11

1 **a** $\sin 70^o \doteq 0.94$ **b** $\cos 35^o \doteq 0.82$

2 M$(\cos 73^o, \sin 73^o) \doteq (0.292, 0.956)$
N$(\cos 190^o, \sin 190^o) \doteq (-0.985, -0.174)$
P$(\cos 307^o, \sin 307^o) \doteq (0.602, -0.799)$

3 $\theta \doteq 102.8^o$ **4** **a** 60^o **b** 15^o **c** 85^o

5 **a** 133^o **b** 172^o **c** 94^o

6 **a** 0.358 **b** -0.035 **c** 0.259 **d** -0.731 **e** 0.766

7 **a** $-0.5, -0.866$ **b** $-0.5, 0.866$ **c** $-0.5, -0.866$

8 **a** 1, 0 **b** -1, 0 **c** 0, -1

9 **a** 79^o **b** 53^o **c** 12^o **10** **a** 84^o **c** 62^o **c** 3^o

11 **a** 0.961 **b** -0.961 **c** -0.961 **d** -0.961

12 **a** -0.743 **b** -0.743 **c** 0.743 **d** -0.743

EXERCISE 12A

1 **a** 28.9 cm^2 **b** 384 km^2 **c** 26.7 cm^2 **2** $x = 19.0$

3 18.9 cm^2 **4** 137 cm^2 **5** 374 cm^2 **6** 7.49 cm

7 11.9 m **8** **a** 48.6^o or 131.4^o **b** 42.1^o or 137.9^o

9 $\frac{1}{4}$ is not covered

10 **a** **i** and **ii** 6 cm^2 **b** **i** $\doteq 21.3$ cm^2 **ii** 30.7 cm^2

EXERCISE 12B

1 **a** **i** 6.535 cm **ii** 29.4 cm^2 **b** **i** 10.5 cm **ii** 25.9 cm^2

2 **a** 3.14 cm **b** 9.305 cm^2

3 **a** 5.91 cm **b** 18.9 cm **4** **a** 39.3^o **b** 34.4^o

5 **a** 11.7 cm **b** 11.7 **c** 37.7 cm **d** 185^o **6** **b** 2 h 24 min

7 **a** 67.7 cm^2 **b** 138 cm^2 **c** 70.6 cm^2

8 **a** $\alpha = 18.43$ **b** $\theta = 143.1$ **c** 387.3 m^2 **9** 227 m^2

10 **a** $\alpha = 5.739$ **b** $\theta = 168.5$ **c** $\phi = 191.5$ **d** 71.62 cm

EXERCISE 12C

1 **a** 28.8 cm **b** 3.38 km **c** 14.2 m

2 $\measuredangle A = 52.0^o, \measuredangle B = 59.3^o, \measuredangle C = 68.7^o$ **3** 112^o

4 **a** 40.3^o **b** 107^o **5** **a** $\cos\theta = 0.65$ **b** $x = 3.81$

EXERCISE 12D.1

1 **a** $x = 28.4$ **b** $x = 13.4$ **c** $x = 3.79$

2 **a** $a = 21.25$ **b** $b = 76.9$ **c** $c = 5.095$

EXERCISE 12D.2

1 $\measuredangle C = 62.1^o$ or $\measuredangle C = 117.9^o$

2 **a** $\measuredangle A = 49.5^o$ **b** $\measuredangle B = 72.05^o$ or 107.95^o **c** $\measuredangle C = 44.3^o$

3 No, $\dfrac{\sin 85^o}{11.4} \neq \dfrac{\sin 27^o}{9.8}$

4 $\measuredangle ABC = 66^o$ BD $= 4.55$ cm **5** $x = 17.7$ $y = 33.1$

EXERCISE 12E

1 17.7 m **2** 207 m **3** 23.9^o **4** 77.5 m

5 13.2^o **6** 69.1 m **7** **a** 38.0 m **b** 94.0

8 55.1^o **9** AC $= 11.7$ km BC $= 8.49$ km

10 **a** 74.9 km^2 **b** 7490 hectares **11** 9.12 km

12 $\doteq 85$ mm **13** 10.1 km **14** 29.2 m

REVIEW SET 12A

1 21.1 km^2 **2** **a** 118 cm^2 **b** 44.9 cm^2

3 perimeter $= 34.1$ cm, area $= 66.5$ cm^2

4 radius $= 8.79$ cm, area $= 81.0$ cm^2

5 $x = 67.4$ or 112.6

6 $x = 47.5$, AC $= 14.3$ cm **or** $x = 132.5$, AC $= 28.1$ cm

7 36.8 cm^2 **8** **a** 10 600 m^2 **b** 1.06 ha **9** 26.6 m^2

REVIEW SET 12B

1 **a** $x = 34.1$ **b** $x = 18.9$ **2** **a** $x = 41.5$ **b** $x = 15.4$

3 AC $= 12.55$ cm, $\measuredangle A = 48.6^o$, $\measuredangle C = 57.4^o$

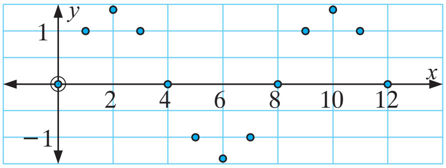
4 113 cm^2 **5** 7.32 m **6** 204 m

7 530 m on a bearing of 077.2^o

8 179 km on a bearing of 352^o

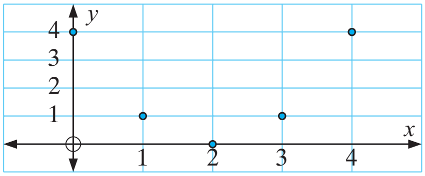
EXERCISE 13A

1 **a**



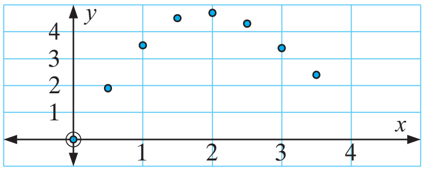
Data exhibits periodic behaviour.

b



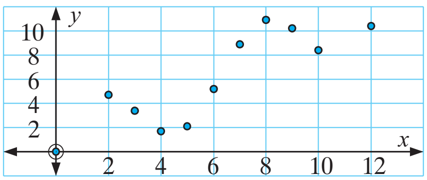
Not enough information to say data is periodic. It may in fact be quadratic.

c



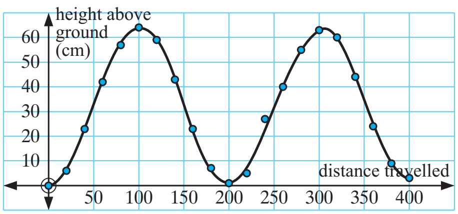
Not enough information to say data is periodic. It may in fact be quadratic.

d



Not enough information to say data is periodic.

2 **a**



b The data is periodic. **i** $y = 32$ (approx.) **ii** $\doteqdot 64$ cm **iii** $\doteqdot 200$ cm **iv** $\doteqdot 32$ cm

c A curve can be fitted to the data.

3 **a** periodic **b** periodic **c** periodic **d** not periodic **e** periodic **f** periodic

EXERCISE 13B.1

1 **a** 45^o **b** -45^o **c** 270^o **d** -270^o **e** 60^o **f** 30^o

2 **a** $\frac{\pi}{6}$ units **b** $\frac{\pi}{3}$ units **c** $\frac{5\pi}{6}$ units **d** $\frac{3\pi}{4}$ units **e** $\frac{4\pi}{3}$ units **f** $-\frac{\pi}{4}$ units **g** $-\frac{3\pi}{4}$ units **h** $-\frac{3\pi}{2}$ units

3 **a** $\theta = 60$ **b** $d = \frac{\pi}{3}$ **c** decrease **d** $\theta = 57.3$

EXERCISE 13B.2

1 **a** $\frac{\pi}{2}^c$ **b** $\frac{\pi}{3}^c$ **c** $\frac{\pi}{6}^c$ **d** $\frac{\pi}{10}^c$ **e** $\frac{\pi}{20}^c$ **f** $\frac{3\pi}{4}^c$ **g** $\frac{5\pi}{4}^c$ **h** $\frac{3\pi}{2}^c$ **i** $2\pi^c$ **j** $4\pi^c$ **k** $\frac{7\pi}{4}^c$ **l** $3\pi^c$ **m** $\frac{\pi}{5}^c$ **n** $\frac{4\pi}{9}^c$ **o** $\frac{23\pi}{18}^c$

2 **a** 0.641^c **b** 2.39^c **c** 5.55^c **d** 3.83^c **e** 6.92^c

3 **a** 36^o **b** 108^o **c** 135^o **d** 10^o **e** 20^o **f** 140^o **g** 18^o **h** 27^o **i** 150^o **j** 22.5^o

4 **a** 114.59^o **b** 87.66^o **c** 49.68^o **d** 182.14^o **e** 301.78^o

5 **a**

Degrees	0	45	90	135	180	225
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$

Degrees	270	315	360
Radians	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π

Degrees	210	240	270	300	330	360
Radians	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

6 **b** **i** 9.6 cm **ii** 14 cm **iii** $\doteqdot 47.1$ cm

c **i** 0.75^c **ii** 1.68^c **iii** 2.32^c

d **i** 8.75 cm^2 **ii** 36.2 cm^2 **iii** 62.8 cm^2

e arc length = 10 cm, area = 25 cm^2 **f** 65 cm^2

EXERCISE 13C.1

1 **a** 0, 1 **b** 1, 0 **c** 0, −1 **d** 0, −1

2 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$ **d** $\cos\theta = 0$

3 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$ **d** $\sin\theta = \pm 1$

4 **a**

Quad-rant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve
2	$90 < \theta < 180$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve
3	$180 < \theta < 270$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve
4	$270 < \theta < 360$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve

b **i** 1 and 4 **ii** 2 and 3 **iii** 3 **iv** 2

5 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$

d $\sin\theta = -\frac{12}{13}$ **e** $\sin\theta = -\frac{\sqrt{3}}{2}$ **f** $\cos\theta = -\frac{1}{\sqrt{2}}$

EXERCISE 13C.2

1

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	−1	$-\frac{1}{\sqrt{2}}$

2

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

3 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** 3 **d** $\frac{1}{4}$ **e** $-\frac{1}{4}$ **f** 1 **g** $\sqrt{2}$ **h** $\frac{1}{2}$ **i** $\frac{1}{2}$

4 **a** $30^o, 150^o$ **b** $60^o, 120^o$ **c** $45^o, 315^o$ **d** $120^o, 240^o$ **e** $135^o, 225^o$ **f** $240^o, 300^o$

5 **a** $30^o, 330^o, 390^o, 690^o$ **b** $210^o, 330^o, 570^o, 690^o$ **c** $270^o, 630^o$

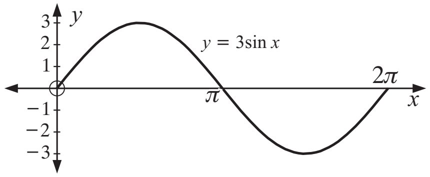
6 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$ **d** $\theta = \frac{\pi}{2}$

e $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ **g** $\theta = 0, \pi, 2\pi$

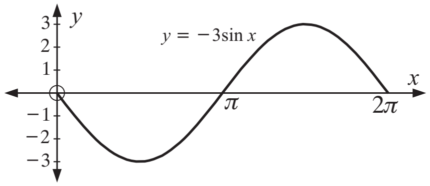
h $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

EXERCISE 13D.1

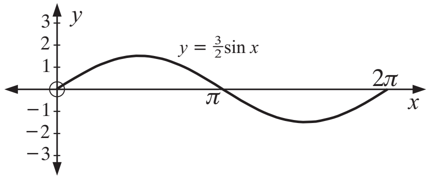
1 **a**



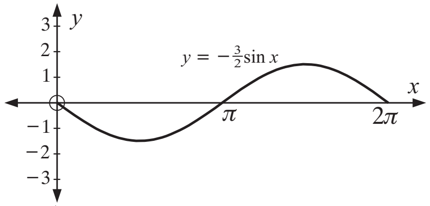
b



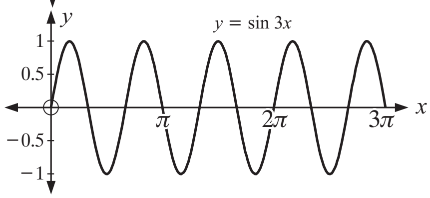
c



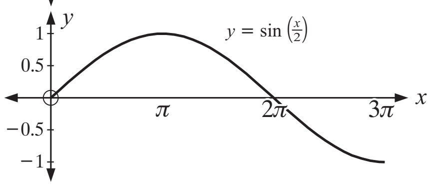
d



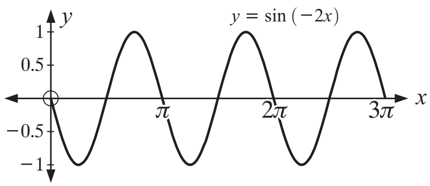
2 **a**



b



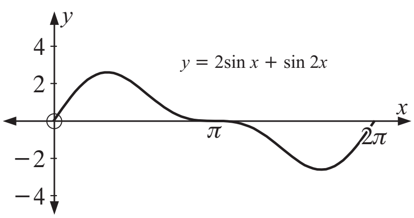
c



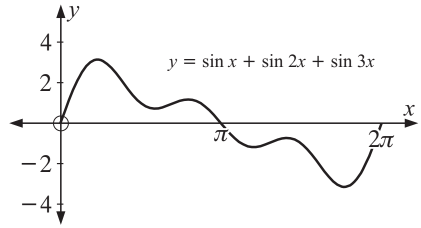
3 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{2}$ **c** 6π **d** $\frac{10\pi}{3}$

4 **a** $B = \frac{2}{5}$ **b** $B = 3$ **c** $B = \frac{1}{6}$ **d** $B = \frac{\pi}{2}$ **e** $B = \frac{\pi}{50}$

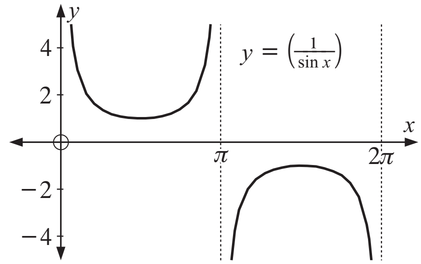
5 **a**



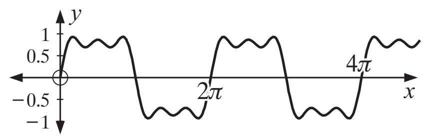
b



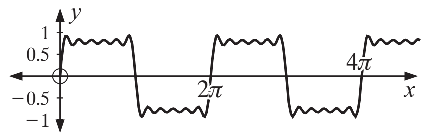
c

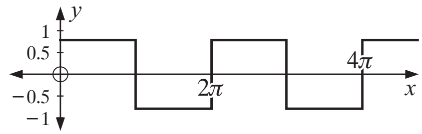


6 **a**



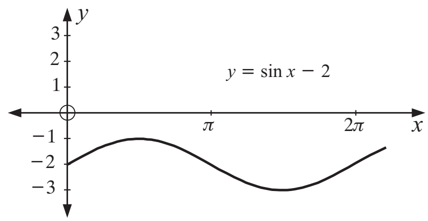
b



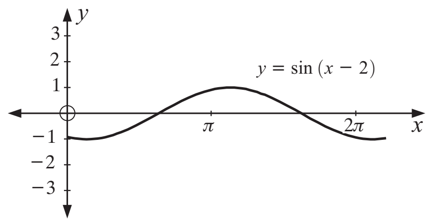


EXERCISE 13D.2

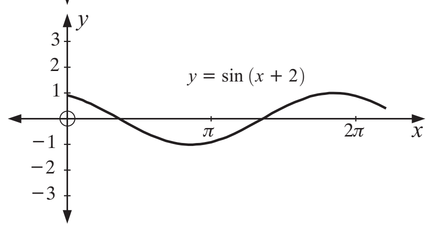
1 **a**



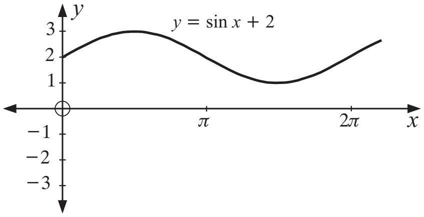
b



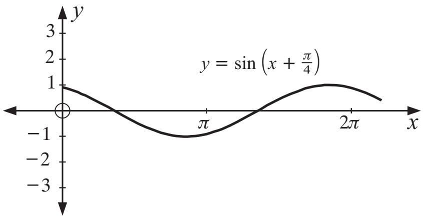
c



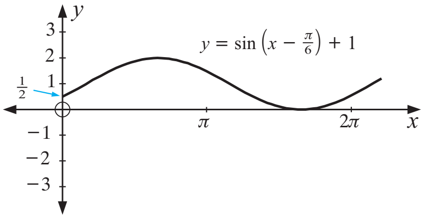
d



e



f



3 **a** $\frac{2\pi}{5}$ **b** 8π **c** π **4** **a** $\frac{2}{3}$ **b** 20 **c** $\frac{1}{50}$ **d** $\frac{\pi}{25}$

5 **a** vertical translation -1 **b** horizontal translation $\frac{\pi}{4}$ right
c vertical dilation, factor 2 **d** horizontal dilation, factor $\frac{1}{4}$
e vertical dilation, factor $\frac{1}{2}$ **f** horizontal dilation, factor 4
g reflection in the x-axis **h** translation $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$
i vertical dilation, factor 2, followed by a horizontal dilation, factor $\frac{1}{3}$
j translation $\begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$

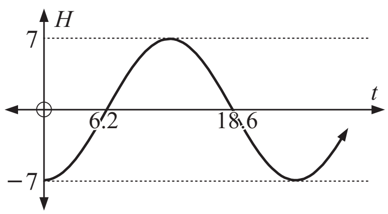
EXERCISE 13E

1 **a** $T \doteq 7.5\sin\frac{\pi}{6}(t - 9.9) + 20.5$

2 **a** $T \doteq 4.5\sin\frac{\pi}{6}(t - 10.5) + 11.5$

3 $T \doteq 13.1\sin(0.345)(t + 6.87) - 5.43$

5 **a** $H \doteq 7\sin 0.253(t - 6.2)$
b



6 $H = 10\sin\frac{\pi}{50}(t - 25) + 12$

EXERCISE 13F.1

1 **a** $x = 0.3, 2.8, 6.6, 9.1, 12.9$ **b** $x = 5.9, 9.8, 12.2$

2 **a** $x = \left.\begin{matrix} 0.4 \\ 1.2 \end{matrix}\right\} + k\pi$ **b** $x = \left.\begin{matrix} 1.7 \\ 3.0 \end{matrix}\right\} + k\pi$

EXERCISE 13F.2

1 **a** $x = 0.4268, 2.715, 6.710$ **b** $x = 3.880, 5.545$
c no solutions
d $x = 0.0814, 1.489, 3.223, 4.631, 6.365, 7.773$
e $x = 7.585$ **f** $x = 1.076, 4.348, 7.360$
g $x = 2.347, 3.394$ **h** $x = 1.831, 3.405$
i $x = 6.532, 7.605$

EXERCISE 13F.3

1 **a** $x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$ **b** $x = -\frac{\pi}{3}, \frac{5\pi}{3}$
c $x = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$
d $x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}, \frac{17\pi}{6}, \frac{10\pi}{3}, \frac{23\pi}{6}$

2 **a** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$
b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ **c** $x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$
d $x = -\frac{15\pi}{4}, -\frac{13\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}$ **e** $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$
f $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ **g** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{25\pi}{12}, \frac{29\pi}{12}$
h $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$
i $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{13\pi}{6}, \frac{7\pi}{3}$
j $x = -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}, \frac{-13\pi}{6}, \frac{-3\pi}{2}$

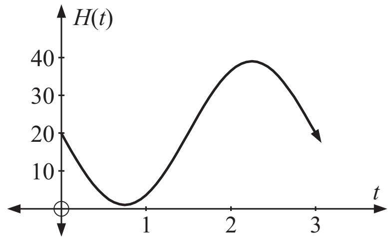
3 **a** $x = \frac{\pi}{2}$ **b** $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **c** $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
d $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

4 **a** $x = 0, \frac{\pi}{2}, \pi$ **b** $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

EXERCISE 13F.4

1 **a** **i** 7500 **ii** 10 272 **b** 10 500, when $t = 4$ weeks
c **i** at $t = 1\frac{1}{3}$ wks and $6\frac{2}{3}$ wks **ii** at $t = 9\frac{1}{3}$ wks
d $2.51 \leqslant t \leqslant 5.49$

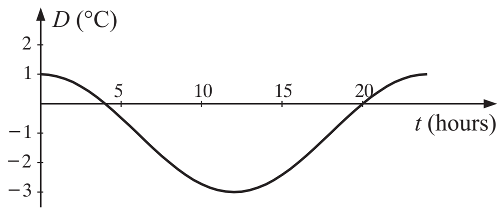
2 **a** 20 m **b** at $t = \frac{3}{4}$ minute **c** 3 minutes
d



3 **a** 400 **b** **i** 577 **ii** 400
c 650 It is the maximum population.
d 150, after 3 years **e** $0.26 < t < 1.74$

4 **a** **i** true **ii** true **b** 116.8 cents/L
c on the 5th and 11th days
d 98.6 cents/L on the 15th day

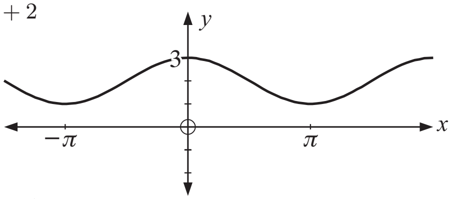
5 **a** inside temperature $= 25.6^o$C,
outside temperature $= 28.3^o$C
b $D = -2\sin\frac{\pi}{12}(t - 6) - 1$



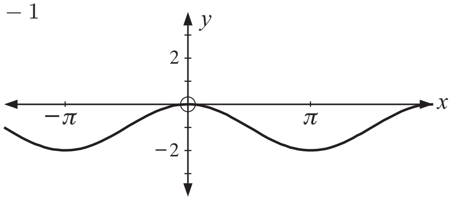
c 4 am and 8 pm

EXERCISE 13G

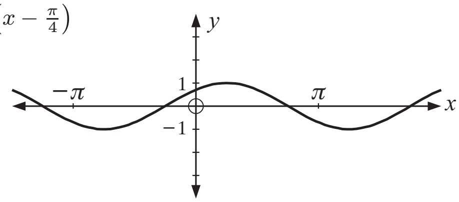
1 **a** $y=\cos x+2$



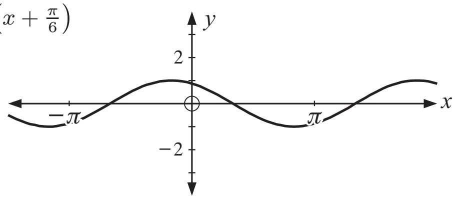
b $y=\cos x-1$



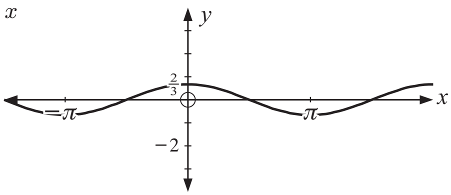
c $y=\cos\left(x-\frac{\pi}{4}\right)$



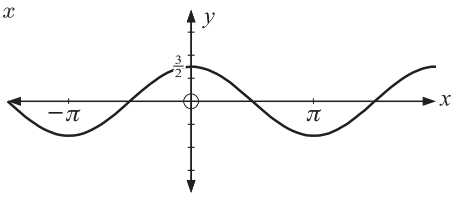
d $y=\cos\left(x+\frac{\pi}{6}\right)$



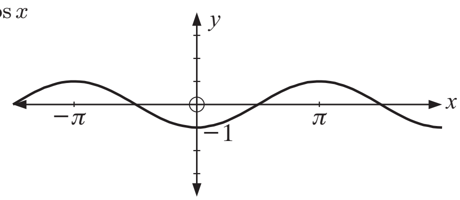
e $y=\frac{2}{3}\cos x$



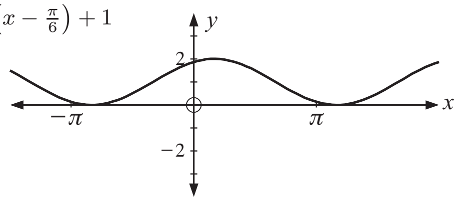
f $y=\frac{3}{2}\cos x$



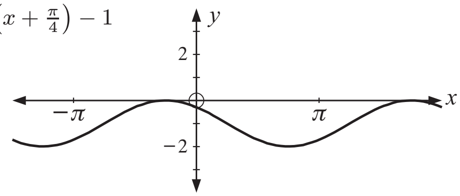
g $y=-\cos x$



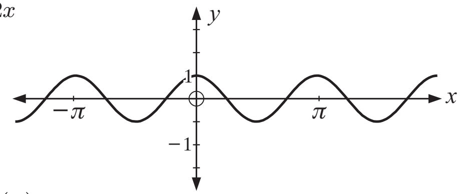
h $y=\cos\left(x-\frac{\pi}{6}\right)+1$



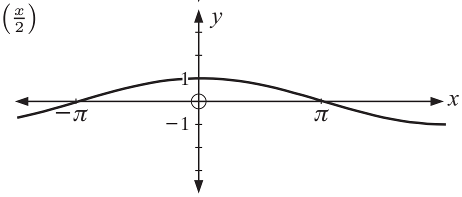
i $y=\cos\left(x+\frac{\pi}{4}\right)-1$



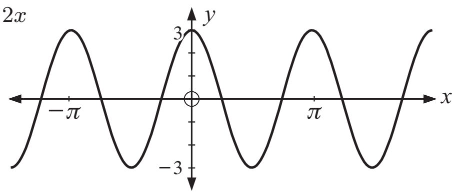
j $y=\cos 2x$



k $y=\cos\left(\frac{x}{2}\right)$



l $y=3\cos 2x$



2 **a** $\frac{2\pi}{3}$ **c** 6π **c** 100

3 A: amplitude, B: $\dfrac{2\pi}{\text{period}}$, C: horizontal translation, D: vertical translation

4 **a** $y=2\cos 2x$ **b** $y=\cos\left(\frac{x}{2}\right)+2$
c $y=3\cos\left(\frac{\pi}{4}x\right)$ **d** $y=-5\cos\left(\frac{\pi}{3}x\right)$

EXERCISE 13H

1 **a** $x=1.2,\ 5.1,\ 7.4$ **b** $x=4.4,\ 8.2,\ 10.7$
c $x=0.5,\ 5.8,\ 6.7$

2 **a** $x=\left.\begin{array}{c}0.5\\2.6\end{array}\right\}+k\pi$ **b** $x=\left.\begin{array}{c}0.9\\2.2\end{array}\right\}+k\pi$

3 **a** $x=0.975,\ 5.308,\ 7.258$
b $x=0.336,\ 2.805,\ 3.478,\ 5.947$
c $x=3.526,\ 5.358,\ 9.809,\ 11.641$
d $x=0.608,\ 1.487,\ 2.702,\ 3.581,\ 4.797$
e $x=\left.\begin{array}{c}0.9912\\2.150\end{array}\right\}+k\pi$

4 **a** $x=\frac{\pi}{4},\ \frac{7\pi}{4},\ \frac{9\pi}{4},\ \frac{15\pi}{4}$ **b** $x=\frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3},\ \frac{10\pi}{3},\ \frac{14\pi}{3}$
c $x=\frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{17\pi}{6}$ **d** $x=-\frac{5\pi}{3},\ -\pi,\ \frac{\pi}{3},\ \pi$
e $x=\pi,\ \frac{3\pi}{2},\ 3\pi$ **f** $x=\frac{\pi}{2},\ \frac{3\pi}{2}$
g $x=\frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{8\pi}{9}$ **h** $x=0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi,\ \frac{8\pi}{3}$

5 **a** $H(t)=3\cos(\frac{\pi t}{2})+4$ **b** $t \doteq 1.46$ sec

EXERCISE 13I

1 **a** $2\sin\theta$ **b** $3\cos\theta$ **c** $2\sin\theta$ **d** $\sin\theta$ **e** $-2\cos\theta$
f $-3\cos\theta$

2 **a** 3 **b** -2 **c** -1 **d** $3\cos^2\theta$ **e** $4\sin^2\theta$ **f** $\sin\theta$
g $-\sin^2\theta$ **h** $-\cos^2\theta$ **i** $-2\sin^2\theta$ **j** 1 **k** $\sin\theta$ **l** $\sin\theta$

3 **a** $1+2\sin\theta+\sin^2\theta$ **b** $\sin^2\alpha-4\sin\alpha+4$

c $\cos^2\alpha - 2\cos\alpha + 1$ **d** $1 + 2\sin\theta\cos\theta$
e $1 - 2\sin\beta\cos\beta$ **f** $-4 + 4\cos\alpha - \cos^2\alpha$

4 **a** $(1-\sin\theta)(1+\sin\theta)$ **b** $(\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha)$
c $(\cos\alpha+1)(\cos\alpha-1)$ **d** $\sin\beta(2\sin\beta-1)$
e $\cos\phi(2+3\cos\phi)$ **f** $3\sin\theta(\sin\theta-2)$
g $(\sin\theta+3)(\sin\theta+2)$ **h** $(2\cos\theta+1)(\cos\theta+3)$
i $(3\cos\alpha+1)(2\cos\alpha-1)$

5 **a** $1+\sin\alpha$ **b** $\cos\beta-1$ **c** $\cos\phi-\sin\phi$
d $\cos\phi+\sin\phi$ **e** $\dfrac{1}{\sin\alpha-\cos\alpha}$ **f** $\dfrac{\cos\theta}{2}$

EXERCISE 13J

1 **a** $\frac{24}{25}$ **b** $-\frac{7}{25}$ **2** $-\frac{7}{9}$ **3** $\frac{1}{9}$

4 **a** $\cos\alpha = \frac{-\sqrt{5}}{3}$, $\sin 2\alpha = \frac{4\sqrt{5}}{9}$
b $\sin\beta = \frac{-\sqrt{21}}{5}$, $\sin 2\beta = \frac{-4\sqrt{21}}{25}$

5 **a** $\frac{1}{3}$ **b** $\frac{2\sqrt{2}}{3}$

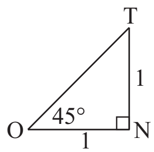
6 **a** $\sin 2\alpha$ **b** $2\sin 2\alpha$ **c** $\frac{1}{2}\sin 2\alpha$ **d** $\cos 2\beta$ **e** $-\cos 2\phi$
f $\cos 2N$ **g** $-\cos 2M$ **h** $\cos 2\alpha$ **i** $-\cos 2\alpha$ **j** $\sin 4A$
k $\sin 6\alpha$ **l** $\cos 8\theta$ **m** $-\cos 6\beta$ **n** $\cos 10\alpha$ **o** $-\cos 6D$
p $\cos 4A$ **q** $\cos\alpha$ **r** $-2\cos 6P$

8 **a** $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **b** $x = \frac{\pi}{2}, \frac{3\pi}{2}$
c $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **d** $x = \frac{\pi}{3}, \frac{5\pi}{3}$
e $x \doteq 3.33, 6.10$ **f** $x = 0, \pi, 2\pi$

EXERCISE 13K.1

1 **a** 0 **b** 0.27 **c** 0.36 **d** 0.47 **e** 0.70 **f** 1 **g** 1.19
h 1.43

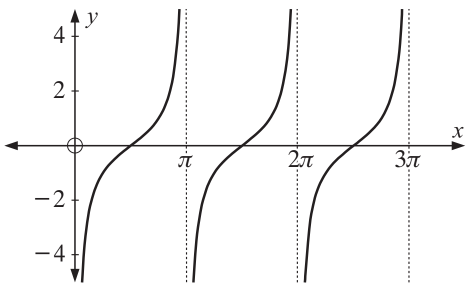
3 triangle TON is isosceles, ON = TN



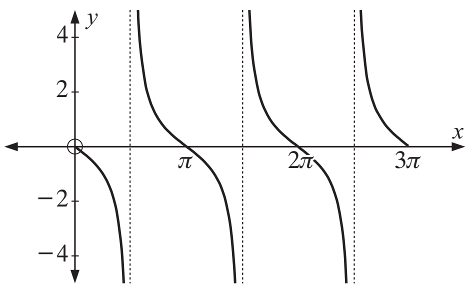
4 $\tan 85^o$ is too large; $\tan 85^o = 11.43$

EXERCISE 13K.2

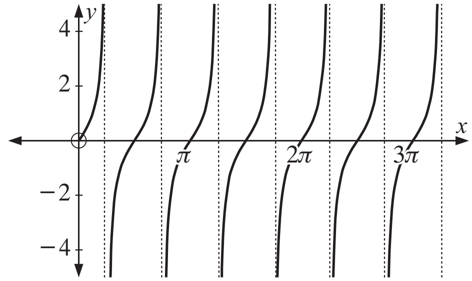
1 **a** **i** $y = \tan(x - \frac{\pi}{2})$



ii



iii



2 **a** translation through $\begin{bmatrix}1\\0\end{bmatrix}$ **b** reflection in x-axis
c horizontal dilation, factor $k = 2$

3 **a** **i** 1.6 **ii** -1.1 **b** **i** 1.557 **ii** -1.119
c **i** $x = 1.1, 4.2, 7.4$ **ii** $x = 2.2, 5.3$

4 **a** π **b** $\frac{\pi}{2}$ **c** $\frac{\pi}{n}$

EXERCISE 13L.1

1 $X \doteq 1.11 + k\pi$ **a** $x \doteq 0.554 + k\frac{\pi}{2}$
b $x \doteq 3.32 + k3\pi$ **c** $x \doteq -0.0929 + k\pi$

2 $X \doteq -1.25 + k\pi$ **a** $x \doteq 0.751 + k\pi$
b $x \doteq -0.416 + k\frac{\pi}{3}$ **c** $x \doteq -2.50 + k2\pi$

3 $X = \frac{\pi}{3} + k\pi$ **a** $x = \frac{\pi}{2} + k\pi$ **b** $x = \frac{\pi}{12} + k\frac{\pi}{4}$
c $x = \left.\begin{array}{c}\frac{\pi}{3}\\ -\frac{\pi}{3}\end{array}\right\} + k\pi$

EXERCISE 13L.2

1 **a** 0 **b** 1 **c** $\frac{1}{\sqrt{3}}$ **d** $\sqrt{3}$ **e** undefined **f** -1
g $-\sqrt{3}$ **h** undefined **i** $-\sqrt{3}$ **j** -1

2 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$ **d** $0, \pi, 2\pi$
e $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

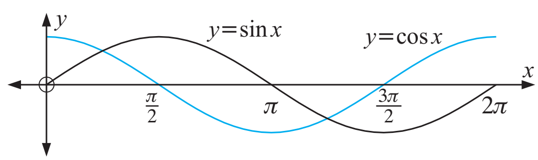
3 **a** $2\tan x$ **b** $-3\tan x$ **c** $\sin x$ **d** $\cos x$ **e** $5\sin x$ **f** $\dfrac{2}{\cos x}$

4 **a** $-\frac{1}{2\sqrt{2}}$ **b** $-2\sqrt{6}$ **c** $\frac{1}{\sqrt{2}}$ **d** $-\frac{\sqrt{7}}{3}$

5 **a** $\sin x = \frac{2}{\sqrt{13}}$, $\cos x = \frac{3}{\sqrt{13}}$ **b** $\sin x = \frac{4}{5}$, $\cos x = -\frac{3}{5}$
c $\sin x = -\sqrt{\frac{5}{14}}$, $\cos x = -\frac{3}{\sqrt{14}}$
d $\sin x = -\frac{12}{13}$, $\cos x = \frac{5}{13}$

EXERCISE 13M

1 **a**



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ **c** $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

2 **a** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

4 **a** $x \doteq 1.37, 4.515, 7.66$ **b** $x \doteq 2.50, 5.64, 8.78$

REVIEW SET 13A

1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π

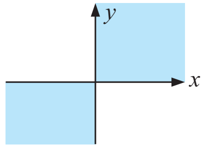
2 **a** 1.239^c **b** 2.175^c **c** -2.478^c **d** -0.4416^c

3 **a** 72^o **b** 225^o **c** 140^o **d** 330^o

4 **a** 171.89^o **b** 83.65^o **c** 24.92^o **d** -302.01^o

5 **a** $(0.766, -0.643)$ **b** $(-0.956, 0.292)$

6



7 **a** $\sin 120^o = \frac{\sqrt{3}}{2}$, $\cos 120^o = -\frac{1}{2}$

b $\sin 480^o = \frac{\sqrt{3}}{2}$, $\cos 480^o = -\frac{1}{2}$

8 **a** $\sin 150^o = \frac{1}{2}$, $\cos 150^o = -\frac{\sqrt{3}}{2}$

b $\sin 300^o = -\frac{\sqrt{3}}{2}$, $\cos 300^o = \frac{1}{2}$

9 **Hint:** $\tan\theta = -1$ when $\cos\theta = -\sin\theta$

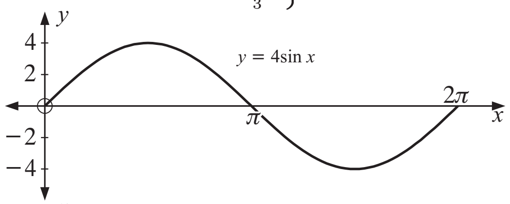
REVIEW SET 13B

1 **a** $0, -1$ **b** $0, -1$ **2** $\pm\frac{\sqrt{7}}{4}$ **3** $\frac{\sqrt{7}}{4}$ **4** **a** $\frac{3}{4}$ **b** $-\sqrt{2}$

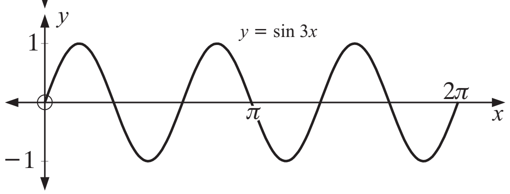
5 **a** $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$ **c** $\frac{1}{2}$ **6** **a** $150^o, 210^o$ **b** $45^o, 135^o$

7 **a** $\theta = \pi + k2\pi$ **b** $\theta = \left.\begin{matrix}\frac{\pi}{3}\\ \frac{2\pi}{3}\end{matrix}\right\} + k\pi$

8



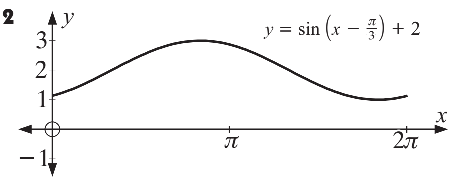
9



REVIEW SET 13C

1 **a** 6π **b** $\frac{\pi}{2}$

2



3 $T \doteqdot 7.05\sin\frac{\pi}{6}(t - 4.2) + 24.7$

4 **a** $x \doteqdot 0.392, 2.75, 6.675$ **b** $x \doteqdot 7.235$

5 **a** $x \doteqdot 3.25, 4.69$ **b** $x \doteqdot 1.445, 5.89, 7.73$

6 **a** $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **b** $x = \frac{-7\pi}{4}, \frac{-5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

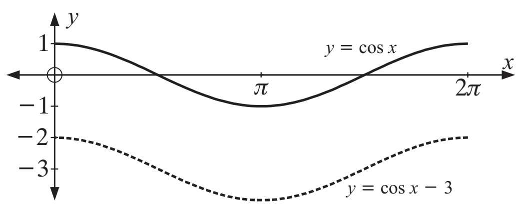
7 **a** $x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **b** $x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

8 **a** $x = \frac{3\pi}{2} + k2\pi$ **b** $x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k\pi$

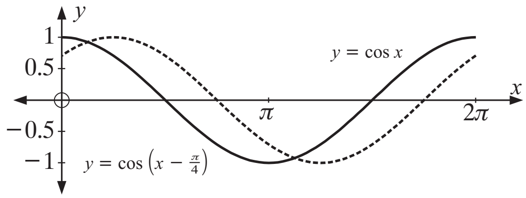
9 **a** 5000 **b** 3000, 7000 **c** $0.5 \leqslant t \leqslant 2.5$ and $6.5 \leqslant t \leqslant 8$

REVIEW SET 13D

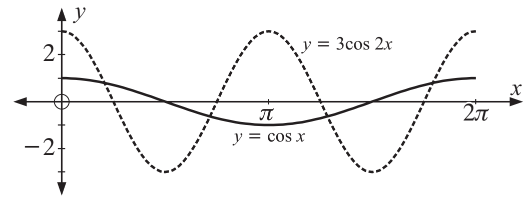
1 **a**



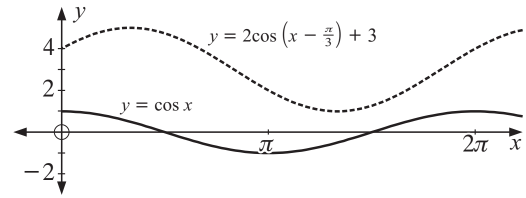
b



c



d



2 **a** 28 milligrams per m^3 **b** 8.00 am Monday

3 **a** $y = -4\cos 2x$ **b** $y = \cos\frac{\pi}{4}x + 2$

4 **a** $x \doteqdot 1.12, 5.17, 7.40$ **b** $x \doteqdot 0.184, 4.616$

5 **a** $x = \left.\begin{matrix}0.317\\ 1.254\end{matrix}\right\} + k\frac{\pi}{2}$ **b** $x \doteqdot 0.912, 2.23, 4.05$

6 **a** $x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$ **b** $x = -\pi, -\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

7 **a** $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi$ **b** $x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k\pi$

8 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $2\cos\theta$ **d** $5\cos^2\theta$ **e** $-\cos\theta$

9 **a** $4\sin^2\alpha - 4\sin\alpha + 1$ **b** $1 - \sin 2\alpha$

REVIEW SET 13E

1 **a** $1 - \cos\theta$ **b** $\dfrac{1}{\sin\alpha + \cos\alpha}$ **c** $\dfrac{-\cos\alpha}{2}$

3 **a** $\frac{120}{169}$ **b** $\frac{119}{169}$ **4** $\cos\alpha = \frac{-\sqrt{7}}{4}$, $\sin 2\alpha = \frac{3\sqrt{7}}{8}$

5 $\sin x = \frac{\sqrt{7}}{2\sqrt{2}}$

6 **a** **i** $x = 1.33 + k\pi$ **ii** $5.30 + k4\pi$ **iii** $2.83 + k\pi$

b **i** $x = \frac{\pi}{2} + k\pi$ **ii** $x = \frac{\pi}{3} + k\frac{\pi}{2}$ **iii** $x = \left.\begin{matrix}\frac{\pi}{3}\\ \frac{2\pi}{3}\end{matrix}\right\} + k\pi$

c $x = 0.612 + k\pi$

7 $\sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = -\frac{3}{\sqrt{13}}$

EXERCISE 14A

1 **a** 1×4 **b** 2×1 **c** 2×2 **d** 3×3

2 **a** $\begin{bmatrix} 2 & 1 & 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{bmatrix}$ **c** total cost of groceries

3 $\begin{bmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{bmatrix}$ **4** $\begin{bmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{bmatrix}$

EXERCISE 14B

1 **a** $\begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 4 \\ -6 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 0 \\ -11 & -3 \end{bmatrix}$

2 **a** $\begin{bmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{bmatrix}$ **b** $\begin{bmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{bmatrix}$ **c** $\begin{bmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{bmatrix}$

3 **a** Friday $\begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix}$ Saturday $\begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix}$ **b** $\begin{bmatrix} 187 \\ 229 \\ 101 \end{bmatrix}$

4 **a** **i** $\begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix}$ **ii** $\begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix}$ **b** subtract cost price from selling price **c** $\begin{bmatrix} 0.07 \\ 0.90 \\ 0.41 \\ 0.28 \\ 0.05 \end{bmatrix}$

5 **a** $\begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array}$ $\begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array}$ $\begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{bmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{fr} \\ \text{st} \\ \text{mi} \end{array}$

6 **a** $x = -2,\ y = -2$ **b** $x = 0,\ y = 0$

7 **a** $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ $\mathbf{B} + \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$

8 **a** $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix}$ $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix}$

EXERCISE 14C

1 **a** $\begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} -3 & -6 \\ -12 & -3 \end{bmatrix}$

2 **a** $\begin{bmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

c $\begin{bmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{bmatrix}$ **d** $\begin{bmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{bmatrix}$

3 **a** $\begin{bmatrix} 12 \\ 24 \\ 120 \\ 60 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ 6 \\ 30 \\ 15 \end{bmatrix}$ **c** $\begin{bmatrix} 9 \\ 18 \\ 90 \\ 45 \end{bmatrix}$

4 **a** $\begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{bmatrix} \end{array}$ **b** $\begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{bmatrix} \end{array}$

5 **a** $\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} \begin{array}{l} \leftarrow \text{VHS} \\ \leftarrow \text{DVD} \\ \leftarrow \text{gam.} \end{array}$ $\begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} \begin{array}{l} \leftarrow \text{VHS} \\ \leftarrow \text{DVD} \\ \leftarrow \text{gam.} \end{array}$ **b** $\begin{bmatrix} 211 \\ 70 \\ 231 \end{bmatrix} \begin{array}{l} \leftarrow \text{VHS} \\ \leftarrow \text{DVD} \\ \leftarrow \text{gam.} \end{array}$

c total weekly average hirings **6** 12**F**

EXERCISE 14D

1 **a** 3**A** **b** **O** **c** $-$**C** **d** **O** **e** 2**A** + 2**B**
f $-$**A** $-$ **B** **g** $-$2**A** + **C** **h** 4**A** $-$ **B** **i** 3**B**

2 **a** $\mathbf{X} = \mathbf{A} - \mathbf{B}$ **b** $\mathbf{X} = \mathbf{C} - \mathbf{B}$ **c** $\mathbf{X} = 2\mathbf{C} - 4\mathbf{B}$
d $\mathbf{X} = \frac{1}{2}\mathbf{A}$ **e** $\mathbf{X} = \frac{1}{3}\mathbf{B}$ **f** $\mathbf{X} = \mathbf{A} - \mathbf{B}$
g $\mathbf{X} = 2\mathbf{C}$ **h** $\mathbf{X} = \frac{1}{2}\mathbf{B} - \mathbf{A}$ **i** $\mathbf{X} = \frac{1}{4}(\mathbf{A} - \mathbf{C})$

3 **a** $\mathbf{X} = \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix}$ **b** $\mathbf{X} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$ **c** $\mathbf{X} = \begin{bmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{bmatrix}$

EXERCISE 14E.1

1 **a** $[11]$ **b** $[22]$ **c** $[16]$ **2** $\begin{bmatrix} w & x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

3 **a** $\mathbf{P} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix}$ $\mathbf{Q} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

b total cost $= \begin{bmatrix} 27 & 35 & 39 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \$291$

4 **a** $\mathbf{P} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix}$ $\mathbf{N} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$

b total points $= \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix} = 56$ points

EXERCISE 14E.2

1 Number of columns in A does not equal number of rows in B.

2 **a** $m = n$ **b** 2×3 **c** B has 3 columns, A has 2 rows

3 **a** $\begin{bmatrix} 28 & 29 \end{bmatrix}$ **b** **i** $\begin{bmatrix} 8 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{bmatrix}$

4 **a** $\begin{bmatrix} 3 & 5 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

5 **a** $\mathbf{C} = \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix}$ $\mathbf{N} = \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix}$

b $\begin{bmatrix} 78\,669.5 \\ 65\,589 \end{bmatrix}$ income from adult rides and children's rides **c** \$144 258.50

6 **a** $\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$ **b** $\mathbf{P} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix}$ **c** $\begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix}$

d My costs at store A are \$48, my friend's costs at store B are \$76. **e** store A

7 $\mathbf{F} = \begin{bmatrix} 6 & 7 & 9 \\ 5 & 8 & 4 \\ 4 & 7 & 2 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} 18 \\ 15 \\ 13 \end{bmatrix}$ $\mathbf{FC} = \begin{bmatrix} 330 \\ 262 \\ 203 \end{bmatrix}$ total cost = \$795

EXERCISE 14F

1 **a** $\begin{bmatrix} 16 & 18 & 15 \\ 13 & 21 & 16 \\ 10 & 22 & 24 \end{bmatrix}$ **b** $\begin{bmatrix} 10 & 6 & -7 \\ 9 & 3 & 0 \\ 4 & -4 & -10 \end{bmatrix}$

c $\begin{bmatrix} 22 & 0 & 132 & 176 & 198 \\ 44 & 154 & 88 & 110 & 0 \\ 176 & 44 & 88 & 88 & 132 \end{bmatrix}$ **d** $\begin{bmatrix} 115 \\ 136 \\ 46 \\ 106 \end{bmatrix}$

2 **a** $\begin{bmatrix} 3 & 3 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix}$ **c** $\begin{bmatrix} 657 & 730 & 670 \end{bmatrix}$

d $\begin{bmatrix} 369 & 420 & 385 \end{bmatrix}$ **e** $\begin{bmatrix} 657 & 730 & 670 \\ 369 & 420 & 385 \end{bmatrix}$

3 \$224 660 **4** \$3398.10

5 **a** $[\,125\ \ 195\ \ 225\,] \times \begin{bmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{bmatrix}$

$-[\,85\ \ 120\ \ 130\,] \times \begin{bmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{bmatrix}$

$= \$7125$

b $[\,125\ \ 195\ \ 225\,] \times \begin{bmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{bmatrix}$

$-[\,85\ \ 120\ \ 130\,] \times \begin{bmatrix}20&20&20&20&20&20&20\\15&15&15&15&15&15&15\\5&5&5&5&5&5&5\end{bmatrix}$

$= -\$9030$, i.e., a loss of $9030

c $([\,125\ \ 195\ \ 225\,] - [\,85\ \ 120\ \ 130\,])$

$\times \begin{bmatrix}15&12&13&11&14&16&8\\4&3&6&2&0&4&7\\3&1&4&4&3&2&0\end{bmatrix}$

EXERCISE 14G

1 $\mathbf{AB} = \begin{bmatrix}-1&1\\-1&7\end{bmatrix}$ $\mathbf{BA} = \begin{bmatrix}0&2\\3&6\end{bmatrix}$ $\mathbf{AB} \neq \mathbf{BA}$

2 $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ **4** **b** $\mathbf{I} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

5 **a** $\begin{bmatrix}7&0\\0&7\end{bmatrix}$ **b** $\begin{bmatrix}97&-59\\118&38\end{bmatrix}$

6 **a** $\mathbf{A}^2$ does not exist **b** when $\mathbf{A}$ is a square matrix

8 **a** $\mathbf{A}^2 + \mathbf{A}$ **b** $\mathbf{B}^2 + 2\mathbf{B}$ **c** $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}$
d $\mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A}$ **e** $\mathbf{AC} + \mathbf{AD} + \mathbf{BC} + \mathbf{BD}$
f $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$ **g** $\mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2$
h $\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}$ **i** $9\mathbf{I} - 6\mathbf{B} + \mathbf{B}^2$

9 **a** $\mathbf{A}^3 = 3\mathbf{A} - 2\mathbf{I}$ $\mathbf{A}^4 = 4\mathbf{A} - 3\mathbf{I}$
b $\mathbf{B}^3 = 3\mathbf{B} - 2\mathbf{I}$ $\mathbf{B}^4 = 6\mathbf{I} - 5\mathbf{B}$ $\mathbf{B}^5 = 11\mathbf{B} - 10\mathbf{I}$
c $\mathbf{C}^3 = 13\mathbf{C} - 12\mathbf{I}$ $\mathbf{C}^5 = 121\mathbf{C} - 120\mathbf{I}$

10 **a** **i** $\mathbf{I} + 2\mathbf{A}$ **ii** $2\mathbf{I} - 2\mathbf{A}$ **iii** $10\mathbf{A} + 6\mathbf{I}$
b $\mathbf{A}^2 + \mathbf{A} + 2\mathbf{I}$ **c** **i** $-3\mathbf{A}$ **ii** $-2\mathbf{A}$ **iii** $\mathbf{A}$

11 **a** $\mathbf{AB} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ **b** $\mathbf{A}^2 = \begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$

c false as $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I} = \mathbf{O}$

d $\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}a&b\\\dfrac{a-a^2}{b}&1-a\end{bmatrix}, b \neq 0$

12 For example, $\mathbf{A} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, gives $\mathbf{A}^2 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

13 **a** $a = 3, b = -4$ **b** $a = 1, b = 8$

14 $p = -2, q = 1$ **a** $\mathbf{A}^3 = 5\mathbf{A} - 2\mathbf{I}$ **b** $\mathbf{A}^4 = -12\mathbf{A} + 5\mathbf{I}$

EXERCISE 14H

1 **a** $\begin{bmatrix}3&0\\0&3\end{bmatrix} = 3\mathbf{I}$, $\begin{bmatrix}1&-2\\-\frac{2}{3}&\frac{5}{3}\end{bmatrix}$

b $\begin{bmatrix}10&0\\0&10\end{bmatrix} = 10\mathbf{I}$, $\begin{bmatrix}0.2&0.4\\-0.1&0.3\end{bmatrix}$

c $\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} = 2\mathbf{I}$, $\begin{bmatrix}-\frac{11}{2}&\frac{9}{2}&\frac{15}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\4&-3&-5\end{bmatrix}$

2 **a** -2 **b** -1 **c** 0 **d** 1

3 **a** 26 **b** 6 **c** -1 **d** $a^2 + a$

4 **a** -3 **b** 9 **c** -12 **5** **Hint:** Let $\mathbf{A} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$

6 **a** $|\mathbf{A}| = ad - bc$ $|\mathbf{B}| = wz - xy$

b $\mathbf{AB} = \begin{bmatrix}aw+by&ax+bz\\cw+dy&cx+dz\end{bmatrix}$ $|\mathbf{AB}| = (ad - bc)(wz - xy)$

7 **a** **i** -2 **ii** -8 **iii** -2 **iv** -9 **v** 2

8 **a** $\frac{1}{14}\begin{bmatrix}5&-4\\1&2\end{bmatrix}$ **b** $\begin{bmatrix}1&0\\1&-1\end{bmatrix}$ **c** does not exist

d $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ **e** does not exist **f** $-\frac{1}{15}\begin{bmatrix}7&-2\\-4&-1\end{bmatrix}$

g $\frac{1}{10}\begin{bmatrix}2&-4\\1&3\end{bmatrix}$ **h** $\begin{bmatrix}-3&-1\\2&1\end{bmatrix}$

EXERCISE 14I

1 **a** $\begin{bmatrix}x+2y\\3x+4y\end{bmatrix}$ **b** $\begin{bmatrix}2a+3b\\a-4b\end{bmatrix}$

2 **a** $\begin{bmatrix}3&-1\\2&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix}$ **b** $\begin{bmatrix}4&-3\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}11\\-5\end{bmatrix}$

c $\begin{bmatrix}3&-1\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}6\\-4\end{bmatrix}$

3 **a** $x = \frac{32}{7}$, $y = \frac{22}{7}$ **b** $x = -\frac{37}{23}$, $y = -\frac{75}{23}$
c $x = \frac{17}{13}$, $y = -\frac{37}{13}$ **d** $x = \frac{59}{13}$, $y = -\frac{25}{13}$
e $x = -40$, $y = -24$ **f** $x = \frac{1}{34}$, $y = \frac{55}{34}$

4 **b** **i** $\mathbf{X} = \begin{bmatrix}-1&3\\2&4\end{bmatrix}$ **ii** $\mathbf{X} = \begin{bmatrix}\frac{13}{7}&\frac{3}{7}\\-\frac{2}{7}&-\frac{8}{7}\end{bmatrix}$

5 **a** $\mathbf{A}^{-1} = \dfrac{1}{2k+6}\begin{bmatrix}2&-1\\6&k\end{bmatrix}$, $k \neq -3$

b $\mathbf{A}^{-1} = \dfrac{1}{3k}\begin{bmatrix}k&1\\0&3\end{bmatrix}$, $k \neq 0$

c $\mathbf{A}^{-1} = \dfrac{1}{(k+2)(k-1)}\begin{bmatrix}k&-2\\-1&k+1\end{bmatrix}$, $k \neq -2$ or 1

6 **a** $\mathbf{AB} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ **b** $\mathbf{A}$ and $\mathbf{B}$ are not inverses since they are not square matrices

7 $\mathbf{X} = \begin{bmatrix}\frac{1}{4}&\frac{3}{4}\\1&0\end{bmatrix}$ **8** **a** **i** $\begin{bmatrix}2&-3\\4&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8\\11\end{bmatrix}$, $|\mathbf{A}| = 10$
ii Yes, $x = 2.5$, $y = -1$

b **i** $\begin{bmatrix}2&k\\4&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8\\11\end{bmatrix}$, $|\mathbf{A}| = -2 - 4k$
ii $k \neq -\frac{1}{2}$, $x = \dfrac{8+11k}{2+4k}$ $y = \dfrac{5}{1+2k}$
iii $k = -\frac{1}{2}$, no solutions

9 **b** $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&-1\\-1&0\end{bmatrix}$

10 **a** $\mathbf{A}^{-1} = \begin{bmatrix}0&-1\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$, $(\mathbf{A}^{-1})^{-1} = \begin{bmatrix}1&2\\-1&0\end{bmatrix}$

b $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = (\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{I}$
c $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$ are inverses

11 **a** **i** $\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}\end{bmatrix}$ **ii** $\begin{bmatrix}\frac{3}{2}&\frac{1}{2}\\1&0\end{bmatrix}$ **iii** $\begin{bmatrix}\frac{5}{6}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}\end{bmatrix}$

iv $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ **v** $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ **vi** $\begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

c $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

d $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$

$\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses

12 $(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$

$k\mathbf{A}$ and $\frac{1}{k}\mathbf{A}^{-1}$ are inverses

13 **a** $\mathbf{X} = \mathbf{ABZ}$ **b** $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X}$

14 **a** $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$ **b** $\mathbf{A}^{-1} = 5\mathbf{I} + \mathbf{A}$ **c** $\mathbf{A}^{-1} = \frac{3}{2}\mathbf{A} - 2\mathbf{I}$

15 $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, $\mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$

17 If $\mathbf{A}^{-1}$ exists, i.e., $|\mathbf{A}| \neq 0$.

EXERCISE 14J

1 **a** 41 **b** -8 **c** 0 **d** 6 **e** -6 **f** -12 **g** 11 **h** 0 **i** 87

2 **a** abc **b** 0 **c** $3abc - a^3 - b^3 - c^3$ **3** $k \neq -3$

4 for all values of k except $\frac{1}{2}$ or -9

5 **a** $k = \frac{5}{2}$ or 2 **b** $k = 1$ or $\dfrac{-1 \pm \sqrt{33}}{2}$ **6** **a** 16 **b** -34

7 **a** -17, $\begin{bmatrix} -\frac{5}{17} & \frac{2}{17} & \frac{14}{17} & -\frac{1}{17} \\ -\frac{11}{17} & \frac{1}{17} & \frac{7}{17} & \frac{8}{17} \\ -\frac{3}{17} & \frac{8}{17} & \frac{5}{17} & -\frac{4}{17} \\ \frac{10}{17} & -\frac{4}{17} & -\frac{11}{17} & \frac{2}{17} \end{bmatrix}$ **b** 2, $\begin{bmatrix} \frac{7}{2} & -4 & -\frac{5}{2} & \frac{11}{2} & 0 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 2 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ -3 & 3 & 2 & -4 & 1 \end{bmatrix}$

8 **a** $\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} o \\ a \\ p \\ c \\ l \end{bmatrix} = \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{bmatrix}$ **b** $|\mathbf{A}| = 0$

c oranges 50 cents, apples 80 cents, pears 70 cents, cabbages \$2.00, lettuces \$1.50

EXERCISE 14K

1 **a** $\begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix}$ **b** $\begin{bmatrix} -5.5 & 4.5 & 7.5 \\ -0.5 & 0.5 & 0.5 \\ 4 & -3 & -5 \end{bmatrix}$

2 **a** $\begin{bmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{bmatrix}$ **b** $\begin{bmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2 & -1.086 & -0.396 \end{bmatrix}$

EXERCISE 14L

1 **a** $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix}$

c $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -2 \end{bmatrix}$

2 **Hint:** Show $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

3 $\mathbf{AB} = \mathbf{I}$, $a = 2, b = -1, c = 3$

4 $\mathbf{MN} = 4\mathbf{I}$, $u = -1, v = 3, w = 5$

5 **a** $x = 2.3, y = 1.3, z = -4.5$ **b** $x = -\frac{1}{3}$, $y = -\frac{95}{21}$, $z = \frac{2}{21}$

c $x = 4\frac{2}{3}$, $y = 2$, $z = -4\frac{1}{3}$

6 **a** $x = 2, y = -1, z = 5$ **b** $x = 4, y = -2, z = 1$

c $x = 4, y = -3, z = 2$ **d** $x = 4, y = 6, z = -7$

e $x = 3$, $y = 11$, $z = -7$

f $x \doteqdot 0.33$, $y \doteqdot 7.65$, $z \doteqdot 4.16$

7 **a** $x = 14$, $y = 11$, $z = 17$

b x represents the cost per cricket ball in dollars, y represents the cost per softball in dollars, z represents the cost per netball in dollars

c 12 netballs

8 **a** $\begin{aligned} 2x + 3y + 8z &= 352 \\ x + 5y + 4z &= 274 \\ x + 2y + 11z &= 351 \end{aligned}$ **b** $x = 42, y = 28, z = 23$ **c** \$1 201 000

9

x	-3	-2	-1	0	1	2	3
y	-23	-15	-9	-5	-3	-3	-5

10 \$11.80 per kg

11 **a** $\begin{aligned} 5p + 5q + 6r &= 405 \\ 15p + 20q + 6r &= 1050 \\ 15p + 20q + 36r &= 1800 \end{aligned}$ **b** $p = 24$, $q = 27$, $r = 25$

12 **a** $a = 50\,000$, $b = 100\,000$, $c = 240\,000$ **b** yes

c 2003, $\doteqdot$ \$284 000, 2005, $\doteqdot$ \$377 000

REVIEW SET 14A

1 **a** $\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} -2 & 0 \\ 4 & -8 \end{bmatrix}$

d $\begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$ **e** $\begin{bmatrix} -5 & -4 \\ -2 & 6 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & 6 \\ 4 & -11 \end{bmatrix}$

g $\begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}$ **h** $\begin{bmatrix} 3 & 2 \\ -6 & -8 \end{bmatrix}$ **i** $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{bmatrix}$

j $\begin{bmatrix} 9 & 4 \\ 0 & 1 \end{bmatrix}$ **k** $\begin{bmatrix} -3 & -10 \\ 6 & 8 \end{bmatrix}$ **l** $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{bmatrix}$

2 **a** $a = 0, b = 5, c = 1, d = -4$

b $a = 2, b = -1, c = 3, d = 8$

3 **a** $\mathbf{Y} = \mathbf{B} - \mathbf{A}$ **b** $\mathbf{Y} = \frac{1}{2}(\mathbf{D} - \mathbf{C})$ **c** $\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$

d $\mathbf{Y} = \mathbf{CB}^{-1}$ **e** $\mathbf{Y} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$ **f** $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$

4 **a** $x = 0, y = -\frac{1}{2}$ **b** $x = \frac{12}{7}, y = \frac{13}{7}$ **c** $\mathbf{X} = \begin{bmatrix} -1 & 8 \\ -2 & 6 \end{bmatrix}$

d $\mathbf{X} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$ **e** $\mathbf{X} = \begin{bmatrix} \frac{14}{3} \\ \frac{1}{3} \end{bmatrix}$ **f** $\mathbf{X} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 **a** $\begin{bmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 11 & 12 \end{bmatrix}$ **d** **BA** does not exist.

6 **a** $\begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{bmatrix}$

REVIEW SET 14B

1 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **2** **a** $\begin{bmatrix} 10 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 15 & 18 & 21 \end{bmatrix}$

d **CA** does not exist **e** $\begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$

3 **a** $\begin{bmatrix} \frac{7}{2} & -4 \\ \frac{-5}{2} & 3 \end{bmatrix}$ **b** does not exist **c** $\begin{bmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{bmatrix}$

4 \$56.30 **5** **b** $2\mathbf{A} - \mathbf{I}$ **6** $\mathbf{AB} = \mathbf{I}$, $\mathbf{BA} = \mathbf{I}$, $\mathbf{A}^{-1} = \mathbf{B}$

REVIEW SET 14C

1 Unique solution for $k \neq -3$ or 1. If $k = -3$, infinitely many solutions exist. If $k = 1$, no solutions exist.

2 $x = -1$, 2 or -4 {using technology}

3 **a** $\begin{bmatrix} 10 & -12 \\ -10 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{bmatrix}$

c not possible **d** $\begin{bmatrix} 2.9 & -0.3 \\ -0.3 & 2.1 \end{bmatrix}$

4 $\mathbf{X} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{bmatrix}$ **5** $\mathbf{A}^{-1} = \frac{5}{3}\mathbf{A} - 2\mathbf{I}$

6 **a** **i** $|\mathbf{B}| \neq 0$ **ii** $\mathbf{AB} = \mathbf{BA}$

b $k \in \mathbb{R}$, but $k \neq 3, -2, 2$ {technology}

REVIEW SET 14D

1 $x = 1$, $y = -1$, $z = 2$

3 $\mathbf{A}^3 = 27\mathbf{A} + 10\mathbf{I}$, $\mathbf{A}^4 = 145\mathbf{A} + 54\mathbf{I}$,
$\mathbf{A}^5 = 779\mathbf{A} + 290\mathbf{I}$, $\mathbf{A}^6 = 4185\mathbf{A} + 1558\mathbf{I}$

4 **a** $d = 80$ **b** $a = 2$, $b = 8$, $c = 10$

5 $\mathbf{X} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$ **6** **a** $x = 6$, $y = -2$, $z = 1$

b $x = \frac{3}{2}$, $y = -\frac{7}{6}$, $z = -\frac{7}{6}$

REVIEW SET 14E

1 **a** $\begin{aligned} 3x + 2y + 5z &= 267 \\ 2x + 3y + z &= 145 \\ x + 5y + 4z &= 230 \end{aligned}$ **b** Opera \$32, Play \$18, Concert \$27 **c** \$200

2 $x = 2$, $y = 1$, $z = 3$

3 **a** $\begin{bmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -10 & -6 \\ 5 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{bmatrix}$

d not possible **e** $\begin{bmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{bmatrix}$

5 **a** $s(t) = -3t^2 + 18t + 48$ m **b** 48 m **c** 8 seconds

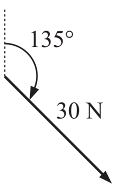
6 $\mathbf{A}^3 = -\mathbf{I}$, $\mathbf{A}^4 = -\mathbf{A}$, $\mathbf{A}^5 = -\mathbf{A} + \mathbf{I}$, $\mathbf{A}^6 = \mathbf{I}$,
$\mathbf{A}^7 = \mathbf{A}$, $\mathbf{A}^8 = \mathbf{A} - \mathbf{I}$

a $\mathbf{A}^{6n+3} = (\mathbf{A}^6)^n\mathbf{A}^3 = -\mathbf{I}$, $\mathbf{A}^{6n+5} = -\mathbf{A} + \mathbf{I}$

b $\mathbf{A}^{-1} = -\mathbf{A} + \mathbf{I}$

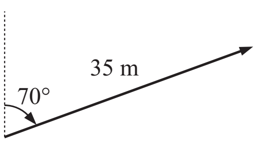
EXERCISE 15A.1

1 **a**

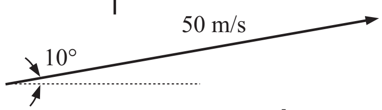


b

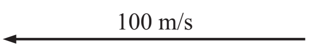
c



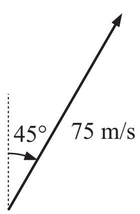
d



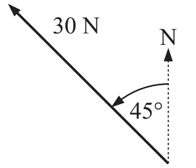
2 **a**



b

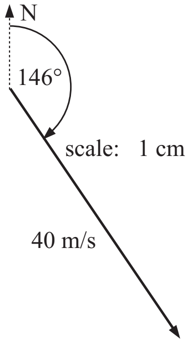


3 **a**



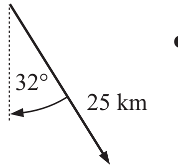
scale: 1 cm ≡ 10 N

b



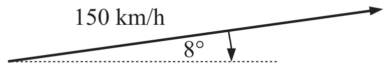
scale: 1 cm ≡ 10 m/s

c



Scale: 1 cm ≡ 10 km

d



Scale: 1 cm ≡ 30 km/h

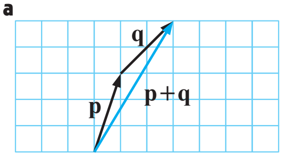
EXERCISE 15A.2

1 **a** **p**, **q**, **s**, **t** **b** **p**, **q**, **r**, **t** **c** **p** and **r**, **q** and **t** **d** **q**, **t**

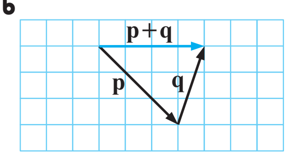
2 **a** true **b** true **c** false **d** false **e** true **f** false

EXERCISE 15B.1

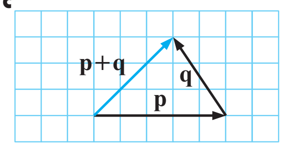
1 **a**



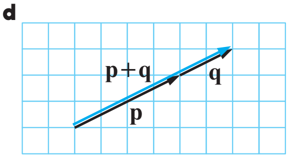
b



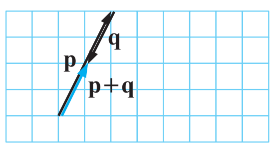
c



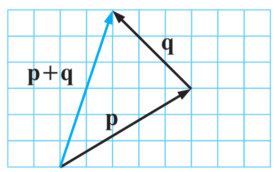
d



e

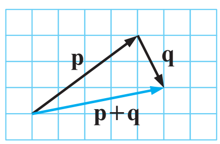


f

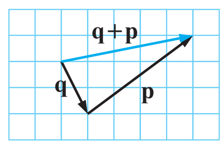


2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{AD}$ **d** $\overrightarrow{AD}$

3 **a** **i**



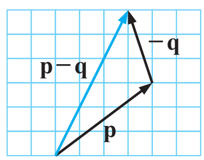
ii



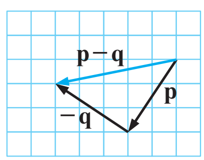
b yes

EXERCISE 15B.2

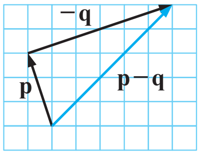
1 **a**



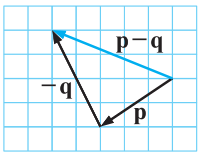
b



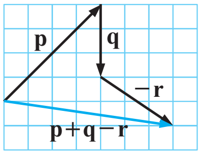
c



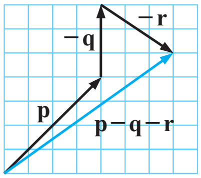
d



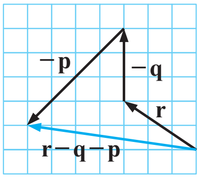
2 **a**



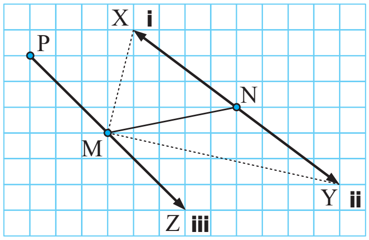
b



c



3 **a**



b a parallelogram

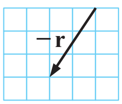
4 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{AB}$ **c** $\mathbf{0}$ **d** $\overrightarrow{AD}$ **e** $\mathbf{0}$ **f** $\overrightarrow{AD}$

5 **a** $\mathbf{t}=\mathbf{r}+\mathbf{s}$ **b** $\mathbf{r}=-\mathbf{s}-\mathbf{t}$ **c** $\mathbf{r}=-\mathbf{p}-\mathbf{q}-\mathbf{s}$
d $\mathbf{r}=\mathbf{q}-\mathbf{p}+\mathbf{s}$ **e** $\mathbf{p}=\mathbf{t}+\mathbf{s}+\mathbf{r}-\mathbf{q}$
f $\mathbf{p}=-\mathbf{u}+\mathbf{t}+\mathbf{s}-\mathbf{r}-\mathbf{q}$

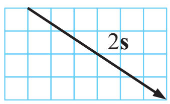
6 **a** **i** $\mathbf{r}+\mathbf{s}$ **ii** $-\mathbf{t}-\mathbf{s}$ **iii** $\mathbf{r}+\mathbf{s}+\mathbf{t}$
b **i** $\mathbf{p}+\mathbf{q}$ **ii** $\mathbf{q}+\mathbf{r}$ **iii** $\mathbf{p}+\mathbf{q}+\mathbf{r}$

EXERCISE 15B.3

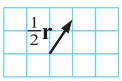
1 **a**



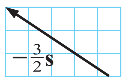
b



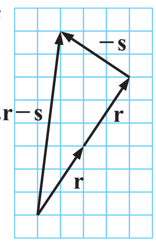
c



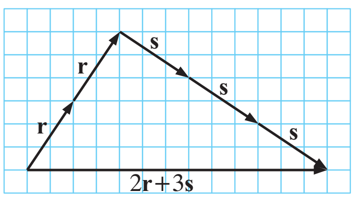
d



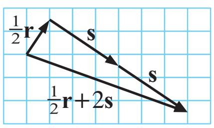
e



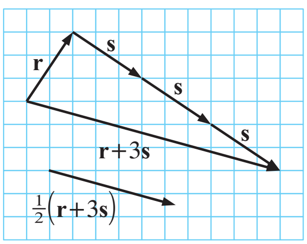
f



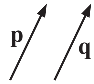
g



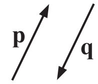
h



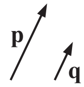
2 **a**



b

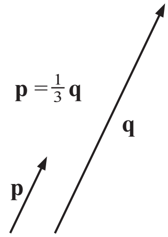


c



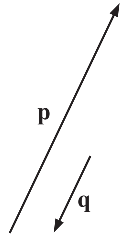
$\mathbf{p}=2\mathbf{q}$

d



$\mathbf{p}=\frac{1}{3}\mathbf{q}$

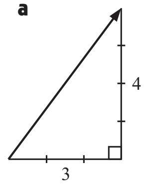
e



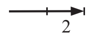
$\mathbf{p}=-3\mathbf{q}$

EXERCISE 15C.1

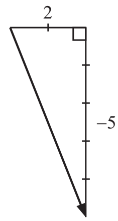
1 **a**



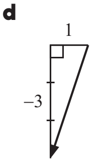
b



c



d



2 **a** $\begin{bmatrix}7\\3\end{bmatrix}$ **b** $\begin{bmatrix}-6\\0\end{bmatrix}$ **c** $\begin{bmatrix}2\\-5\end{bmatrix}$ **d** $\begin{bmatrix}0\\6\end{bmatrix}$ **e** $\begin{bmatrix}-6\\3\end{bmatrix}$ **f** $\begin{bmatrix}-5\\-5\end{bmatrix}$

EXERCISE 15C.2

1 **a** $\begin{bmatrix}-2\\6\end{bmatrix}$ **b** $\begin{bmatrix}-2\\6\end{bmatrix}$ **c** $\begin{bmatrix}-1\\-1\end{bmatrix}$ **d** $\begin{bmatrix}-1\\-1\end{bmatrix}$ **e** $\begin{bmatrix}-5\\-3\end{bmatrix}$ **f** $\begin{bmatrix}-5\\-3\end{bmatrix}$
g $\begin{bmatrix}-6\\4\end{bmatrix}$ **h** $\begin{bmatrix}-4\\1\end{bmatrix}$

2 **a** $\begin{bmatrix}-3\\7\end{bmatrix}$ **b** $\begin{bmatrix}-4\\-3\end{bmatrix}$ **c** $\begin{bmatrix}-8\\-1\end{bmatrix}$ **d** $\begin{bmatrix}-6\\9\end{bmatrix}$ **e** $\begin{bmatrix}0\\-5\end{bmatrix}$ **f** $\begin{bmatrix}6\\-9\end{bmatrix}$

3 **a** $\begin{bmatrix}-5\\4\end{bmatrix}$ **b** $\begin{bmatrix}1\\2\end{bmatrix}$ **c** $\begin{bmatrix}6\\-5\end{bmatrix}$

EXERCISE 15C.3

1 **a** $\begin{bmatrix}-3\\-15\end{bmatrix}$ **b** $\begin{bmatrix}-1\\2\end{bmatrix}$ **c** $\begin{bmatrix}0\\14\end{bmatrix}$ **d** $\begin{bmatrix}5\\-3\end{bmatrix}$ **e** $\begin{bmatrix}\frac{5}{2}\\\frac{11}{2}\end{bmatrix}$ **f** $\begin{bmatrix}-7\\7\end{bmatrix}$
g $\begin{bmatrix}5\\11\end{bmatrix}$ **h** $\begin{bmatrix}3\\\frac{17}{3}\end{bmatrix}$ **2** **a** $\begin{bmatrix}8\\-1\end{bmatrix}$ **b** $\begin{bmatrix}8\\-1\end{bmatrix}$ **c** $\begin{bmatrix}8\\-1\end{bmatrix}$

EXERCISE 15C.4

1 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units **d** $\sqrt{10}$ units
e $\sqrt{29}$ units

2 **a** $\sqrt{10}$ units **b** $2\sqrt{10}$ units **c** $2\sqrt{10}$ units **d** $3\sqrt{10}$ units
e $3\sqrt{10}$ units **f** $2\sqrt{5}$ units **g** $8\sqrt{5}$ units **h** $8\sqrt{5}$ units
i $\sqrt{5}$ units **j** $\sqrt{5}$ units

EXERCISE 15D

1 **a** $\mathbf{x}=\frac{1}{2}\mathbf{q}$ **b** $\mathbf{x}=2\mathbf{n}$ **c** $\mathbf{x}=-\frac{1}{3}\mathbf{p}$ **d** $\mathbf{x}=\frac{1}{2}(\mathbf{r}-\mathbf{q})$
e $\mathbf{x}=\frac{1}{5}(4\mathbf{s}-\mathbf{t})$ **f** $\mathbf{x}=3(4\mathbf{m}-\mathbf{n})$

2 **a** $\mathbf{y}=\begin{bmatrix}-1\\\frac{3}{2}\end{bmatrix}$ **b** $\mathbf{y}=\begin{bmatrix}2\\4\end{bmatrix}$ **c** $\mathbf{y}=\begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}$ **d** $\mathbf{y}=\begin{bmatrix}\frac{5}{4}\\\frac{3}{4}\end{bmatrix}$

EXERCISE 15E

1 **a** $\begin{bmatrix}-1\\8\end{bmatrix}$, $\sqrt{65}$ units **b** $\begin{bmatrix}-3\\-10\end{bmatrix}$, $\sqrt{109}$ units **c** $\begin{bmatrix}4\\2\end{bmatrix}$, $2\sqrt{5}$ units

2 **a** M(1, 4) **b** $\overrightarrow{CA}=\begin{bmatrix}7\\5\end{bmatrix}$, $\overrightarrow{CM}=\begin{bmatrix}5\\3\end{bmatrix}$, $\overrightarrow{CB}=\begin{bmatrix}3\\1\end{bmatrix}$

3 **a** B(−1, 10) **b** B(−2, −9) **c** B(7, 4)

4 C(5, 1), D(8, −1), E(11, −3)

5 **a** parallelogram **b** not parallelogram **c** parallelogram

6 **a** D(9, −1) **b** R(3, 1) **c** X(2, −1)

7 **a** $r=2,\ s=-5$ **b** $r=4,\ s=-1$

EXERCISE 15F

1 **a** and **f**, **b** and **e**, **c** and **d**

2 **a** $t=12$ **b** $t=8$ **c** $t=12$

EXERCISE 15G

1 **a** unit vector **b** unit vector **c** not a unit vector
d unit vector **e** not a unit vector

2 **a** $2\mathbf{i}-\mathbf{j}$ **b** $-3\mathbf{i}-4\mathbf{j}$ **c** $-3\mathbf{i}$ **d** $7\mathbf{j}$ **e** $\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$

3 **a** $\begin{bmatrix}3\\5\end{bmatrix}$ **b** $\begin{bmatrix}5\\-4\end{bmatrix}$ **c** $\begin{bmatrix}-4\\0\end{bmatrix}$ **d** $\begin{bmatrix}0\\3\end{bmatrix}$ **e** $\begin{bmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix}$

4 **a** $k=\pm1$ **b** $k=\pm1$ **c** $k=0$ **d** $k=\pm\frac{\sqrt{3}}{2}$ **e** $k=\pm\frac{\sqrt{5}}{3}$

5 **a** 5 units **b** 5 units **c** $\sqrt{53}$ units **d** $\doteqdot 6.12$ units

6 **a** $\frac{1}{\sqrt{5}}(\mathbf{i}+2\mathbf{j})$ **b** $\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ **c** $\frac{1}{\sqrt{29}}(-2\mathbf{i}-5\mathbf{j})$

EXERCISE 15H

1 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

2 **a** $t=6$ **b** $t=-6$ **c** $t=\pm2\sqrt{3}$ **d** $t=-8$ **e** $t=0$ or 2

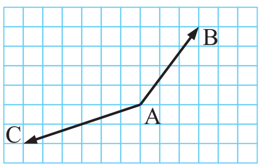
3 **a** $\measuredangle$BAC is a right angle **b** not a right angle
c $\measuredangle$BAC is a right angle **d** $\measuredangle$ACB is a right angle

4 **a** $k\begin{bmatrix}-2\\5\end{bmatrix},\ k\neq0$ **b** $k\begin{bmatrix}-2\\1\end{bmatrix},\ k\neq0$ **c** $k\begin{bmatrix}1\\3\end{bmatrix},\ k\neq0$
d $k\begin{bmatrix}3\\4\end{bmatrix},\ k\neq0$ **e** $k\begin{bmatrix}0\\1\end{bmatrix},\ k\neq0$

5 **a** 5 **b** -9

6 **a** **i** $\mathbf{a}=\begin{bmatrix}4\\4\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}7\\-1\end{bmatrix}$ **ii** $\theta\doteqdot53.1^\circ$
b **i** $\mathbf{a}=\begin{bmatrix}-2\\6\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}9\\-1\end{bmatrix}$ **ii** $\theta\doteqdot114.8^\circ$

7 **a**



b $\doteqdot 145.3^\circ$

8 **a** 71.57° **b** 176.6° **c** 18.43° **d** 45°

9 **a** (23, 14) **b** $\doteqdot64.4^\circ$ **c** 138 units^2

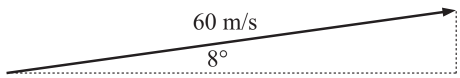
10 **a** 18.43° **b** 85.60°

11 **a** $\measuredangle A=18.4^\circ,\ \measuredangle B=8.1^\circ,\ \measuredangle C=153.4^\circ$
b $\measuredangle A=66.8^\circ,\ \measuredangle B=31.3^\circ,\ \measuredangle C=81.9^\circ$

12 **a** 101.3° or 78.7° **b** 116.6° or 63.4°
c 63.4° or 116.6° **d** 71.6° or 108.4°

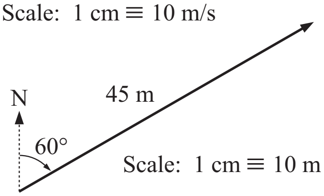
REVIEW SET 15A

1 **a**



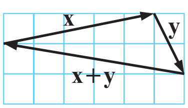
Scale: 1 cm ≡ 10 m/s

b

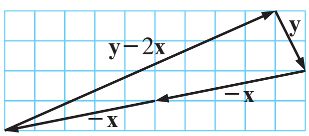


Scale: 1 cm ≡ 10 m

2 **a**



b



3 **a** $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$ **4** 4.845 km, 208° **5** **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AD}$

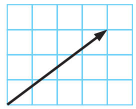
6 **a** AB $=\frac{1}{2}$CD, AB $\parallel$ CD **b** C is midpoint AB

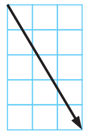
7 **a** $\mathbf{p}+\mathbf{r}=\mathbf{q}$ **b** $\mathbf{l}+\mathbf{m}=\mathbf{k}-\mathbf{j}+\mathbf{n}$

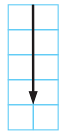
8 **a** $\mathbf{r}+\mathbf{q}$ **b** $-\mathbf{p}+\mathbf{r}+\mathbf{q}$ **c** $\mathbf{r}+\frac{1}{2}\mathbf{q}$ **d** $-\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{r}$

REVIEW SET 15B

1 **a** $\begin{bmatrix}4\\3\end{bmatrix}$ **b** $\begin{bmatrix}3\\-5\end{bmatrix}$ **c** $\begin{bmatrix}0\\-4\end{bmatrix}$







2 **a** $\begin{bmatrix}5\\2\end{bmatrix}$ **b** $\begin{bmatrix}2\\-4\end{bmatrix}$ **3** **a** $\begin{bmatrix}-4\\-2\end{bmatrix}$ **b** $\begin{bmatrix}-1\\-13\end{bmatrix}$ **c** $\begin{bmatrix}-4\\8\end{bmatrix}$

4 **a** $\begin{bmatrix}-4\\3\end{bmatrix}$ **b** $\begin{bmatrix}-4\\6\end{bmatrix}$ **c** $\begin{bmatrix}-5\\3\end{bmatrix}$ **5** $\begin{bmatrix}2\\1\end{bmatrix}$ **6** $\begin{bmatrix}1\\4\end{bmatrix}$

7 **a** $\sqrt{17}$ units **b** $\sqrt{13}$ units **c** $\sqrt{10}$ units **d** $\sqrt{109}$ units

8 **a** $\mathbf{p}+\mathbf{q}$ **b** $\frac{3}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}$

REVIEW SET 15C

1 **a** $\mathbf{x}=\begin{bmatrix}-1\\\frac{1}{3}\end{bmatrix}$ **b** $\mathbf{x}=\begin{bmatrix}1\\-10\end{bmatrix}$

2 **a** $\begin{bmatrix}5\\-6\end{bmatrix}$ **b** $\begin{bmatrix}4\\-2\end{bmatrix}$ **c** $2\sqrt{5}$ units **4** $r=4,\ s=7$

5 **a** $\mathbf{q}+\mathbf{r}$ **b** $\mathbf{r}+\mathbf{q}$, DB = AC, DB $\parallel$ AC

6 **a** $\begin{bmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{bmatrix}$ **b** $\sqrt{5}$ units, 297°

7 **a** $\begin{bmatrix}-2\\11\end{bmatrix}$ **b** $\sqrt{10}$ units **c** $\begin{bmatrix}0\\-1\end{bmatrix}$

8 **a** $k=\pm\frac{7}{\sqrt{33}}$ **b** $k=\pm\frac{1}{\sqrt{2}}$

REVIEW SET 15D

1 **a** -13 **b** -36 **3** $t=\frac{2}{3}$ or -3 **4** $k=6$

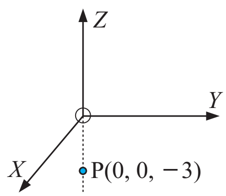
5 $k\begin{bmatrix}5\\4\end{bmatrix},\ k\neq0$ **6** $\measuredangle K=64.44^\circ,\ \measuredangle L=56.89^\circ,\ \measuredangle M=58.67^\circ$

7 72.35° or 107.65°

8 **a** **i** (1) $\mathbf{p}+\mathbf{q}$ (2) $\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}$
b **i** $\overrightarrow{AC}=-\mathbf{p}+\mathbf{r}$, $\overrightarrow{BC}=-\mathbf{q}+\mathbf{r}$

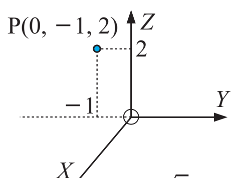
EXERCISE 16A

1 **a**



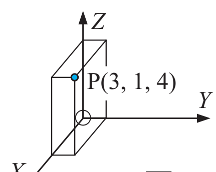
OP = 3 units

b



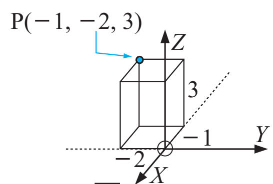
OP = $\sqrt{5}$ units

c



OP = $\sqrt{26}$ units

d



OP = $\sqrt{14}$ units

2 **a** **i** $\sqrt{14}$ units **ii** $(-\frac{1}{2},\frac{1}{2},2)$ **b** **i** $\sqrt{14}$ units **ii** $(1,-\frac{1}{2},\frac{3}{2})$
c **i** $\sqrt{21}$ units **ii** $(1,-\frac{1}{2},0)$ **d** **i** $\sqrt{14}$ units **ii** $(1,\frac{1}{2},-\frac{3}{2})$

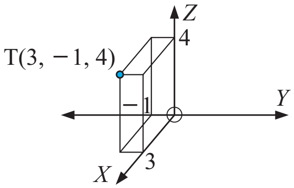
4 **a** isosceles **b** right angled **c** right angled **d** straight line

5 $(0,\ 3,\ 5)$, $r = \sqrt{3}$ units

6 **a** $(0,\ y,\ 0)$ **b** $(0,\ 2,\ 0)$ and $(0,\ -4,\ 0)$

EXERCISE 16B.1

1 **a**



b $\overrightarrow{OT} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ **c** $OT = \sqrt{26}$ units

2 **a** $\overrightarrow{AB} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$, $\overrightarrow{BA} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ **b** $AB = \sqrt{26}$ units, $BA = \sqrt{26}$ units

3 $\overrightarrow{OA} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

4 **a** $\overrightarrow{NM} = \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$ **b** $\overrightarrow{MN} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$ **c** $MN = \sqrt{42}$ units

5 **a** $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$, $OA = \sqrt{30}$ units **b** $\overrightarrow{AC} = \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix}$, $AC = \sqrt{30}$ units

c $\overrightarrow{CB} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$, $CB = \sqrt{35}$ units

6 **a** $\sqrt{13}$ units **b** $\sqrt{14}$ units **c** 3 units

EXERCISE 16B.2

1 **a** $a = 5,\ b = 6,\ c = -6$ **b** $a = 4,\ b = 2,\ c = 1$

2 **a** $a = \frac{1}{3},\ b = 2,\ c = 1$ **b** $a = 1,\ b = 2$

c $a = 1,\ b = -1,\ c = 2$

3 **a** $r = 2,\ s = 4,\ t = -7$ **b** $r = -4,\ s = 0,\ t = 3$

4 **a** $\overrightarrow{AB} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$, $\overrightarrow{DC} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ **b** ABCD is a parallelogram

5 **a** $S = (-2,\ 8,\ -3)$

EXERCISE 16C

1 **a** $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}$ **d** $\begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix}$ **e** $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

f $\begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{bmatrix}$ **g** $\begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$ **h** $\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

2 **a** $\sqrt{11}$ units **b** $\sqrt{14}$ units **e** $\begin{bmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{bmatrix}$ **f** $\begin{bmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{bmatrix}$

c $\sqrt{38}$ units **d** $\sqrt{3}$ units

3 **a** $\mathbf{x} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$ **b** $\mathbf{x} = \frac{1}{3}(\mathbf{b} - 2\mathbf{a})$ **c** $\mathbf{x} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

4 **a** $\mathbf{x} = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}$ **b** $\mathbf{x} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$ **c** $\mathbf{x} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{bmatrix}$

5 $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, $AB = \sqrt{29}$ units

7 **a** $\overrightarrow{BD} = \frac{1}{2}\mathbf{a}$ **b** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **c** $\overrightarrow{BA} = -\mathbf{b} + \mathbf{a}$

d $\overrightarrow{OD} = \mathbf{b} + \frac{1}{2}\mathbf{a}$ **e** $\overrightarrow{AD} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ **f** $\overrightarrow{DA} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

8 **a** $\begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$ **c** $\begin{bmatrix} -3 \\ 6 \\ -5 \end{bmatrix}$

EXERCISE 16D

1 $r = 3,\ s = -9$

2 $a = -6,\ b = -4$

3 **a** $\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$

4 **a** $\overrightarrow{AB} \parallel \overrightarrow{CD}$, $AB = 3\,CD$

b $\overrightarrow{RS} \parallel \overrightarrow{KL}$, $RS = \frac{1}{2}KL$ opposite direction

c A, B and C are collinear and $AB = 2\,BC$

d A, B and C are collinear and $AC = 3\,BC$

5 **a** $\overrightarrow{PR} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$, $\overrightarrow{QS} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$ **b** $PR = \frac{1}{2}\,QS$

EXERCISE 16E

1 **a** $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\sqrt{3}$ units **b** $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\sqrt{11}$ units

c $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$, $\sqrt{26}$ units **d** $\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $\frac{1}{\sqrt{2}}$ units

2 **a** $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ **b** $2\mathbf{i} - 5\mathbf{k}$ **c** $-3\mathbf{k}$ **d** $-3\mathbf{i} + \mathbf{k}$

3 **a** $\sqrt{2}$ units **b** 1 unit **c** $\sqrt{3}$ units **d** $\sqrt{5}$ units

4 **a** $2\mathbf{i} + 3\mathbf{k}$ **b** $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ **c** $13\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$

d $-9\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

5 **a** $\frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ **b** $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

EXERCISE 16F

1 **a** $-7 : 2$ **b** $-1 : 2$

2 **a** $a = 7,\ b = -1$ **b** $a = -\frac{7}{2},\ b = -\frac{21}{2}$

EXERCISE 16G

1 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

2 **a** 1 **b** 1 **c** 0 **4** $t = -\frac{3}{2}$ **6** **b** $t = -\frac{5}{6}$

8 **b** $AB = \sqrt{14}$ units, $BC = \sqrt{14}$ units ABCD is a rhombus

c 0, the diagonals of a rhombus are perpendicular.

9 **a** 1 **b** 109.5^o (acute 70.5^o) **c** $\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$ **d** $\frac{1}{\sqrt{3}}$

10 $\angle ABC \doteqdot 62.5^o$, the exterior angle 117.5^o

11 **a** 43.6^o **b** 11.0^o **c** 105.0^o **d** 90^o

12 **a** 54.7^o **b** 60^o **c** 35.3^o

13 **a** 30.3^o **b** 54.2^o **14** **a** $M(\frac{3}{2},\ \frac{5}{2},\ \frac{3}{2})$ **b** 51.5^o

15 **a** $t = 0$ or -3 **b** $r = -2,\ s = 5,\ t = -4$

16 **a** 74.5^o **b** 72.45^o

REVIEW SET 16A

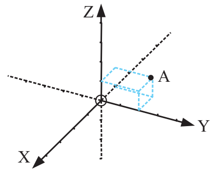
1 **a** $\overrightarrow{PQ} = \begin{bmatrix} -3 \\ 12 \\ 3 \end{bmatrix}$ **b** $\sqrt{162}$ units **c** $\sqrt{51}$ units

2 **a** $\begin{bmatrix}3\\-3\\11\end{bmatrix}$ **b** $\begin{bmatrix}7\\-3\\-26\end{bmatrix}$ **c** $\sqrt{74}$ units **3** $\begin{bmatrix}8\\-8\\7\end{bmatrix}$

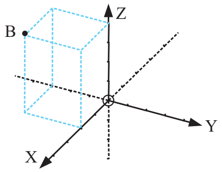
4 $m=5,\ n=-\frac{1}{2}$ **5** $2:3$ **6** $t=2\pm\sqrt{2}$ **7** 80.3^o **8** 40.7^o

REVIEW SET 16B

1 **a**



b



2 **a** $\begin{bmatrix}-6\\1\\3\end{bmatrix}$ **b** $\sqrt{46}$ units **c** $(-1,\ 3\frac{1}{2},\ \frac{1}{2})$

3 **a** -1 **b** $\begin{bmatrix}4\\-1\\7\end{bmatrix}$ **c** 60^o

4 $\measuredangle K \doteqdot 123.7^o$, $\measuredangle L \doteqdot 11.3^o$, $\measuredangle M \doteqdot 45.0^o$

5 $t=\frac{3}{2}$ or -2 **7** 63.95^o **8** $c=\frac{50}{3}$

EXERCISE 17A

1 **a** **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-4\end{bmatrix}+t\begin{bmatrix}1\\4\end{bmatrix}$ **ii** $x=3+t$
$y=-4+4t,\quad t\in\mathcal{R}$

b **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}+t\begin{bmatrix}-8\\2\end{bmatrix}$ **ii** $x=5-8t$
$y=2+2t,\quad t\in\mathcal{R}$

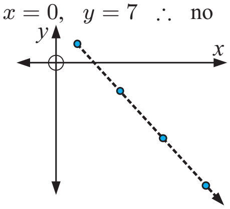
c **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6\\0\end{bmatrix}+t\begin{bmatrix}3\\7\end{bmatrix}$ **ii** $x=-6+3t$
$y=7t,\quad t\in\mathcal{R}$

d **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\11\end{bmatrix}+t\begin{bmatrix}-2\\1\end{bmatrix}$ **ii** $x=-1-2t$
$y=11+t,\quad t\in\mathcal{R}$

2 $x=-1+2\lambda,\quad y=4-\lambda,\quad t\in\mathcal{R}$
Points are: $(-1,4)$, $(1,3)$, $(5,1)$, $(-3,5)$, $(-9,8)$

3 **a** When $t=1$, $x=3$, $y=-2$ $\therefore$ yes **b** $k=-5$
When $t=-2$, $x=0$, $y=7$ $\therefore$ no

4 **a** $(1,2)$ **b**



c $\sqrt{29}$ cm/s

EXERCISE 17B

1 **a** **i** $(-4,3)$ **ii** $\begin{bmatrix}12\\5\end{bmatrix}$ **iii** 13 m/s

b **i** $(0,-6)$ **ii** $\begin{bmatrix}3\\-4\end{bmatrix}$ **iii** 5 m/s

c **i** $(-2,-7)$ **ii** $\begin{bmatrix}-6\\-4\end{bmatrix}$ **iii** $\sqrt{52}$ m/s

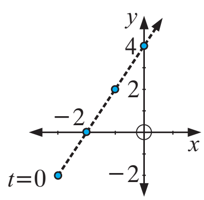
2 **a** **i** $\begin{bmatrix}8\\4\end{bmatrix}$ **ii** $\sqrt{80}$ km/h **b** **i** $\begin{bmatrix}6\\2\end{bmatrix}$ **ii** $\sqrt{40}$ km/h

c **i** $\begin{bmatrix}7\\24\end{bmatrix}$ **ii** 25 km/h

3 **a** $\begin{bmatrix}120\\-90\end{bmatrix}$ **b** $\begin{bmatrix}12\\3.5\end{bmatrix}$ **c** $\begin{bmatrix}20\sqrt{5}\\10\sqrt{5}\end{bmatrix}$ **d** $\begin{bmatrix}-60\\80\end{bmatrix}$

EXERCISE 17C

1 **a** $\begin{bmatrix}-3+2t\\-2+4t\end{bmatrix}$ **d**



b $(2,8)$

c **i** $t=1.5$ sec
ii $t=0.5$ sec

2 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-10\end{bmatrix}+t\begin{bmatrix}4\\-3\end{bmatrix},\ t\in\mathcal{R}$ **b** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\6\end{bmatrix}+\frac{t}{2.5}\begin{bmatrix}20\\15\end{bmatrix},\ t\in\mathcal{R}$

c $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-5\\0\end{bmatrix}+t\begin{bmatrix}2\\1\end{bmatrix},\ t\in\mathcal{R}$ **d** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-8\\-16\end{bmatrix}+t\begin{bmatrix}9\\12\end{bmatrix},\ t\in\mathcal{R}$

3 **a** 37.7 km **b** 13 km/h **c** 7:40 am

4 **a** A is at $(4,5)$, B is at $(1,-8)$

b For A it is $\begin{bmatrix}1\\-2\end{bmatrix}$. For B it is $\begin{bmatrix}2\\1\end{bmatrix}$.

c For A, speed is $\sqrt{5}$ km/h. For B, speed is $\sqrt{5}$ km/h.

d 10:12 am

e $\begin{bmatrix}1\\-2\end{bmatrix}\bullet\begin{bmatrix}2\\1\end{bmatrix}=0$, $\therefore$ direction vectors are $\perp$

5 **a** $\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}-5\\4\end{bmatrix}+t\begin{bmatrix}3\\-1\end{bmatrix}$ $\therefore\ x_1(t)=-5+3t,\ \ y_1(t)=4-t$

b speed $=\sqrt{10}$ km/min

c a minutes later, $(t-a)$ min have elapsed.

$\therefore\ \begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}15\\7\end{bmatrix}+(t-a)\begin{bmatrix}-4\\-3\end{bmatrix}$

$\therefore\ x_2(t)=15-4(t-a),\quad y_2(t)=7-3(t-a)$

d Torpedo is fired at 1:35:28 pm and the explosion occurs at 1:37:42 pm.

EXERCISE 17D

1 **a** $(-3+2t,\ -2+4t)$ **b** **i** $t=0.5$ sec **ii** $t=1.5$ sec

c $(0,4)$ and $(-2,0)$

2 **a** $(-2-t,\ 1-3t)$ **b** $t=\frac{1}{3}$ sec **c** $(-2\frac{1}{3},\ 0)$

3 **a** $-6\mathbf{i}+8\mathbf{j}$ **b** $\begin{bmatrix}6-6t\\-6+8t\end{bmatrix}$ **c** when $t=\frac{3}{4}$ hour

d $t=0.84$ and position is $(0.96,\ 0.72)$

4 **a** **i** $-8\mathbf{i}-5\mathbf{j}$ **ii** $3\mathbf{i}+3\mathbf{j}$ **iii** $(-8+3t)\mathbf{i}+(-5+3t)\mathbf{j}$

b $t=2\frac{1}{6}$ hours, i.e., 2 h 10 min

c shortest distance is $\frac{3}{2}\sqrt{2}\doteqdot 2.12$ km $\therefore$ breaking the law

5 **a** $(3,6)$ **b** 2.5 m/s **c** $t=2$ sec **d** $4x-3y=-6$

6 **a** $\begin{bmatrix}-120\\-40\end{bmatrix}$ **b** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}200\\100\end{bmatrix}+t\begin{bmatrix}-120\\-40\end{bmatrix}$ **c** $\begin{bmatrix}80\\60\end{bmatrix}$

d $\left|\begin{bmatrix}80\\60\end{bmatrix}\right|=100$ km **e** $t=1\frac{3}{4}$ hours and $d_{\min}\doteqdot 31.6$ km

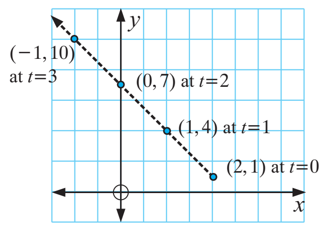
f at $t=2\frac{1}{2}$ hours

7 **a** A(18, 0) and B(0, 12) **b** R is at $\left(x,\ \dfrac{36-2x}{3}\right)$

c $\overrightarrow{PR}=\begin{bmatrix}x-4\\\frac{36-2x}{3}\end{bmatrix}$ and $\overrightarrow{AB}=\begin{bmatrix}-18\\12\end{bmatrix}$

d $(\frac{108}{13},\ \frac{84}{13})$ and distance $\doteqdot 7.766$ km

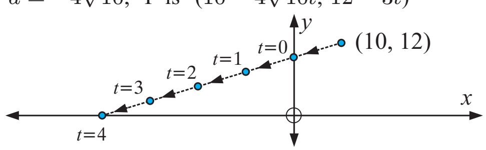
8 **a** P(2, 1) **b**



c $\frac{3}{\sqrt{10}}$ cm
when $t=2.9$ sec

9 **a** $(10,12)$ **b** $a=\pm4\sqrt{10}$

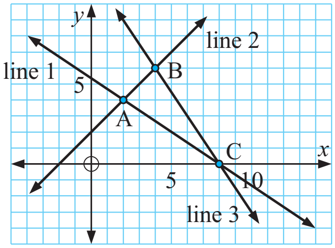
c for $a=-4\sqrt{10}$, P is $(10-4\sqrt{10}t,\ 12-3t)$



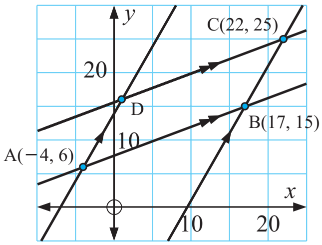
10 **a** A(3, −4) and B(4, 3) **b** For A $\begin{bmatrix}-1\\2\end{bmatrix}$, for B $\begin{bmatrix}-3\\-2\end{bmatrix}$

c 97.1^o **d** at $t=1.5$ hours

EXERCISE 17E

1 **a**


b A(2, 4), B(4, 6), C(8, 0)
c BC = CA $= \sqrt{52}$ units $\therefore$ isosceles Δ

2 **a**


b B(17, 15), C(22, 25), D(1, 16)

3 **a** A is at (2, 3), B(8, 6), C(5, 0) **b** AB = BC = $\sqrt{45}$ units

4 **a** P is at (10, 4), Q(3, −1), R(20, −10)
b $\overrightarrow{PQ} = \begin{bmatrix}-7\\-5\end{bmatrix}$, $\overrightarrow{PR} = \begin{bmatrix}10\\-14\end{bmatrix}$, $\overrightarrow{PQ} \bullet \overrightarrow{PR} = 0$
c $\angle QPR = 90^o$ **d** 74 units^2

5 **a** B is at (18, 9), C(14, 25), D(−2, 21)
b $\overrightarrow{AC} = \begin{bmatrix}12\\20\end{bmatrix}$ and $\overrightarrow{DB} = \begin{bmatrix}20\\-12\end{bmatrix}$
i $\sqrt{544}$ units **ii** $\sqrt{544}$ units **iii** 0
c Diagonals are perpendicular and equal in length, and as their midpoints are the same, i.e., (8, 15), ABCD is a square.

EXERCISE 17F

1 **a** $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\3\\-7\end{bmatrix} + t\begin{bmatrix}2\\1\\3\end{bmatrix}$ $t \in \mathcal{R}$

b $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\-2\end{bmatrix}$, $t \in \mathcal{R}$

c $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2\\2\\1\end{bmatrix} + t\begin{bmatrix}1\\0\\0\end{bmatrix}$, $t \in \mathcal{R}$

2 **a** $x = 5 - t,\ y = 2 + 2t,\ z = -1 + 6t,\ t \in \mathcal{R}$
b $x = 2t,\ y = 2 - t,\ z = -1 + 3t,\ t \in \mathcal{R}$
c $x = 3,\ y = 2,\ z = -1 + t,\ t \in \mathcal{R}$

3 **a** $x = 1 - 2t,\ y = 2 + t,\ z = 1 + t,\ t \in \mathcal{R}$
b $x = 3t,\ y = 1,\ z = 3 - 4t,\ t \in \mathcal{R}$
c $x = 1,\ y = 2 - 3t,\ z = 5,\ t \in \mathcal{R}$
d $x = 5t,\ y = 1 - 2t,\ z = -1 + 4t,\ t \in \mathcal{R}$

4 **a** $(-\frac{1}{2}, \frac{9}{2}, 0)$ **b** (0, 4, 1) **c** (4, 0, 9)

5 **a** (0, 7, 3) and $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$

6 **a** (1, 2, 3) **b** $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$ **7** **a** $3\sqrt{3}$ units **b** $\sqrt{\frac{3}{2}}$ units

EXERCISE 17G

1 **a** They intersect at (1, 2, 3), angle $\doteqdot 10.9^o$.
b Lines are skew, angle $\doteqdot 62.7^o$.
c They are parallel, $\therefore$ angle $= 0^o$.
d They are skew, angle $\doteqdot 11.4^o$.
e They intersect at (−4, 7, −7), angle $\doteqdot 40.2^o$.
f They are parallel, $\therefore$ angle $\doteqdot 0^o$.

REVIEW SET 17A

1 **a** $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix} + t\begin{bmatrix}4\\-3\end{bmatrix}$ **b** $x = -6 + 4t,\ y = 3 - 3t,\ t \in \mathcal{R}$

2 $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\8\end{bmatrix} + t\begin{bmatrix}5\\4\end{bmatrix}$, $t \in \mathcal{R}$ **3** $m = 10$

4 **a** (−4, 3) **b** (28, 27) **c** 10 m/s **d** $\begin{bmatrix}8\\6\end{bmatrix}$ **5** $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$

6 **a** **i** $-6\mathbf{i} + 10\mathbf{j}$ **ii** $-5\mathbf{i} - 15\mathbf{j}$ **iii** $(-6-5t)\mathbf{i} + (10-15t)\mathbf{j}$
b $t = 0.48$ h **c** shortest dist. $\doteqdot 8.85$ km, so, will miss reef.

7 **a** X23, $x_1 = 2 + t,\ y_1 = 4 - 3t,\ t \in \mathcal{R}$
b Y18, $x_2 = 11 - t,\ y_2 = 3 + at,\ t \in \mathcal{R}$
c interception occurred at 2 : 21 : 30 pm
d $\theta = 70.2^o$, $\doteqdot 8.86$ km/min

8 **a** KL is parallel to MN as $\begin{bmatrix}5\\-2\end{bmatrix}$ is parallel to $\begin{bmatrix}-5\\2\end{bmatrix}$
b KL is perpendicular to NK as $\begin{bmatrix}5\\-2\end{bmatrix} \bullet \begin{bmatrix}4\\10\end{bmatrix} = 0$
and NK is perpendicular to MN as $\begin{bmatrix}4\\10\end{bmatrix} \bullet \begin{bmatrix}-5\\2\end{bmatrix} = 0$
c K(7, 17), M(33, −5), N(3, 7) **d** 261 units^2

REVIEW SET 17B

1 **a** $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\2\\-1\end{bmatrix} + t\begin{bmatrix}-4\\0\\5\end{bmatrix}$, $t \in \mathcal{R}$ **b** (−5, 2, 9) or (11, 2, −11)

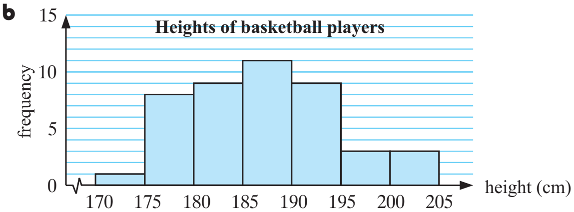
2 **a** $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix} + t\begin{bmatrix}-3\\3\\-2\end{bmatrix}$, $t \in \mathcal{R}$ **b** $P(\frac{6}{7}, \frac{8}{7}, -\frac{3}{7})$

3 **a** $\overrightarrow{PQ} = \begin{bmatrix}5\\-2\\-4\end{bmatrix}$ **b** $\doteqdot 41.8^o$ **4** **b** $\doteqdot 28.6^o$ **5** $(\frac{8}{3}, \frac{7}{3}, \frac{4}{3})$

6 **a** $\overrightarrow{PQ} = \begin{bmatrix}1\\4\\-3\end{bmatrix}$, PQ $= \sqrt{26}$ units
b $x = 2 + t,\ y = 4t,\ z = 1 - 3t,\ t \in \mathcal{R}$

7 **a** intersecting **b** $\frac{10}{3\sqrt{14}}$ **8** $\doteqdot 26.4^o$

EXERCISE 18A

1 **a** Heights can take any value from 170 cm to 205 cm, e.g., 181.37 cm
b


c The modal class is (185-) cm, as this occurred the most frequently.
d slightly positively skewed

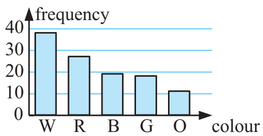
2 **a** Continuous numerical, but has been rounded to become discrete numerical data.
b

Stem	*Leaf*
0	3 6 8 8 8 8
1	0 0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

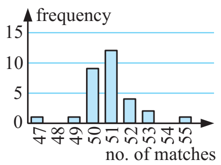
1 | 2 means 12 minutes

c positively skewed
d The modal travelling time was between 10 and 20 minutes.

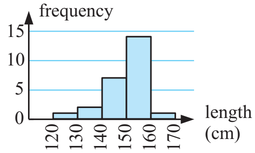
3 **a** column graph



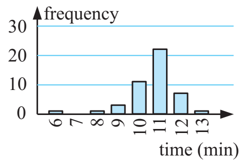
b column graph



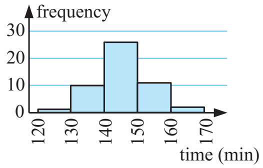
c histogram



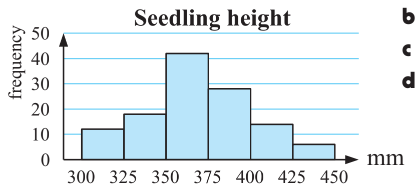
d column graph



e histogram



4 **a**



b 20 **c** 58.33% **d** **i** 1218 **ii** 512

5 **a** 140 **b** 65 **c** $\doteqdot 53.6\%$

EXERCISE 18B.1

1 **a** **i** 5.61 **ii** 6 **iii** 6 **b** **i** 16.3 **ii** 17 **iii** 18
c **i** 24.8 **ii** 24.9 **iii** 23.5
d **i** 128.6 **ii** 128 **iii** 115 and 127

2 **a** $A: 6.46$ $B: 6.85$ **b** $A: 7$ $B: 7$
c The data sets are the same except for the last value, and the last value of A is less than the last value of B, so the mean of A is less than the mean of B.
d The middle value of the data sets is the same, so the median is the same.

3 **a** mean: \$29 300, median: \$23 500, mode: \$23 000
b The mode is the lowest value, so does not take the higher values into account.
c No, since the data is positively skewed, the median is not in the centre.

4 **a** mean: 3.3, median: 0, mode: 0
b The data is very positively skewed so the median is not in the centre.
c The mode is the lowest value so does not take the higher values into account.
d yes, 21 and 42 **e** no

5 **a** 44 **b** 44 **c** 40.2 **d** increase mean to 40.3 **6** 116

7 3144 km **8** \$185 604 **9** $x = 15$ **10** $a = 5$ **11** 37

12 **a** 1280 km **b** 516 km **c** 224.5 km **13** 14.77

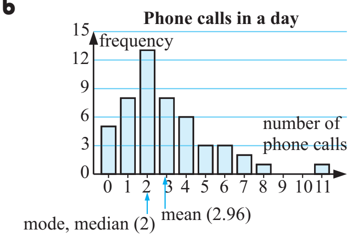
14 27.3 **15** 9 and 7

EXERCISE 18B.2

1 **a** 1 **b** 1 **c** 1.43

2 **a** **i** 2.96 **ii** 2 **iii** 2

b



c positively skewed with data value 11 as an outlier
d The mean takes into account the larger numbers of phone calls.
e the mean

3 **a**

Donation	*Frequency*
1	7
2	9
3	2
4	4
5	8

b 30
c **i** \$2.90 **ii** \$2 **iii** \$2
d the mode

4 **a** **i** 49 **ii** 49 **iii** 49.03 **b** no
c The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

5 **a** **i** 2.61 **ii** 2 **iii** 2 **b** This school has more children per family than the average Australian family. **c** positive
d The mean is larger than the median and the mode.

6 **a** **i** 69.1 **ii** 67 **iii** 73 **b** **i** 5.86 **ii** 5.8 **iii** 6.7

7 **a** **i** 5.63 **ii** 6 **iii** 6 **b** **i** 6.79 **ii** 7 **iii** 7
c the mean **d** yes

8 Team A: 91.25, Team B: 91.75, $\therefore$ Team B

9 **a** 49 **b** 144 and 147 (bi-modal) **c** 25

10 **a** 29 **b** 107 **c** 149.5

11 **a** 0 **b** 1.7 **c** 1.5 **12** $\doteqdot 81.2$ mm

13 **a** mean = \$163 770, median = \$147 200 (differ by \$16 570)
b **i** mean selling price **ii** median selling price

14 mean $\doteqdot 34.6$, mode = 35, median = 35 **15** $\doteqdot 17.7$

16 **a** $\doteqdot 70.9$ g **b** $\doteqdot 210$ g **c** 139 g **17** 10.1 cm

18 17.25 goals per game **19** 6 and 12

20 **a** mean for A $\doteqdot 50.8$, mean for B $\doteqdot 49.9$
b No, as to the nearest match, A is 51 and B is 50.

21 For X, mean $\doteqdot 399$ g. For Y, mean $\doteqdot 402$ g
The magazine will recommend Y.

22 **a** **i** \$31 500 **ii** \$28 000 **iii** \$33 300
b The mean as it is the highest measure.

EXERCISE 18B.3

1 **a** $\overline{x} \doteqdot 13.5$ **b** $\overline{x} \doteqdot 50.5$ **2** 31.7

3 **a** $\doteqdot 13.6$ goals **b** **i** 13.5 goals **ii** 13.6 goals
c The approximations are about the same. **4** $\doteqdot 495$ mm

5 **a** 70 **b** $\doteqdot 411\,000$ litres, i.e., $\doteqdot 411$ kL **c** $\doteqdot 5870$ L

6 **a** 125 people **b** $\doteqdot 119$ people **c** $\frac{3}{25}$ **d** 137 marks

7 **a** 95 **b** 59.6 kg **c** 25 **d** 36.8% **e** $\frac{9}{19}$ or 47.4%

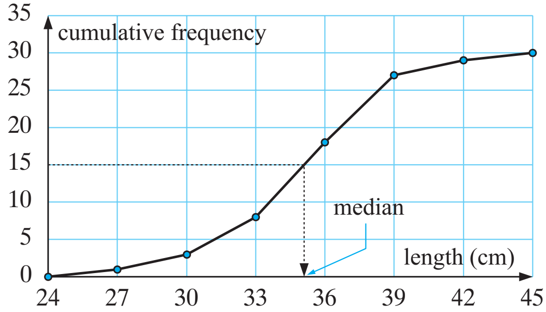
EXERCISE 18C

1 **a** 2 **b** 8 **2** 1 error

3 **a**

Length (x cm)	*Frequency*	*C. frequency*
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

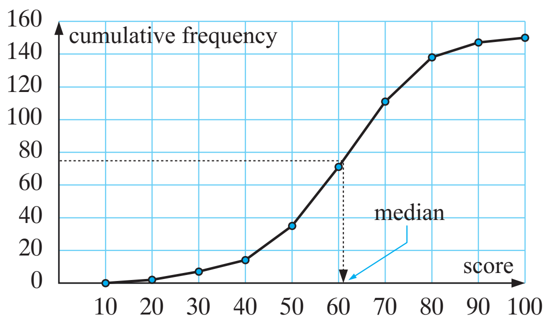
b



c median $\doteqdot$ 35 cm

d actual median = 34.5, i.e., a good approximation

4



a $\doteqdot$ 61 students **b** $\doteqdot$ 91 students **c** $\doteqdot$ 76 students
d 24 (or 25) students **e** 76 marks

5 **a** 8 **b** **i** 40 **ii** 40

6 **a** 28.8 min **b** 35 **c** 26.5 min

7 **a** 26 years **b** 36% **c** **i** 0.527 **ii** 0.030

8 **a** 2270 h **b** 69.3% **c** 62 or 63 **d** 0.0854

9 **a** 47.8 m **b** 12 or 13 **c** 50 or 51 **d** 35 **e** 0.0733

EXERCISE 18D.1

1 **a** **i** 6 **ii** $Q_1 = 4$, $Q_3 = 7$ **iii** 7 **iv** 3
b **i** 17.5 **ii** $Q_1 = 15$, $Q_3 = 19$ **iii** 14 **iv** 4
c **i** 24.9 **ii** $Q_1 = 23.5$, $Q_3 = 26.1$ **iii** 7.7 **iv** 2.6
d **i** 128 **ii** $Q_1 = 121$, $Q_3 = 140.5$ **iii** 45 **iv** 19.5

2 **a** median = 2.45, $Q_1 = 1.45$, $Q_3 = 3.8$
b range = 5.2, IQR = 2.35
c **i** ...... greater than 2.45 min **ii** ...... less than 3.8 min
iii The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes. The waiting times were spread over 5.2 minutes.

3 **a** 3 **b** 42 **c** 20 **d** 13 **e** 29 **f** 39 **g** 16

4 **a** **i** 124 cm **ii** $Q_1 = 116$ cm, $Q_3 = 130$ cm
b **i** ...... 124 cm **ii** ...... 130 cm tall
c **i** 29 cm **ii** 14 cm **d** .... over 14 cm

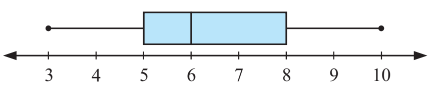
5 **a** **i** 7 **ii** 6 **iii** 5 **iv** 6.5 **v** 1.5
b **i** 10 **ii** 7 **iii** 6 **iv** 8 **v** 2

EXERCISE 18D.2

1 **a** **i** 35 **ii** 78 **iii** 13 **iv** 53 **v** 26 **b** **i** 65 **ii** 27

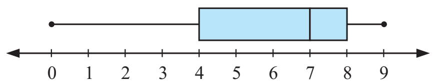
2 **a** ...... was 98 **b** ...... was 25
c .... greater than or equal to 70 **d** .... at least 85 marks
e ...... between 55 and 85 ...... **f** 73 **g** 30

3 **a** **i** min = 3, $Q_1 = 5$, median = 6, $Q_3 = 8$, max = 10
ii



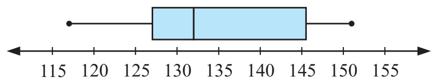
iii range = 7 **iv** IQR = 3

b **i** min = 0, $Q_1 = 4$, median = 7, $Q_3 = 8$, max = 9
ii



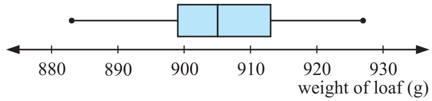
iii range = 9 **iv** IQR = 4

c **i** min = 117, $Q_1 = 127$, median = 132, $Q_3 = 145.5$, max = 151
ii



iii range = 34 **iv** IQR = 18.5

4 **a** median = 905 g, $Q_1 = 899$ g, $Q_3 = 913$ g, max weight = 927 g, min weight = 883 g
b



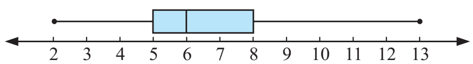
c **i** IQR = 14 g **ii** range = 44 g
d **i** ...... at least 905 g **ii** 25% of the loaves ......
iii ...... spread over 14 g
iv ...... a weight of 899 g or less
e a little negatively skewed

5 **a**

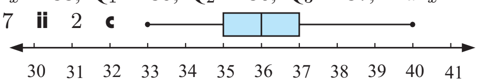
Statistic	*Year 9*	*Year 12*
min value	1	6
Q_1	5	10
median	7.5	14
Q_3	10	16
max value	12	17.5

b **i** Year 9: 11, Year 12: 11.5
ii Year 9: 5, Year 12: 6
c **i** true
ii true

6 **a** median = 6, $Q_1 = 5$, $Q_3 = 8$ **b** 3
c



7 **a** $\text{Min}_x = 33$, $Q_1 = 35$, $Q_2 = 36$, $Q_3 = 37$, $\text{Max}_x = 40$
b **i** 7 **ii** 2 **c**



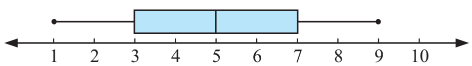
8 **a** 10 **b** $\doteqdot$ 28.3% **c** 7 cm **d** IQR $\doteqdot$ 2.6 cm
e 10 cm, which means that 90% of the seedlings have a height of 10 cm or less.

9 **a** 27 min **b** 29 min **c** $31\frac{1}{2}$ min **d** IQR $\doteqdot 4\frac{1}{2}$ min
e 28 min 10 sec

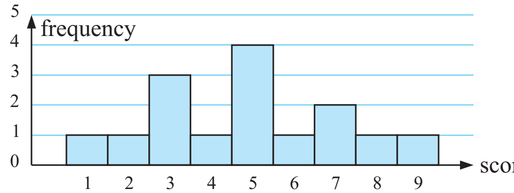
10 **a** 480 **b** 120 marks **c** 84 **d** IQR $\doteqdot$ 28 **e** 107 marks

EXERCISE 18E.1

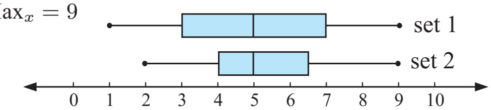
1 **a** $\overline{x} \doteqdot 4.87$, $\text{Min}_x = 1$, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 7$, $\text{Max}_x = 9$
b



c

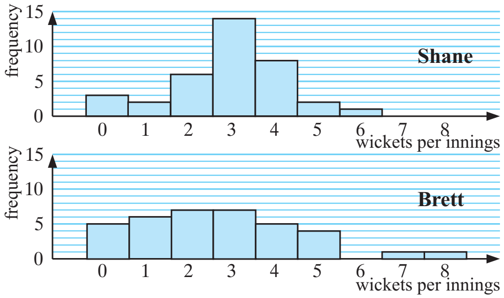


d $\overline{x} \doteqdot 5.24$, $\text{Min}_x = 2$, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6.5$, $\text{Max}_x = 9$



EXERCISE 18E.2

1 **a** discrete **c**



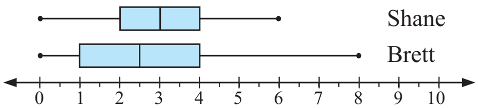
d There are no outliers for Shane. Brett has outliers of 7 and 8 which must not be removed.

e Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.

f Shane has a higher mean ($\doteq$ 2.89 wickets) compared with Brett ($\doteq$ 2.67 wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has a bi-modal distribution of 2 and 3 (7 times each).

g Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.

h



j Generally, Shane takes more wickets than Brett and is a more consistent bowler.

2 **a** continuous

c For the 'New type' globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d

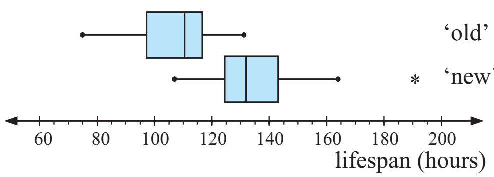
	Old type	*New type*
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The mean and median are $\doteq$ 25% and $\doteq$ 19% higher for the 'new type' of globe compared with the 'old type'.

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).

The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed. For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

3

	Set 1	*Set 2*
Mean	21.95	21.81
Min_x	21.6	21.5
Q_1	21.9	21.7
Median	21.9	21.8
Q_3	22.0	21.9
Max_x	22.2	22.2
Range	0.6	0.7
IQR	0.1	0.2

Set 2 has more data points than *set 1*.
The 5-number summary for each set of data is reasonably similar, however the spread of the middle 50% of data for *set 2* is double that for *set 1*.

EXERCISE 18F

1 **a** Sally: $\overline{x} = 25$, $s = 4.98$; Joanne: $\overline{x} = 30.5$, $s = 12.56$
b The standard deviation is an indicator of consistency; lower s means better consistency.

2 **a** Glen: range $= 11$, $\overline{x} = 5.7$;
Shane: range $= 11$, $\overline{x} = 5.7$
b We suspect Glen's, he has two zeros.
c Glen: $s = 3.9$ $\leftarrow$ greater variability
Shane: $s = 3.29$
d standard deviation

3 **a** We suspect variability in standard deviation since the factors may change every day.
b **i** sample mean **ii** sample standard deviation
c less variability

4 **a** $\overline{x} = 69$, $s = 6.05$ **b** $\overline{x} = 79$, $s = 6.05$
c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.

5 **a** $\overline{x} = 1.01$ kg; $s = 0.17$ **b** $\overline{x} = 2.02$ kg; $s = 0.34$
c Doubling the values doubles the mean and the standard deviation.

6 $a = 6$, $b = 5$ **7** $\doteq$ 1.67 children

8 **a** $\overline{x} \doteq 5.1$, $s \doteq 1.81$ **b** $\overline{x} \doteq 14.5$, $s \doteq 1.75$

9 $\overline{x} \doteq 37.3$, $s \doteq 1.45$ **10** $\overline{x} \doteq 47.8$ cm, $s \doteq 2.66$ cm

11 $\overline{x} \doteq \$390.30$, $s \doteq \$15.87$

12 **a** $\overline{x} = 26.5$, $s \doteq 2.29$ **b** $\overline{x} \doteq 13.2$, $s \doteq 4.65$
c $\overline{x} \doteq 170$, $s \doteq 9.84$ **d** $\overline{x} = 0$, $s = 2$

13 $\overline{x} = 55$ L, $s \doteq 10.9$ L

14 **a** $\overline{x} \doteq 33.7$, $s \doteq 1.11$ **b** $\overline{x} \doteq 17.4$, $s \doteq 0.123$
c $\overline{x} \doteq 34.6$, $s \doteq 10.4$

15 $\overline{x} \doteq \$18.60$, $s \doteq \$8.33$

EXERCISE 18G

1 **a** 16% **b** 84% **c** 97.4% **d** 0.15%

2 3 years **3** **a** 5 **b** 32 **c** 136

4 **a** 458 babies **b** 444 babies

REVIEW SET 18A

1 **a** **i** many **ii** n.a. **iii** categorical
b **i** many **ii** breaths per minutes **iii** quantitative discrete
c **i** infinitely many **ii** centimetres, metres, etc.
iii quantitative continuous

2 **a** Diameter of bacteria colonies

0	4 8 9
1	3 5 5 7
2	1 1 5 6 8 8
3	0 1 2 3 4 5 5 6 6 7 7 9
4	0 1 2 7 9

leaf unit: 0.1 cm

b **i** 3.15 cm
ii 4.5 cm

c The distribution is slightly negatively skewed.

3 **a**

	Girls	*Boys*
shape	pos. skewed	approx. symm.
centre (median)	36.3 sec	34.9 sec
spread (range)	7.7 sec	4.9 sec

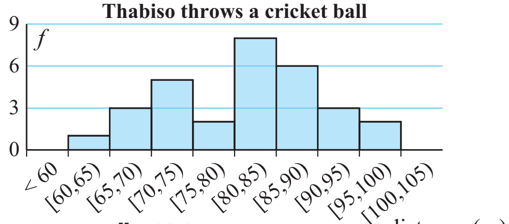
b The girls' distribution is positively skewed and boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 **a** highest = 97.5 m, lowest = 64.6 m
b use groups 60 -, 65 -, 70 -, etc.

c

A frequency distribution table for distances thrown by Thabiso		
distance (m)	tally	freq. (f)
60 -	|	1
65 -	|||	3
70 -	~~||||~~	5
75 -	||	2
80 -	~~||||~~ |||	8
85 -	~~||||~~ |	6
90 -	|||	3
95 < 100	||	2
	Total	30

d **i / ii** **Frequency histogram displaying the distance Thabiso throws a cricket ball**



e **i** $\doteqdot 81.1$ m **ii** 83.0 m

5 **a**

	A	B
Min	11	11.2
Q_1	11.6	12
Median	12	12.6
Q_3	12.6	13.2
Max	13	13.8

b

		A	B
i	Range	2	2.6
ii	IQR	1	1.2

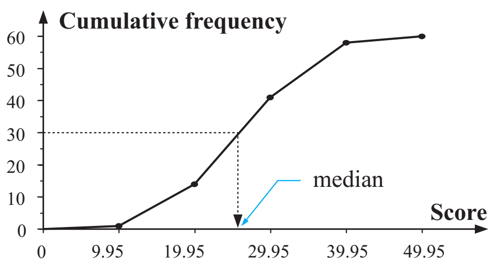
c **i** The members of squad A generally ran faster times.
ii The times in squad B were more varied.

6 **a** **i** 101.5 **ii** 98 **iii** 105.5
b **i** 7.5 **c** $\overline{x} = 100.2$, $s \doteqdot 7.59$

7 **a** The mean length is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

8 $\overline{x} \doteqdot 33.1$ L, $s \doteqdot 7.63$ L

9 **a**

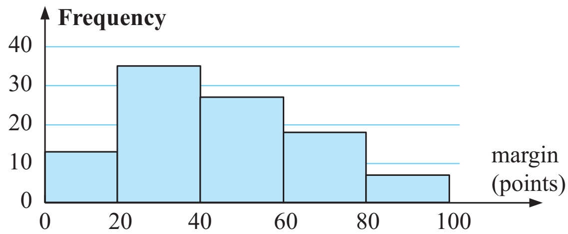


b $\doteqdot 25.9$ **c** $\doteqdot 12.0$ **d** $\overline{x} \doteqdot 26.0$, $s \doteqdot 8.31$

10 $x = 4$, $y = 9$

REVIEW SET 18B

1



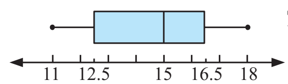
2 **a** $\overline{x} \doteqdot 3.69$, mode = 4, median = 4
b $\overline{x} = 56$, bi-modal (58 and 63), median = 58

3 $\overline{x} \doteqdot 49.6$, $s \doteqdot 1.60$ Does not justify claim. A much greater sample is needed.

4 $\doteqdot 414$ customers

5 range = 19, lower quartile = 119, upper quartile = 130, $s \doteqdot 6.38$

6



7 $\overline{x} = \$103.50$, $s \doteqdot \$19.40$

8 **a** 68% **b** 95% **c** 81.5% **d** 13.5%

9 **a** $\overline{x} \doteqdot 46.04$, $s \doteqdot 1.497$ **b** $\overline{x} \doteqdot 149.5$, $s \doteqdot 14.04$

10 **a** 2.5% **b** 84% **c** 81.5% **11** $a = 8$, $b = 4$

EXERCISE 19A

1 **a** 0.78 **b** 0.22 **2** **a** 0.487 **b** 0.051 **c** 0.731

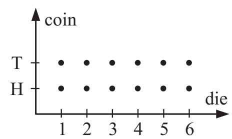
3 **a** 43 days **b** **i** $\doteqdot 0.047$ **ii** $\doteqdot 0.186$ **iii** 0.465

4 **a** $\doteqdot 0.089$ **b** $\doteqdot 0.126$

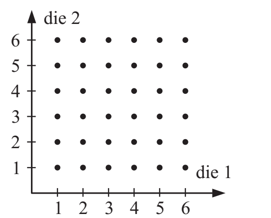
EXERCISE 19B

1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

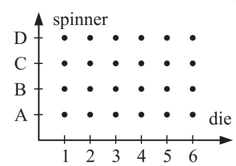
2 **a**



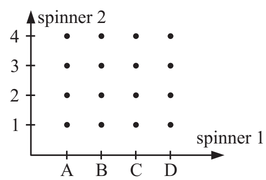
b



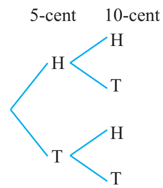
c



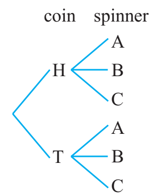
d



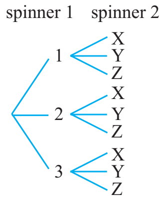
3 **a**



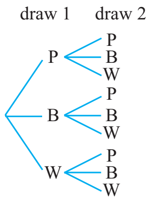
b



c



d



EXERCISE 19C

1 **a** $\frac{1}{5}$ **b** $\frac{1}{3}$ **c** $\frac{7}{15}$ **d** $\frac{4}{5}$ **e** $\frac{1}{5}$ **f** $\frac{8}{15}$

2 **a** 4 **b** **i** $\frac{2}{3}$ **ii** $\frac{1}{3}$ **3** **a** 0 **b** $\frac{1}{4}$ **c** $\frac{19}{32}$ **d** $\frac{5}{32}$

4 **a** $\frac{1}{4}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{1}{36}$ **e** $\frac{1}{18}$ **f** $\frac{1}{6}$

5 **a** $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{124}{1461}$ **d** $\frac{237}{1461}$ {remember leap years}

6 {AKN, ANK, KAN, KNA, NAK, NKA} **a** $\frac{1}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

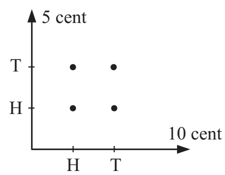
7 **a** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
b **i** $\frac{1}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{8}$ **iv** $\frac{3}{8}$ **v** $\frac{1}{2}$ **vi** $\frac{7}{8}$

8 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

9 {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, THTH, HTTH, TTHH, TTTH, TTHT, THTT, HTTT, TTTT}
a $\frac{1}{16}$ **b** $\frac{3}{8}$ **c** $\frac{5}{16}$ **d** $\frac{15}{16}$ **e** $\frac{1}{4}$ **10** $\frac{4}{9}$

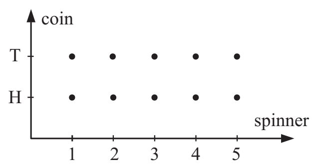
EXERCISE 19D

1



a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{3}{4}$

2 **a**



b 10
c **i** $\frac{1}{10}$ **ii** $\frac{1}{5}$ **iii** $\frac{3}{5}$ **iv** $\frac{3}{5}$

3 **a** $\frac{1}{36}$ **b** $\frac{1}{18}$ **c** $\frac{5}{9}$ **d** $\frac{11}{36}$ **e** $\frac{5}{18}$ **f** $\frac{25}{36}$ **g** $\frac{1}{6}$
h $\frac{5}{18}$ **i** $\frac{2}{9}$ **j** $\frac{13}{18}$

EXERCISE 19E.1

1 **a** $\frac{6}{7}$ **b** $\frac{36}{49}$ **c** $\frac{216}{343}$ **2** **a** $\frac{1}{8}$ **b** $\frac{1}{8}$

3 **a** 0.0096 **b** 0.8096 **4** **a** $\frac{1}{16}$ **b** $\frac{15}{16}$

5 **a** 0.56 **b** 0.06 **c** 0.14 **d** 0.24 **6** **a** $\frac{8}{125}$ **b** $\frac{12}{125}$ **c** $\frac{27}{125}$

EXERCISE 19E.2

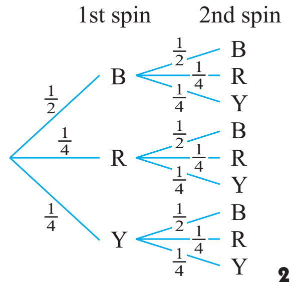
1 **a** $\frac{7}{15}$ **b** $\frac{7}{30}$ **c** $\frac{7}{15}$ **2** **a** $\frac{14}{55}$ **b** $\frac{1}{55}$

3 **a** $\frac{3}{100}$ **b** $\frac{3}{100} \times \frac{2}{99} \doteq 0.0006$
c $\frac{3}{100} \times \frac{2}{99} \times \frac{1}{98} \doteq 0.000\,006$ **d** $\frac{97}{100} \times \frac{96}{99} \times \frac{95}{98} \doteq 0.912$

4 **a** $\frac{3}{7}$ **b** $\frac{1}{7}$

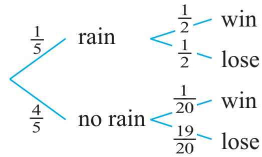
EXERCISE 19F

1 **a**



b $\frac{1}{4}$ **c** $\frac{1}{16}$
d $\frac{5}{8}$ **e** $\frac{3}{4}$

2



$\Pr(\text{win}) = \frac{7}{50}$

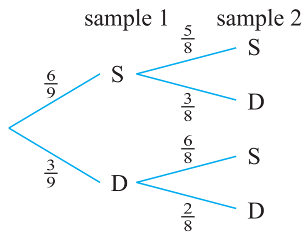
3 0.032 **4** $\frac{17}{40}$ **5** **a** $\frac{11}{30}$ **b** $\frac{19}{30}$

EXERCISE 19G

1 **a** $\frac{20}{49}$ **b** $\frac{10}{21}$ **2** **a** $\frac{3}{10}$ **b** $\frac{1}{10}$ **c** $\frac{3}{5}$

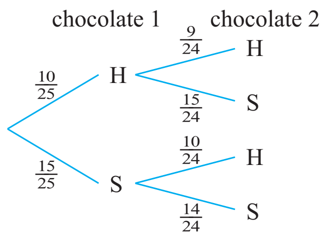
3 **a** $\frac{2}{9}$ **b** $\frac{5}{9}$ **4** **a** $\frac{1}{3}$ **b** $\frac{2}{15}$ **c** $\frac{4}{15}$ **d** $\frac{4}{15}$
These cases cover all possibilities, so their probabilities must sum to 1.

5 **a**



b $\frac{1}{12}$
c $\frac{5}{12}$

6 **a**



b $\frac{3}{20}$ **c** $\frac{7}{20}$

7 **a** $\frac{1}{5}$ **b** $\frac{3}{5}$ **c** $\frac{4}{5}$ **8** $\frac{19}{45}$ **9** **a** $\frac{2}{100} \times \frac{1}{99} \doteq 0.0002$
b $\frac{98}{100} \times \frac{97}{99} \doteq 0.9602$ **c** $1 - \frac{98}{100} \times \frac{97}{99} \doteq 0.0398$

EXERCISE 19H

1 **a** The rule is: *add* the two terms directly above the new row.
b 1 7 21 35 35 21 7 1 **c** 1 7 21 35 35 21 7 1

2

```
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5   10  10   5   1
    1   6   15  20  15   6   1
  1   7   21  35  35  21   7   1
1   8   28  56  70  56  28   8   1
```

a $C^3_2 = 3$ **b** $C^7_5 = 21$
c $C^5_2 = 10$ **d** $C^8_1 = 8$
e $C^4_0 = 1$ **f** $C^6_6 = 1$

3 **a** $C^9_5 = 126$ **b** $C^{14}_3 = 364$ **c** $C^{40}_1 = 40$ **d** $C^3_2 = 3$
e $C^{40}_{10} = 847\,660\,528$ **f** $C^{40}_{20} \doteq 1.3784 \times 10^{11}$

4 **a**

HHHHH	HTHHT	HHTTT	TTHTH
HHHHT	THHHT	HTHTT	TTTHH
HHHTH	HHTTH	THHTT	HTTTT
HHTHH	HTHTH	HTTHT	THTTT
HTHHH	THHTH	THTHT	TTHTT
THHHH	HTTHH	TTHHT	TTTHT
HHHTT	THTHH	HTTTH	TTTTH
HHTHT	TTHHH	THTTH	TTTTT

b **i** 1 **ii** 5 **iii** 10 **iv** 10 **v** 5 **vi** 1

5 **a** $C^{18}_{10} = 43\,758$ **b** $C^{23}_{14} = 817\,190$

6 **a** $p^3 + 3p^2q + 3pq^2 + q^3$ 1 3 3 1
b $p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$ 1 4 6 4 1
c **i** $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
ii $p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$
d 1

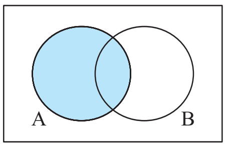
7 **a** $(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$
b $4(\frac{1}{2})^3(\frac{1}{2}) = \frac{1}{4}$

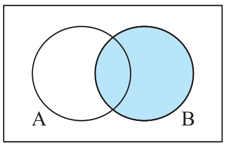
8 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
b **i** $5(\frac{1}{2})^4(\frac{1}{2}) = \frac{5}{32}$ **ii** $10(\frac{1}{2})^2(\frac{1}{2})^3 = \frac{5}{16}$

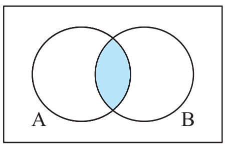
9 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$
b **i** $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ **ii** $6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$ **iii** $\frac{8}{9}$

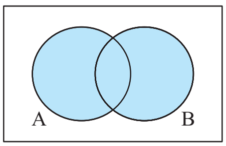
10 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
b **i** $10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{135}{512}$ **ii** $\frac{53}{512}$

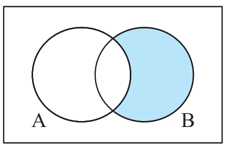
EXERCISE 19I

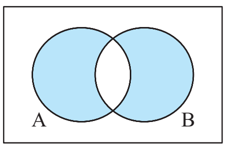
1 **a**


b


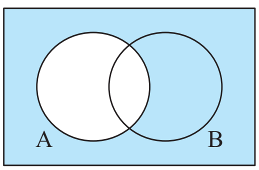
c


d


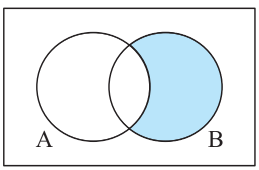
e


f


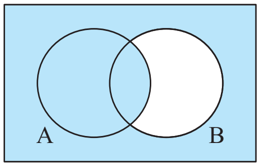
2 **a** 29 **b** 17 **c** 26 **d** 5 **3** **a** 65 **b** 9 **c** 4 **d** 52
4 **a** $\frac{19}{40}$ **b** $\frac{1}{2}$ **c** $\frac{4}{5}$ **d** $\frac{5}{8}$ **e** $\frac{13}{40}$ **f** $\frac{7}{20}$
5 **a** $\frac{19}{25}$ **b** $\frac{13}{25}$ **c** $\frac{6}{25}$ **d** $\frac{7}{19}$
6 **a** $\frac{7}{15}$ **b** $\frac{1}{15}$ **c** $\frac{2}{15}$ **d** $\frac{6}{7}$

7 **a**


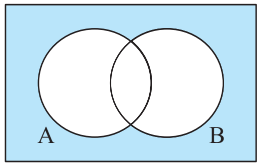
A′

b


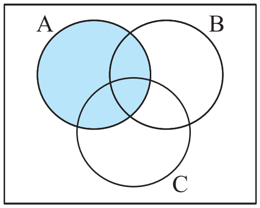
A′∩B

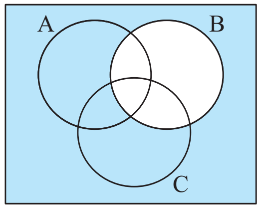
c


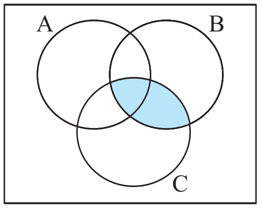
A ∪ B′

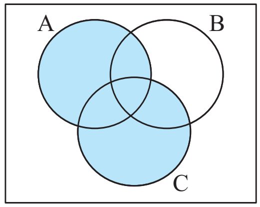
d


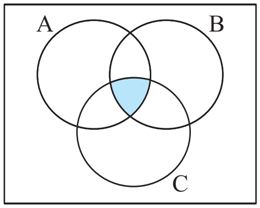
A′∩B′

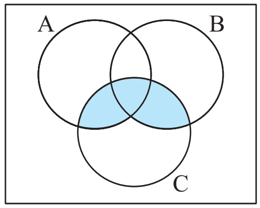
8 **a**


b


c


d


e


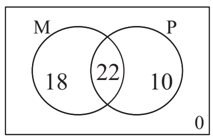
f


9 For each of these draw **two** diagrams, shade the first with the LHS set and the second with the RHS set.

10 **a** A = {7, 14, 21, 28, 35, ..., 98}
B = {5, 10, 15, 20, 25, ..., 95}
i $n(A) = 14$ **ii** $n(B) = 19$ **iii** 2 **iv** 31

12 **a** **i** $\dfrac{b+c}{a+b+c+d}$ **ii** $\dfrac{b}{a+b+c+d}$
iii $\dfrac{a+b+c}{a+b+c+d}$ **iv** $\dfrac{a+b+c}{a+b+c+d}$
b Pr(A or B) = Pr(A) + Pr(B) − Pr(A and B)

EXERCISE 19J

1 **a**


22 study both
b **i** $\frac{9}{25}$ **ii** $\frac{11}{20}$

2 **a** $\frac{3}{8}$ **b** $\frac{7}{20}$ **c** $\frac{1}{5}$ **d** $\frac{15}{23}$
3 **a** $\frac{14}{25}$ **b** $\frac{4}{5}$ **c** $\frac{1}{5}$ **d** $\frac{5}{23}$ **e** $\frac{9}{14}$ **4** **a** $\frac{22}{25}$ **b** $\frac{1}{2}$
5 $\frac{5}{6}$ **6** **a** $\frac{13}{20}$ **b** $\frac{7}{20}$ **c** $\frac{11}{50}$ **d** $\frac{7}{25}$ **e** $\frac{4}{7}$ **f** $\frac{1}{4}$
7 **a** $\frac{3}{5}$ **b** $\frac{2}{3}$ **8** **a** 0.46 **b** $\frac{14}{23}$ **9** $\frac{70}{163}$
10 **a** 0.45 **b** 0.75 **c** 0.65 **11** **a** 0.0484 **b** 0.3926
12 $\frac{2}{3}$

EXERCISE 19K

1 P(R ∩ S) = 0.2 and P(R) × P(S) = 0.2
∴ are independent events

2 **a** $\frac{7}{30}$ **b** $\frac{7}{12}$ **c** $\frac{7}{10}$ No, as P(A ∩ B) ≠ P(A) × P(B)

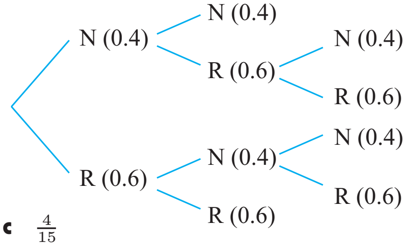
3 **a** 0.35 **b** 0.85 **c** 0.15 **d** 0.15 **e** 0.5
4 $\frac{14}{15}$ **5** **a** $\frac{91}{216}$ **b** 26
6 **Hint:** Show P(A ∩ B′) = P(A) P(B′) using a Venn diagram and P(A ∩ B)

REVIEW SET 19A

1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA
a $\frac{1}{2}$ **b** $\frac{1}{3}$

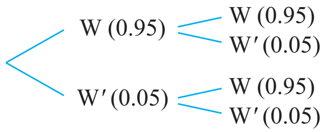
2 **a** $\frac{3}{8}$ **b** $\frac{1}{8}$ **c** $\frac{5}{8}$ **3** **a** $\frac{3}{25}$ **b** $\frac{24}{25}$ **c** $\frac{11}{12}$

4 P(N wins)
$= \frac{44}{125}$
$= 0.352$



5 **a** $\frac{2}{5}$ **b** $\frac{13}{15}$ **c** $\frac{4}{15}$
6 **a** $10(\frac{4}{5})^3(\frac{1}{5})^2 \doteqdot 0.205$ **b** $5(\frac{4}{5})^4(\frac{1}{5}) + (\frac{4}{5})^5 \doteqdot 0.737$
7 **a** $\frac{4}{500} \times \frac{3}{499} \times \frac{2}{498} \doteqdot 0.000\,000\,193$
b $1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498} \doteqdot 0.023\,86$

8



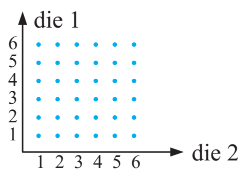
0.9975

9 **a** $\frac{19}{30}$ **b** $\frac{6}{19}$

REVIEW SET 19B

1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGB, GGGG
$\frac{3}{8}$

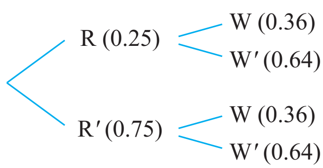
2



a $\frac{2}{9}$
b $\frac{5}{12}$

3 **a** $\frac{1}{4}$ **b** $\frac{37}{40}$ **c** $\frac{10}{25} = \frac{2}{5}$ **4** $\frac{8}{33}$
5 **a** $\frac{5}{33}$ **b** $\frac{19}{66}$ **c** $\frac{5}{11}$ **d** $\frac{16}{33}$
6 **a** Two events are independent if the occurrence of one does not influence the occurrence of the other.
For A and B independent, P(A) × P(B) = P(A and B)
b Two events, A and B, are disjoint if they have no common outcomes. P(A or B) = P(A) + P(B)

7



a 0.09
b 0.52

8 $1 - 0.9 \times 0.8 \times 0.7 = 0.496$ **9** **a** $\frac{31}{70}$ **b** $\frac{21}{31}$

10 **a** $\left(\frac{3}{5}+\frac{2}{5}\right)^4 = \left(\frac{3}{5}\right)^4 + 4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2 + 4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$ **b** **i** $\frac{144}{625}$ **ii** $\frac{328}{625}$

EXERCISE 20A.1

1 **a** Every min. Ozair's heart beats 67 times. **b** 4020
2 **a** 0.0015 errors/word **b** 0.15 errors/100 words
3 Paul $12.35/hour
4 **a** 1.77×10^{-4} mm/km **b** 1.77 mm/10 000 km
5 **a** 76.07 km/h **b** 21.13 m/s

EXERCISE 20A.2

1 **a** 96.17 km/h **b** 65.5 km/h
2 **a** 800 m **b** 125 **c** 125 m/min **d** average walking speed **e** 8 minutes **f** 66.67 m/min **g** 1.6 km
3 509 091 kL/day
4 **a** 1.187 kL/day **b** 1.029 kL/day **c** 1.052 kL/day

EXERCISE 20A.3

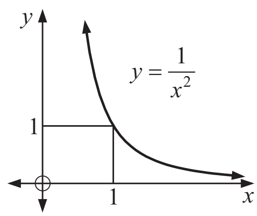
1 **a** 0.1 m/s **b** 0.9 m/s **c** 0.5 m/s
2 **a** **i** 3.2 beetles/gram **ii** 4.5 beetles/gram
b No effect 0 to 1 gram, rapid decrease 1 to 8 grams, rate of decrease decreases for 8 to 14 grams.

EXERCISE 20B.1

1 **a** 1 m/s **b** 3 km/h **c** $50/item **d** −5 bats/week
2 **a** 8200 L **b** 3000 L **c** 8200 L/hour **d** 3000 L/hour

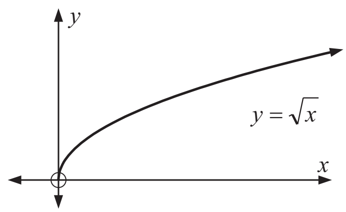
EXERCISE 20B.2

1 3 **2** **b** $1+3h+3h^2+h^3$ **c** $\text{M}\left(1+h,\ (1+h)^3\right)$
d $3+3h+h^2$ **e** 3
3 **b** $8+12h+6h^2+h^3$ **c** $\text{M}\left(2+h,\ (2+h)^3\right)$
d $12+6h+h^2$ **e** 12
4 **a** $\dfrac{-h}{x(x+h)}$ **b** $\text{M}\left(2+h,\ \dfrac{1}{2+h}\right)$ slope MF $= \dfrac{-1}{2(2+h)}$
c $-\frac{1}{4}$ **d** $-\frac{1}{9}$
5 **b** $\dfrac{1}{(2+h)^2} - \dfrac{1}{4}$
$= \dfrac{-4h-h^2}{4(2+h)^2}$
c



d $-\frac{1}{4}$

6 **a** $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$ **b** $\dfrac{\sqrt{9+h}-3}{h} = \dfrac{1}{\sqrt{9+h}+3}$
c



d $\frac{1}{6}$
e $\frac{1}{6}$ thousand/day

REVIEW SET 20

1 **a** 10 m/s **b** 4.6 m/s **2** **b** $\doteqdot 5.9$ **3** 8
4 **a** $4x^2+8xh+4h^2+12x+12h+9$ **b** $8x+12+4h$
c It is the slope of chord AB. **d** **i** $8x+12$ **ii** 20

e **i** $8x+12$ gives us the slope of the tangent at any point with x-coordinate x.
ii 20 is the slope of the tangent at $x=1$.

5 **a** 11 m/s **b** $(8+h)$ m/s
c 8 m/s, the instantaneous velocity at $t=2$ sec

EXERCISE 21A

1 **a** 6 **b** $-\frac{1}{4}$ **2** **b** 12

3 **a** **Hint:** $x-a=\left(\sqrt{x}+\sqrt{a}\right)\left(\sqrt{x}-\sqrt{a}\right)$ **b** $\frac{1}{6}$

EXERCISE 21B

1 **a** -4 **b** 1 **c** -12 **d** 3

2 **a** -1 **b** $\frac{3}{4}$ **c** $-\frac{1}{32}$ **d** -12 **e** -1 **f** $-\frac{45}{289}$

3 **a** $\frac{1}{4}$ **b** 1 **c** $-\frac{1}{27}$ **d** $\frac{1}{4}$

4 **a** 9 **b** 10 **c** $-\frac{2}{25}$ **d** $-\frac{2}{27}$ **e** $\frac{1}{4}$ **f** $-\frac{1}{2}$

5 **a** 12 **b** 108

EXERCISE 21C

1 **a** 1 **b** 0 **c** $3x^2$ **d** $4x^3$

2 **a** 2 **b** $2x-3$ **c** $3x^2-4x$

3 **a** $\dfrac{-1}{(x+2)^2}$ **b** $\dfrac{-2}{(2x-1)^2}$ **c** $-\dfrac{2}{x^3}$ **d** $-\dfrac{3}{x^4}$

4 **a** $\dfrac{1}{2\sqrt{x+2}}$ **b** $-\dfrac{1}{2x\sqrt{x}}$ **c** $\dfrac{1}{\sqrt{2x+1}}$

5

Function	*Derivative*	*Function*	*Derivative*
x	1	x^{-2}	$-2x^{-3}$
x^2	$2x^1$	x^{-3}	$-3x^{-4}$
x^3	$3x^2$	$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
x^4	$4x^3$	$x^{-\frac{1}{2}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$
x^{-1}	$-x^{-2}$	x^n	nx^{n-1}

EXERCISE 21D

1 **a** $3x^2$ **b** $6x^2$ **c** $14x$ **d** $2x+1$ **e** $-4x$
f $2x+3$ **g** $3x^2+6x+4$ **h** $20x^3-12x$ **i** $\dfrac{6}{x^2}$
j $-\dfrac{2}{x^2}+\dfrac{6}{x^3}$ **k** $2x-\dfrac{5}{x^2}$ **l** $2x+\dfrac{3}{x^2}$

2 **a** x^3 **b** $1-\dfrac{1}{x^2}$ **c** $-\dfrac{1}{x^2}$ **d** $-\dfrac{1}{x^2}-\dfrac{15}{x^4}$ **e** $2x-1$
f $-\dfrac{2}{x^3}+\dfrac{3}{\sqrt{x}}$ **g** $-\dfrac{1}{2x\sqrt{x}}$ **h** $8x-4$ **i** $3x^2+12x+12$

3 **a** $6x^2-14x$ **b** $2\pi x$ **c** $-\dfrac{2}{5x^3}$ **d** 100 **e** 10 **f** $12\pi x^2$

4 **a** 6 **b** $\dfrac{3\sqrt{x}}{2}$ **c** $2x-10$ **d** $2-9x^2$ **e** $4+\dfrac{1}{4x^2}$
f $6x^2-6x-5$

5 **a** 4 **b** $-\frac{16}{729}$ **c** -7 **d** $\frac{13}{4}$ **e** $\frac{1}{8}$ **f** -11

6 **a** $\dfrac{2}{\sqrt{x}}+1$ **b** $\dfrac{1}{3\sqrt[3]{x^2}}$ **c** $\dfrac{1}{x\sqrt{x}}$ **d** $2-\dfrac{1}{2\sqrt{x}}$
e $-\dfrac{2}{x\sqrt{x}}$ **f** $6x-\frac{3}{2}\sqrt{x}$ **g** $\dfrac{-25}{2x^3\sqrt{x}}$ **h** $2+\dfrac{9}{2x^2\sqrt{x}}$

7 **a** $\dfrac{dy}{dx}=4+\dfrac{3}{x^2}$, $\dfrac{dy}{dx}$ is the slope function of $y=4x-\dfrac{3}{x}$ from which the slope at any point can be found.

b $\dfrac{dS}{dt}=4t+4$ metres per second, $\dfrac{dS}{dt}$ is the instantaneous rate of change in position at the time t, i.e., it is the velocity function.

c $\dfrac{dC}{dx}=3+0.004x$ dollars per toaster, $\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 21E.1

1 **a** $f(g(x))=(2x+7)^2$ **b** $f(g(x))=2x^2+7$
c $f(g(x))=\sqrt{3-4x}$ **d** $f(g(x))=3-4\sqrt{x}$
e $f(g(x))=\dfrac{2}{x^2+3}$ **f** $f(g(x))=\dfrac{4}{x^2}+3$
g $f(g(x))=2^{3x+4}$ **h** $f(g(x))=3\times 2^x+4$

2 **a** $f(x)=x^3$, $g(x)=3x+10$
b $f(x)=\dfrac{1}{x}$, $g(x)=2x+4$
c $f(x)=\sqrt{x}$, $g(x)=x^2-3x$
d $f(x)=\dfrac{1}{\sqrt{x}}$, $g(x)=5-2x$
e $f(x)=x^4$, $g(x)=x^2+5x-1$
f $f(x)=\dfrac{10}{x^3}$, $g(x)=3x-x^2$

EXERCISE 21E.2

1 **a** u^{-2}, $u=2x-1$ **b** $u^{\frac{1}{2}}$, $u=x^2-3x$
c $2u^{-\frac{1}{2}}$, $u=2-x^2$ **d** $u^{\frac{1}{3}}$, $u=x^3-x^2$
e $4u^{-3}$, $u=3-x$ **f** $10u^{-1}$, $u=x^2-3$

2 **a** $8(4x-5)$ **b** $2(5-2x)^{-2}$ **c** $\frac{1}{2}(3x-x^2)^{-\frac{1}{2}}\times(3-2x)$
d $-12(1-3x)^3$ **e** $-18(5-x)^2$
f $\frac{1}{3}(2x^3-x^2)^{-\frac{2}{3}}\times(6x^2-2x)$ **g** $-60(5x-4)^{-3}$
h $-4(3x-x^2)^{-2}\times(3-2x)$ **i** $6(x^2-\dfrac{2}{x})^2\times(2x+\dfrac{2}{x^2})$

3 **a** $-\frac{1}{\sqrt{3}}$ **b** -18 **c** -8 **d** -4 **e** $-\frac{3}{32}$ **f** 0

4 **a** $\dfrac{dy}{dx}=3x^2$, $\dfrac{dx}{dy}=\frac{1}{3}y^{-\frac{2}{3}}$ **Hint**: Substitute $y=x^3$
b $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=\dfrac{dy}{dy}=1$

EXERCISE 21F.1

1 **a** $2x(2x-1)+2x^2$ **b** $4(2x+1)^3+24x(2x+1)^2$
c $2x(3-x)^{\frac{1}{2}}-\frac{1}{2}x^2(3-x)^{-\frac{1}{2}}$
d $\frac{1}{2}x^{-\frac{1}{2}}(x-3)^2+2\sqrt{x}(x-3)$
e $10x(3x^2-1)^2+60x^3(3x^2-1)$
f $\frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3+3\sqrt{x}(x-x^2)^2(1-2x)$

2 **a** -48 **b** $406\frac{1}{4}$ **c** $\frac{13}{3}$ **d** $\frac{11}{2}$ **3** $x=3$ or $\frac{3}{5}$

EXERCISE 21F.2

1 **a** $\dfrac{3(2-x)+(1+3x)}{(2-x)^2}$ **b** $\dfrac{2x(2x+1)-2x^2}{(2x+1)^2}$
c $\dfrac{(x^2-3)-2x^2}{(x^2-3)^2}$ **d** $\dfrac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)+2\sqrt{x}}{(1-2x)^2}$

e $\dfrac{2x(3x-x^2)-(x^2-3)(3-2x)}{(3x-x^2)^2}$

f $\dfrac{(1-3x)^{\frac{1}{2}}+\frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$

2 **a** 1 **b** 1 **c** $-\frac{7}{324}$ **d** $-\frac{28}{27}$

3 **a** **i** never (note: $\dfrac{dy}{dx}$ is undefined at $x=-1$)
ii $x \leqslant 0$ and $x=1$ **b** **i** $x=-2\pm\sqrt{11}$ **ii** $x=-2$

EXERCISE 21G

1 **a** $y=-7x+11$ **b** $4y=x+8$ **c** $y=-2x-2$
d $y=-2x+6$ **e** $y=-5x-9$ **f** $y=-5x-1$

2 **a** $6y=-x+57$ **b** $7y=-x+26$ **c** $y=-x+6$
d $y=18x-161$ **e** $3y=x+11$ **f** $x+6y=43$

3 **a** $y=21$ and $y=-6$ **b** $(\frac{1}{2}, 2\sqrt{2})$ **c** $k=-5$
d $y=-3x+1$

4 **a** $a=-4,\ b=7$ **b** $a=2,\ b=4$

5 **a** $3y=x+5$ **b** $9y=x+4$ **c** $y=2x-\frac{7}{4}$
d $y=-27x-\frac{242}{3}$

6 $a=4, b=3$ **7** **a** $16y=x-3$ **b** $57y=-4x+1042$
c $y=-4$ **d** $2y=x+1$

8 **a** $(-4,-64)$ **b** $(4,-31)$ **c** $(\frac{1}{3}, -\frac{80}{9})$
d does not meet the curve again

9 **a** $y=(2a-1)x-a^2+9$; $y=5x$, contact at $(3, 15)$
$y=-7x$, contact at $(-3, 21)$
b $y=0,\ y=27x+54$ **c** $y=0,\ y=-\sqrt{14}x+4\sqrt{14}$

EXERCISE 21H

1 **a** 6 **b** $12x-6$ **c** $\dfrac{3}{2x^{\frac{5}{2}}}$ **d** $\dfrac{12-6x}{x^4}$ **e** $24-48x$
f $\dfrac{20}{(2x-1)^3}$

2 **a** $-6x$ **b** $2-\dfrac{30}{x^4}$ **c** $-\frac{9}{4}x^{-\frac{5}{2}}$ **d** $\dfrac{8}{x^3}$
e $6(x^2-3x)(5x^2-15x+9)$ **f** $2+\dfrac{2}{(1-x)^3}$

3 **a** $x=1$ **b** $x=0, \pm\sqrt{6}$

REVIEW SET 21A

1 $y=4x+2$ **2** **a** $6x-4x^3$ **b** $1+\dfrac{1}{x^2}$ **3** $2x+2$

4 $x=1$ **5** $(-2,-25)$ **6** $a=\frac{5}{2},\ b=-\frac{3}{2}$ **7** $a=\frac{1}{2}$

8 $y=16x-\frac{127}{2}$

9 **a** $\dfrac{dM}{dt}=8t(t^2+3)^3$ **b** $\dfrac{dA}{dt}=\dfrac{\frac{1}{2}t(t+5)^{-\frac{1}{2}}-2(t+5)^{\frac{1}{2}}}{t^3}$

10 **a** $-\dfrac{2}{x\sqrt{x}}-3$ **b** $4\left(x-\dfrac{1}{x}\right)^3\left(1+\dfrac{1}{x^2}\right)$
c $\frac{1}{2}(x^2-3x)^{-\frac{1}{2}}(2x-3)$

REVIEW SET 21B

1 **a** $5+3x^{-2}$ **b** $4(3x^2+x)^3(6x+1)$
c $2x(1-x^2)^3-6x(1-x^2)^2(x^2+1)$

2 $y=7,\ y=-25$ **3** $5y=x-11$

4 **a** $\dfrac{3(x+3)^2\sqrt{x}-\frac{1}{2}x^{-\frac{1}{2}}(x+3)^3}{x}$
b $4x^3\sqrt{x^2+3}+x^5(x^2+3)^{-\frac{1}{2}}$

5 **a** $\dfrac{1}{2\sqrt{x}}(1-x)^2-2\sqrt{x}(1-x)$
b $\frac{1}{2}(3x-x^2)^{-\frac{1}{2}}\times(3-2x)$ **c** $\dfrac{1}{(2-x)^2}$

6 **a** $f''(x)=6-\dfrac{2}{x^3}$ **b** $f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$

7 area $=\frac{3267}{152}$ units^2 **Hint:** tangent is $4y=-57x-99$

8 $a=-1,\ b=2$

9 $a=2$ and the tangent is $y=3x-1$ which meets the curve again at $(-4, -13)$

10 $A=-14,\ B=21$

REVIEW SET 21C

1 **a** $\dfrac{dy}{dx}=3x^2(1-x^2)^{\frac{1}{2}}-x^4(1-x^2)^{-\frac{1}{2}}$
b $\dfrac{dy}{dx}=\dfrac{(2x-3)(x+1)^{\frac{1}{2}}-\frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$

2 $5y=x-11$ **3** $x=-\frac{1}{2}, \frac{3}{2}$

4 $\text{BC}=\frac{8\sqrt{10}}{3}$ (**Hint:** normal is $y=-3x+8$)

5 **a** $\dfrac{6x-2x^2}{(3-2x)^2}$ **b** $\frac{1}{2}x^{-\frac{1}{2}}(x^2-x)^3+3x^{\frac{1}{2}}(x^2-x)^2\times(2x-1)$

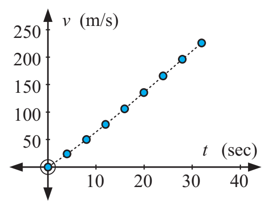
6 **a** $36x^2-\dfrac{4}{x^3}$ **b** $6x+\frac{3}{4}x^{-\frac{5}{2}}$

7 $a=9,\ b=-16$ **8** $A=9,\ B=2,\ f''(-2)=-18$

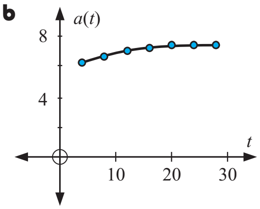
9 $a=64$ **10** $4y=3x+5$

EXERCISE 22A

1 **a**

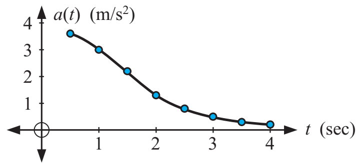


b



c $v'(t) \doteqdot 5.695t^{0.0865t}$ The fit is excellent.

2 **a** **i** 3 m/s^2 **ii** 0.8 m/s^2 **b**



EXERCISE 22B

1 **a** \$118 000 **b** $\dfrac{dP}{dt}=4t-12$ \$1000 per year
c $\dfrac{dP}{dt}$ is the rate of change in profit with time
d **i** $0\leqslant t\leqslant 3$ years **ii** $t\geqslant 3$ years
e minimum profit is \$100 000 when $t=3$
f $\left.\dfrac{dP}{dt}\right]_{t=4}=4$ Profit is increasing at \$4000 per year after 4 years.
$\left.\dfrac{dP}{dt}\right]_{t=10}=28$ Profit is increasing at \$28 000 per year after 10 years.
$\left.\dfrac{dP}{dt}\right]_{t=25}=88$ Profit is increasing at \$88 000 per year after 25 years.

2 **a** 8.5 cm^2 per cm **b** 8 cm^2 per cm

3 **a** $19\,000$ m^3 per minute **b** $18\,000$ m^3 per minute

4 **a** $\dfrac{dV}{dt} = 1.2\ \text{m}^3/\text{min}$ **b** $\dfrac{dV}{dr} = 4\pi r^2$ **d** $0.007\,368$ m/min

5 **a** 1.2 m

b $\dfrac{ds}{dt} = 28.1 - 9.8t$ represents the instantaneous velocity of the ball

c $t = 2.867$ secs. The ball has stopped and reached its maximum height. **d** 41.49 m

e **i** 28.1 m/s **ii** 8.5 m/s **iii** -20.9 m/s

$s'(t) \geqslant 0$ ball travelling upwards

$s'(t) \leqslant 0$ ball travelling downwards

f 5.735 sec **g** $\dfrac{d^2s}{dt^2}$ is the rate of change of $\dfrac{ds}{dt}$, i.e., the instantaneous acceleration.

6 **b** 69.58 m/s

EXERCISE 22C.1

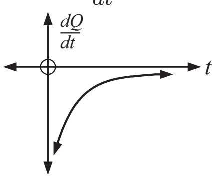
1 **a** **i** $Q = 100$ **ii** $Q = 50$ **iii** $Q = 0$

b **i** decr. 1 unit per year **ii** decr. $\frac{1}{\sqrt{2}}$ units per year

c **Hint**: Consider the graph of $\dfrac{dQ}{dt}$ against t.

$\dfrac{dQ}{dt} = \dfrac{-5}{\sqrt{t}} < 0$

for all $t > 0$



2 **a** 18.2 metres

b $t = 4$; 19 m, $t = 8$; 19.3 m, $t = 12$; 19.5 m

c $t = 0$: 0.36 m/year $t = 5$: 0.09 m/year

$t = 10$: 0.04 m/year

d as $\dfrac{dH}{dt} = \dfrac{9}{(t+5)^2} > 0$, for all $t \geqslant 0$, the tree is always growing, and $\dfrac{dH}{dt} \to 0$ as t increases

3 **a** 0^oC; 20, 20^oC; 24, 40^oC; 32 **b** $\dfrac{dR}{dT} = \frac{1}{10} + \dfrac{T}{100}$

c $\dfrac{dR}{dT} > 0$ (i.e., inc) for all $T > -10^o$C

4 **a** **i** \$4500 **ii** \$8250

b **i** increase of \$100 per km/h

ii increase of \$188.89 per km/h

c $\dfrac{dC}{dt} = 0$ at $v = \sqrt{50}$ i.e., 7.1 km/h

5 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$ $x = \frac{1}{2}$; $\dfrac{dy}{dx} = 0.175$, height of hill is increasing as slope is positive

$x = 1\frac{1}{2}$; $\dfrac{dy}{dx} = -0.225$, height of hill is decreasing as slope is negative

$\therefore$ top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

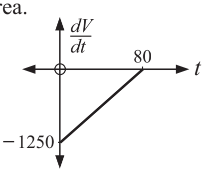
c 2.55 km from the sea, 63.1 m deep

6 **a** $\dfrac{dV}{dx} = 3x^2\ \text{mm}^3/\text{mm}$. This is the rate at which the volume increases as the length of the sides increase.

b $\dfrac{dV}{dx} = 12\ \text{mm}^3/\text{mm}$ at $x = 2$. For every millimetre the sides increase, the volume increases by 12 mm^3.

c For a small change in side length (Δx), the increase in volume is approx. $\Delta x \times$ surface area.

7 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$



b at $t = 0$ (when the tap was first opened)

c $\dfrac{d^2V}{dt^2} = \dfrac{125}{8}$ This shows that the rate of change of V is constantly increasing, i.e., the outflow is decreasing at a constant rate.

8 **a** When $\dfrac{dP}{dt} = 0$, the population is not changing over time, i.e., it is stable.

b 4000 fish **c** 8000 fish

EXERCISE 22C.2

1 **a** $C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = \$56.36$ per pair. This estimates the additional cost of making one more pair of jeans if 220 pairs are currently being made.

c \$56.58 This is the actual increase in cost to make an extra pair of jeans (221 rather than 220).

d $C''(x) = 0.0018x + 0.04$, $C''(x) = 0$ when $x = -22.2$ This is where the rate of change is a minimum, however it is out of the bounds of the model (you cannot make < 0 jeans!)

2 **a** $C'(x) = 0.000\,216x^2 - 0.001\,22x + 0.19$ dollars per item. $C'(x)$ is the instantaneous rate of change in cost with respect to the number of items made.

b $C'(300) = \$19.26$ per item. This estimates the increase in cost to make 301 items, rather than 300.

c \$19.33

d $C''(x) = 0.000\,432x - 0.001\,22$, $x = 2.8$ This is the minimum of the $C'(x)$ curve.

EXERCISE 22D.1

1 **a** 7 m/s **b** $(h + 5)$ m/s **c** 5 m/s $= s'(1)$

d av. velocity $= (2t + h + 3)$ m/s,

$\lim\limits_{h\to 0}(2t + h + 3) = s'(t) \to 2t + 3$ as $h \to 0$

2 **a** -14 cm/s **b** $(-8 - 2h)$ cm/s

c -8 cm/s $= s'(2)$ i.e., velocity $= -8$ cm/s at $t = 2$

d $-4t = s'(t) = v(t)$

3 **a** $\frac{2}{3}$ m/s^2 **b** $\left(\dfrac{2}{\sqrt{1+h}+1}\right)$ m/s^2 **c** 1 m/s^2 $= v'(1)$

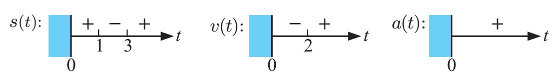
d $\dfrac{1}{\sqrt{t}}$ m/s^2 $= v'(t)$ i.e., the instantaneous accn. at time t.

4 **a** velocity at $t = 3$ **b** acceleration at $t = 5$

c velocity at t, i.e., $v(t)$ **d** acceleration at t, i.e., $a(t)$

EXERCISE 22D.2

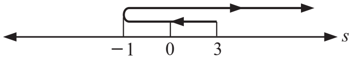
1 **a** $v(t) = 2t - 4$, $a(t) = 2$



b The object is initially 3 cm to the right of the origin and is moving to the left at 4 cm/s. It is accelerating at 2 m/s^2 to the right.

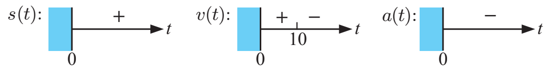
c The object is instantaneously stationary, 1 cm to the left of the origin and is accelerating to the right at 2 m/s^2.

d At $t=2$, $s(2)=1$ cm to the left of the origin.

e 

f $0 \leqslant t \leqslant 2$

2 **a** $v(t) = 98 - 9.8t$, $a(t) = -9.8$



b $s(0) = 0$ m above the ground
$v(0) = 98$ m/s skyward

c $t=5$ Stone is 367.5 m above the ground and moving skyward at 49 m/s. Its speed is decreasing. $t=12$ Stone is 470.4 m above the ground and moving groundward at 19.6 m/s. Its speed is increasing.

d 490 m **e** 20 seconds

3 **a** $v(t) = 12 - 6t^2$, $a(t) = -12t$

b $s(0) = -1$, $v(0) = 12$, $a(0) = 0$

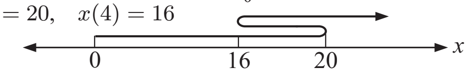
Particle started 1 cm to the left of the origin and was travelling to the right at a constant speed of 12 cm/s.

c $t = \sqrt{2}$, $s(\sqrt{2}) = 8\sqrt{2} - 1 \doteqdot 10.3$

d **i** $t \geqslant \sqrt{2}$ **ii** never

4 **a** $v(t) = 3t^2 - 18t + 24$ $a(t) = 6t - 18$

b $x(2) = 20$, $x(4) = 16$



c **i** $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$ **ii** $0 \leqslant t \leqslant 3$ **d** 28 m

5 **Hint:** $s'(t) = v(t)$ and $s''(t) = a(t) = g$

Show that $a = \frac{1}{2}g$ $b = v(0)$ $c = 0$

EXERCISE 22E.1

1 **a** **i** $x \geqslant 0$ **ii** never **b** **i** never **ii** $-2 < x \leqslant 3$
c **i** $x \leqslant 2$ **ii** $x \geqslant 2$ **d** **i** all real x **ii** never
e **i** $1 \leqslant x \leqslant 5$ **ii** $x \leqslant 1, x \geqslant 5$
f **i** $2 \leqslant x < 4$, $x > 4$ **ii** $x < 0$, $0 < x \leqslant 2$

EXERCISE 22E.2

1 **a** increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

b decreasing for all x

c increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$

d increasing for $x \geqslant 0$, never decreasing

e decreasing for $x > 0$, never increasing

f incr. for $x \leqslant 0$ and $x \geqslant 4$, decr. for $0 \leqslant x \leqslant 4$

g increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$, $x \geqslant \sqrt{\frac{2}{3}}$

h decr. for $x \leqslant -\frac{1}{2}$, $x \geqslant 3$, incr. for $-\frac{1}{2} \leqslant x \leqslant 3$

i increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

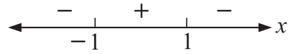
j increasing for $x \geqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$ and $x \leqslant -\frac{3}{2} - \frac{\sqrt{5}}{2}$
decreasing for $-\frac{3}{2} - \frac{\sqrt{5}}{2} \leqslant x \leqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$

k increasing for $x \leqslant 2 - \sqrt{3}$, $x \geqslant 2 + \sqrt{3}$
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

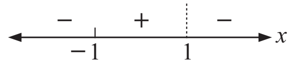
l increasing for $x \geqslant 1$, decreasing for $0 \leqslant x \leqslant 1$

m increasing for $-1 \leqslant x \leqslant 1$, $x \geqslant 2$
decreasing for $x \leqslant -1$, $1 \leqslant x \leqslant 2$

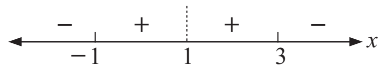
n increasing for $1 - \sqrt{2} \leqslant x \leqslant 1$, $x \geqslant 1 + \sqrt{2}$
decreasing for $x \leqslant 1 - \sqrt{2}$, $1 \leqslant x \leqslant 1 + \sqrt{2}$

2 **a** **i** 

ii increasing for $-1 \leqslant x \leqslant 1$
decreasing for $x \leqslant -1$, $x \geqslant 1$

b **i** 

ii increasing for $-1 \leqslant x < 1$
decreasing for $x \leqslant -1$, $x > 1$

c **i** 

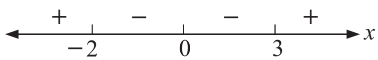
ii increasing for $-1 \leqslant x < 1$, $1 < x \leqslant 3$
decreasing for $x \leqslant -1$, $x \geqslant 3$

3 **a** increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x \leqslant 0$,
$0 \leqslant x < 1$, $1 < x \leqslant \sqrt{3}$

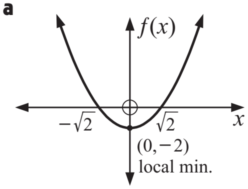
b increasing for $x \geqslant 2$ decreasing for $x < 1$, $1 < x \leqslant 2$

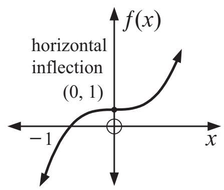
EXERCISE 22E.3

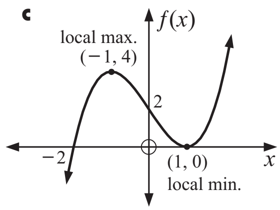
1 **a** A - local min B - local max C - horiz. inflection

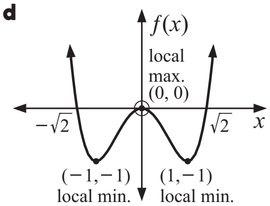
b 

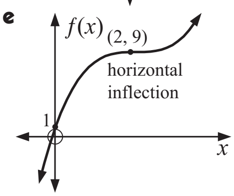
c **i** $x \leqslant -2, x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

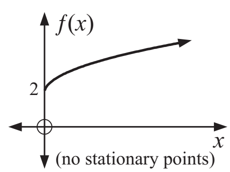
2 **a** 

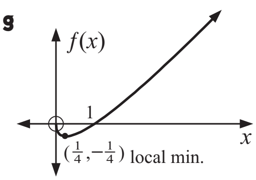
b 

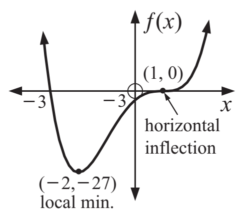
c 

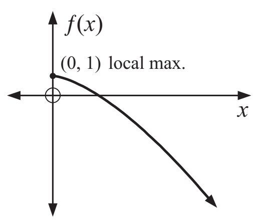
d 

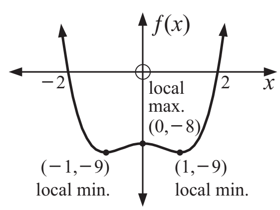
e 

f 

g 

h 

i 

j 

3 $x = -\dfrac{b}{2a}$, local min if $a > 0$, local max if $a < 0$ **4** $a = 9$

5 **a** $a = -12$, $b = -13$
b $(-2, 3)$ local max. $(2, -29)$ local min

6 $P(x) = -9x^3 - 9x^2 + 9x + 2$

7 **a** greatest value $= 63$ (at $x = 5$)
least value $= -18$ (at $x = 2$)
b greatest value $= 4$ (at $x = 3$ and $x = 0$)
least value $= -16$ (at $x = -2$)

8 Maximum hourly cost $=$ \$680.95 when 150 hinges are made per hour. Minimum hourly cost $=$ \$529.80 when 104 hinges are made per hour.

EXERCISE 22F.1

1 **a** $f(x) = 3 - \dfrac{5}{x+1}$ H.A. $y = 3$, V.A. $x = -1$

b $f(x) = \frac{1}{2} - \dfrac{7}{2(2x-1)}$ H.A. $y = \frac{1}{2}$, V.A. $x = \frac{1}{2}$

c $f(x) = -2 + \dfrac{2}{x-1}$ H.A. $y = -2$, V.A. $x = 1$.

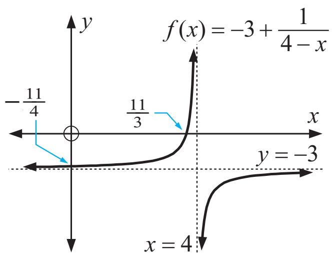
2 **a** **i** H.A. $y = -3$, V.A. $x = 4$

ii $f'(x) = (4-x)^{-2}$

$+ \mid +$
4

iii x-int. is $\frac{11}{3}$, y-int. is $-\frac{11}{4}$.

iv



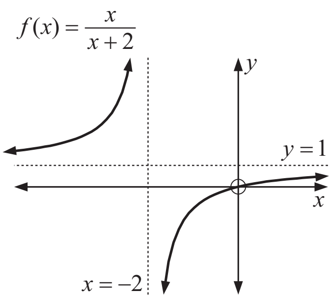
b **i** H.A. $y = 1$, V.A. $x = -2$

ii $f'(x) = \dfrac{2}{(x+2)^2}$

$+ \mid +$
-2

iii x-int. is 0, y-int. is 0.

iv



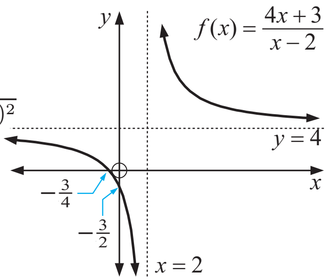
c **i** H.A. $y = 4$, V.A. $x = 2$

ii $f'(x) = \dfrac{-11}{(x-2)^2}$

$- \mid -$
2

iii x-int. is $-\frac{3}{4}$, y-int. is $-\frac{3}{2}$.

iv



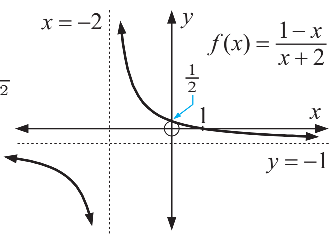
d **i** H.A. $y = -1$, V.A. $x = -2$

ii $f'(x) = \dfrac{-3}{(x+2)^2}$

$- \mid -$
-2

iii x-int. is 1, y-int. is $\frac{1}{2}$.

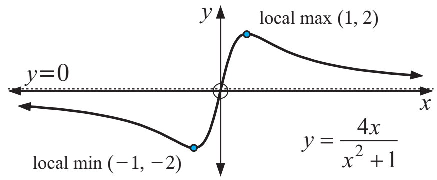
iv



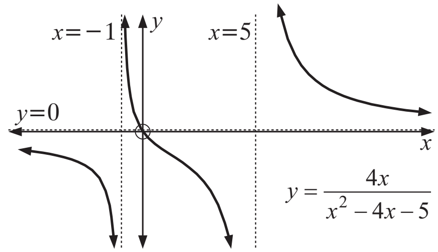
EXERCISE 22F.2

1 **a** H.A. $y = 0$, V.A. $x = 2$ and $x = -2$
b H.A. $y = 0$, V.A. $x = -2$ **c** H.A. $y = 0$, no V.A

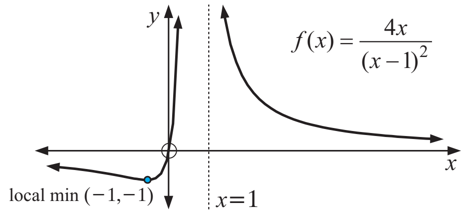
2 **a** **i** H.A. $y = 0$ **ii** $(1, 2)$ is a local maximum, $(-1, -2)$ is a local minimum
iii x-intercept is 0, y-intercept is 0
iv



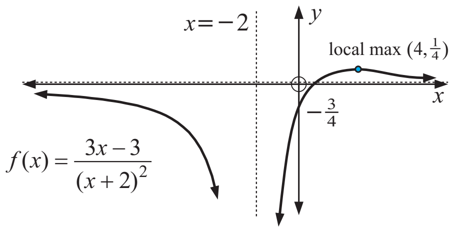
b **i** H.A. $y = 0$, V.A.s $x = 5$ and $x = -1$
ii no stationary points $\left[f'(x) = \dfrac{-4(x^2+5)}{(x-5)^2(x+1)^2} \right]$
iii x-intercept is 0, y-intercept is 0
iv



c **i** H.A. $y = 0$, V.A. $x = 1$
ii $(-1, -1)$ is a local minimum
iii x-intercept is 0, y-intercept is 0
iv



d **i** H.A. $y = 0$, V.A. $x = -2$
ii $(4, \frac{1}{4})$ is a local maximum
iii x-intercept is 1, y-intercept is $-\frac{3}{4}$
iv

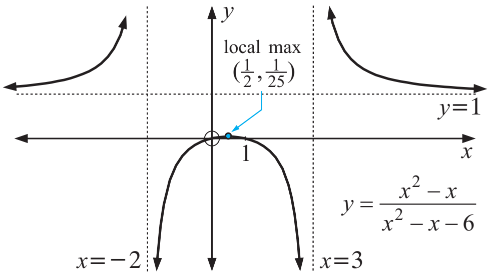


EXERCISE 22F.3

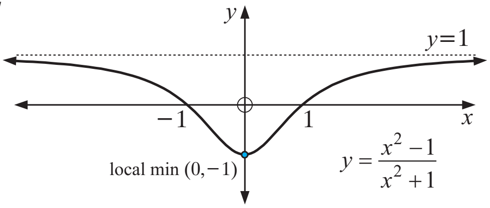
1 **a** H.A. $y = 2$, V.A.s $x = 1$ and $x = -1$
b H.A. $y = -1$, no V.A.
c H.A. $y = 3$, V.A. $x = -2$

2 **a** **i** H.A. $y = 1$, V.A.s $x = 3$ and $x = -2$
ii $\left(\frac{1}{2}, \frac{1}{25}\right)$ is a local maximum
iii x-intercepts are 0 and 1, y-intercept is 0

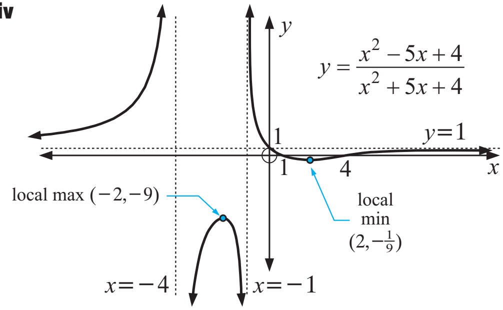
iv



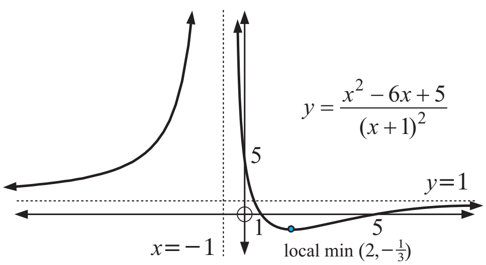
b **i** H.A. $y=1$, no V.A.
ii $(0, -1)$ is a local minimum
iii x-intercepts are 1 and -1, y-intercept is -1
iv



c **i** H.A. $y=1$, V.A.s $x=-4$ and $x=-1$
ii $(2, -\frac{1}{9})$ is a local min., $(-2, -9)$ is a local max.
iii x-intercepts are 4 and 1, y-intercept is 1
iv



d **i** H.A. $y=1$, V.A. $x=-1$
ii $(2, -\frac{1}{3})$ is a local minimum
iii x-intercepts are 5 and 1, y-intercept is 5
iv

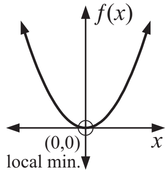


EXERCISE 22G

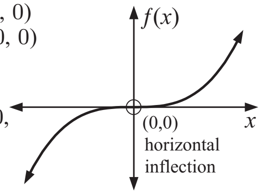
1 **a** no inflection **b** horizontal inflection at $(0, 2)$
c non-horizontal inflection at $(2, 3)$
d horizontal inflection at $(-2, -3)$
e horizontal inflection at $(0, 2)$
non-horizontal inflection at $(-\frac{4}{3}, \frac{310}{27})$ **f** no inflection

2 **a** **i** local minimum at $(0, 0)$ **ii** no points of inflection
iii decreasing for $x \leqslant 0$, increasing for $x \geqslant 0$
iv function is convex for all x

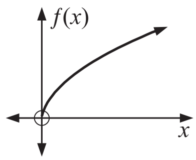
v



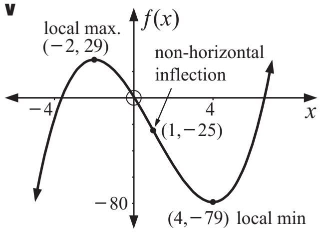
b **i** horizontal inflection at $(0, 0)$
ii horizontal inflection at $(0, 0)$
iii increasing for all real x
iv concave down for $x \leqslant 0$, concave up for $x \geqslant 0$
v



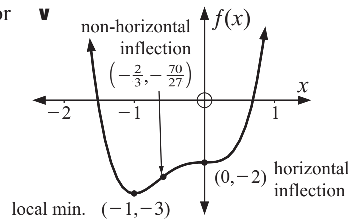
c **i** $f'(x) \neq 0$, no stationary points
ii no points of inflection
iii incr. for $x \geqslant 0$, never decr.
iv concave down for $x \geqslant 0$, never concave up
v



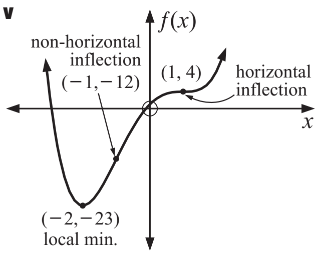
d **i** local max. at $(-2, 29)$ local min at $(4, -79)$
ii non-horizontal inflection at $(1, -25)$
iii increasing for $x \leqslant -2$, $x \geqslant 4$
decreasing for $-2 \leqslant x \leqslant 4$
iv concave down for $x \leqslant 1$, concave up for $x \geqslant 1$
v



e **i** horiz. inflection at $(0, -2)$ local min. at $(-1, -3)$
ii horizontal inflection at $(0, -2)$
non-horizontal inflection at $(-\frac{2}{3}, -\frac{70}{27})$
iii increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$
iv concave down for $-\frac{2}{3} \leqslant x \leqslant 0$
concave up for $x \leqslant -\frac{2}{3}$, $x \geqslant 0$
v



f **i** local min. at $(-2, -23)$ horizontal inflection at $(1, 4)$
ii horizontal inflection at $(1, 4)$
non-horizontal inflection at $(-1, -12)$
iii increasing for $x \geqslant -2$, decreasing for $x \leqslant -2$
iv concave down for $-1 \leqslant x \leqslant 1$,
concave up for $x \leqslant -1$, $x \geqslant 1$
v



g **i** local minimum at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, -1)$, local maximum at $(0, 3)$,

ii non-horizontal inflection at $\left(\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$

non-horizontal inflection at $\left(-\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$

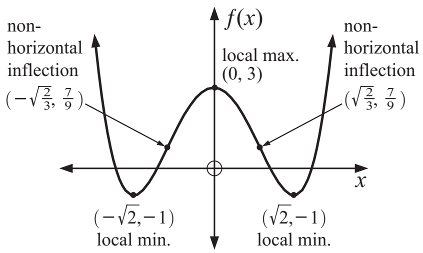
iii increasing for $-\sqrt{2} \leqslant x \leqslant 0,\ x \geqslant \sqrt{2}$

decreasing for $x \leqslant -\sqrt{2},\ 0 \leqslant x \leqslant \sqrt{2}$

iv concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$

concave up for $x \leqslant -\sqrt{\frac{2}{3}},\ x \geqslant \sqrt{\frac{2}{3}}$

v



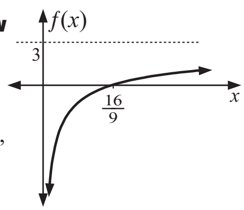
h **i** no stationary points

ii no inflections

iii increasing for $x > 0$, never decreasing

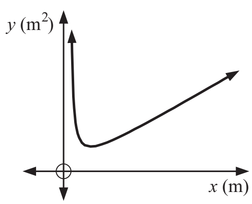
iv concave down for $x > 0$, never concave up

v



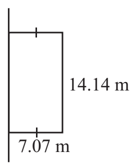
EXERCISE 22H

1 **b**



c $L_{\min} = 28.28$ m, $x = 7.07$ m

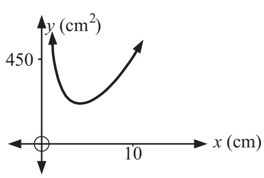
d



2 **a** $2x$ cm **b** $V = 200 = 2x \times x \times h$

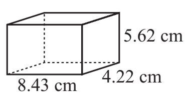
c **Hint**: Show $h = \dfrac{100}{x^2}$ and substitute into the surface area equation.

d



e $SA_{\min} = 213.4$ cm², $x = 4.22$ cm

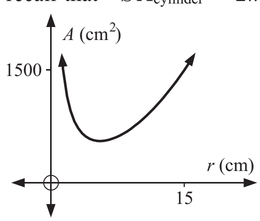
f



3 **a** recall that $V_{\text{cylinder}} = \pi r^2 h$ and that $1 \text{ L} = 1000 \text{ cm}^3$

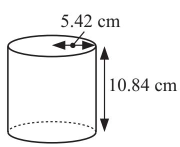
b recall that $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$

c



d $A = 554$ cm², $r = 5.42$ cm

e



4 **b** 6 cm $\times$ 6 cm

5 **a** $0 \leqslant x \leqslant 63.66$

c $x = 63.66$ m, $l = 0$ m (i.e., circular track)

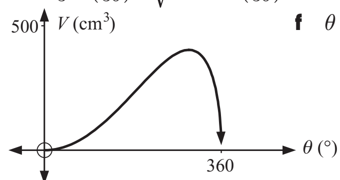
6 **a** **Hint**: Show that $\text{AC} = \frac{\theta}{360} \times 2\pi \times 10$

b **Hint**: Show that $2\pi r = \text{AC}$

c **Hint**: Use the result from **b** and Pythagoras' theorem.

d $V(\theta) = \frac{1}{3}\pi \left(\frac{\theta}{36}\right)^2 \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$

e



f $\theta = 293.9^o$

7 **b** **Hint**: Show $C = 25x^2 + 200xy$ then use the result from **a**. **c** 1.59 m $\times$ 1.59 m $\times$ 0.397 m

8 **a** $2x$ units $\times \dfrac{100}{x^2}$ units **b** **Hint**: Show that $\dfrac{dA}{dx} = -\dfrac{200}{x^2}$

c $P_{\min} = 27.8$ units, 9.28 units $\times$ 4.64 units

9 13.44 cm from left (i.e., uses 13.44 cm for square tubing)

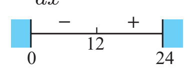
10 **a** For $x < 0$ or $x > 6$, X is not on AC.

c $x = 2.67$ km This is the distance from A to X which minimises the time taken to get from B to C. (Proof: Use sign diagram or second derivative test. Be sure to check the end points.)

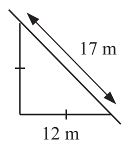
11 3.33 km **12** radius = 31.7 cm, height = 31.7 cm (Note: $100 \text{ L} = 0.1 \text{ m}^3$)

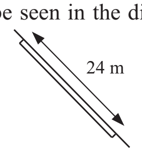
13 4 m from the 40 cp globe

14 **a** $D(x) = \sqrt{x^2 + (24 - x)^2}$ **b** $\dfrac{d[D(x)]^2}{dx} = 4x - 48$



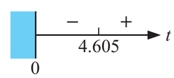
c Smallest $D(x) = 17.0$ Largest $D(x) = 24$, which is not an acceptable solution as can be seen in the diagram.





15 **a** **Hint**: Use the cosine rule.

b 3553 km²



c 5:36 pm

16 **a** $\text{QR} = \left(\dfrac{2 + x}{x}\right)$ m **c** **Hint**: All solutions < 0 can be discarded as $x \geqslant 0$.

d 416 cm

17 between A and N, 2.566 m from N

18 at grid reference (3.544, 8) **19** $\text{A} = (4a, 0)$

20 $\sqrt{\frac{3}{2}} : 1$ **21** **e** 63.7%

EXERCISE 22I

1 **a** cost = \$878 000, average cost = \$1097.50, marginal cost = \$1850

b 195 items, \$639.87

2 **a** $A(x) = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$

$C'(x) = 24 - 0.16x + 0.0024x^2$

b min. average cost = \$26.41 (when 79 items are made)

c min. marginal cost = \$21.33 (when 33 items are made)

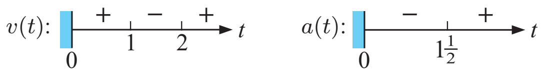
3 50 fittings **4** 250 items **5** 10 blankets

6 **a** $p(x) = 250 - \frac{x}{8}$, $x \geqslant 800$ **b** \$25 **c** \$10

7 25 km/h

REVIEW SET 22A

1 **a** $v(t) = (6t^2 - 18t + 12)$ cm/s
$a(t) = (12t - 18)$ cm/s^2

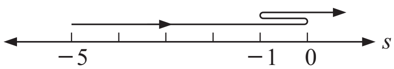


b $s(0) = 5$ cm to left of origin
$v(0) = 12$ cm/s towards origin
$a(0) = -18$ cm/s^2 (reducing speed)

c At $t = 2$, particle is 1 cm to the left of the origin, is stationary and is accelerating towards the origin.

d $t = 1$, $s = 0$ and $t = 2$, $s = -1$

e



f Speed is increasing for $1 \leqslant t \leqslant 1\frac{1}{2}$ and $t \geqslant 2$.

2 **a** **i** \$312 **ii** \$1218.75

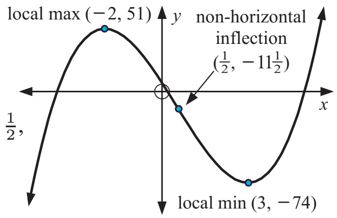
b **i** \$9.10 per km/h **ii** \$7.50 per km/h **c** 3 km/h

3 **a** local maximum at $(-2, 51)$, local minimum at $(3, -74)$
non-horizontal inflection at $(\frac{1}{2}, -11.5)$

b increasing for $x \leqslant -2$, $x \geqslant 3$
decreasing for $-2 \leqslant x \leqslant 3$

c concave down for $x \leqslant \frac{1}{2}$,
concave up for $x \geqslant \frac{1}{2}$

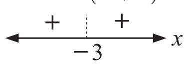
d



4 **a** $x = -3$

b y-intercept at $y = -\frac{2}{3}$, x-intercept at $x = \frac{2}{3}$

c $f'(x) = \frac{11}{(x+3)^2}$ **d** There are no stationary points.

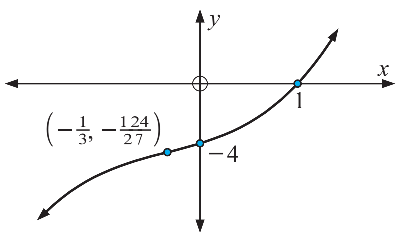


5 **b** $k = 9$

6 **a** $y = -4$ **b** $x = 1$ (only 1 intercept)

c no stat. points, non-horizontal inflection at $(-\frac{1}{3}, -\frac{124}{27})$

d



7 **a** $y = \frac{500}{x}$, $x > 0$ **b** $\frac{dy}{dx} = -\frac{500}{x^2}$ as $x^2 > 0$, $-\frac{500}{x^2} < 0$

c As the width of the rectangle increases, the length decreases.

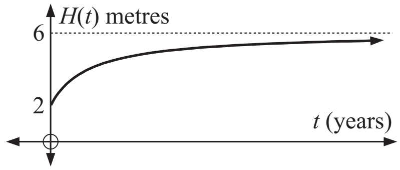
8 **a** 2 m

b $H(3) = 4$ m, $H(6) = 4.67$ m, $H(9) = 5$ m

c $H'(0) = 1.33$ m/year, $H'(3) = 0.333$ m/year,
$H'(6) = 0.148$ m/year, $H'(9) = 0.083$ m/year

d As $H'(t) > 0$, the tree is always growing.

e



9 **a** $y = \frac{1}{x^2}$, $x > 0$

c base is 1.26 cm square, height is 0.630 m

10 **b** $\frac{d[A(x)]^2}{dx} = 5000x - 4x^3$

Area is a maximum when $x \doteqdot 35.4$, $A = 1250$ m^2.

11 **a** $v(t) = 15 + \frac{120}{(t-1)^3}$ cm/s, $a(t) = \frac{-360}{(t-1)^4}$ cm/s^2

b At $t = 3$, particle is 30 cm to the right of the origin, moving to the right at 30 cm/s and decelerating at 22.5 cm/s^2.

c $0 \leqslant t < 1$

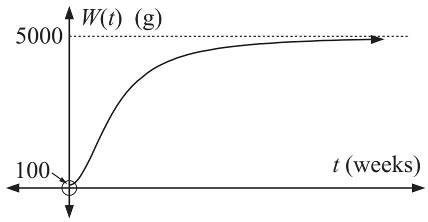
12 6 cm from each end

REVIEW SET 22B

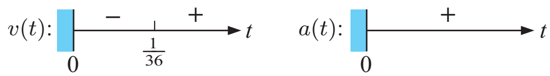
1 **a** 100 g **b** **i** 2550 g **ii** 4711.8 g **iii** 4992.8 g

c **i** 0 **ii** 1514 g/week **iii** 333.5 g/week

d



2 **a** $v(t) = 3 - \frac{1}{2\sqrt{t}}$, $a(t) = \frac{1}{4t\sqrt{t}}$



b $x(0) = 0$, $v(0)$ is undefined, $a(0)$ is undefined

c Particle is 24 cm to the right of the origin and is travelling to the right at 2.83 cm/s. Its speed is increasing.

d Changes direction at $t = \frac{1}{36}$, 0.083 cm to the left of the origin.

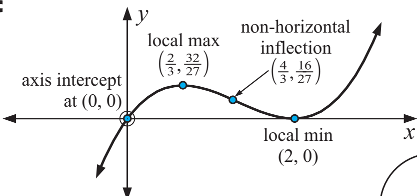
e Particle's speed is decreasing for $0 \leqslant t \leqslant \frac{1}{36}$.

3 **a** $t \geqslant 2$ **b** $t > 17$

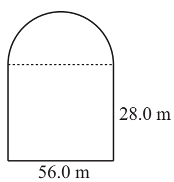
4 **a** y-intercept at $y = 0$, x-intercept at $x = 0$ and $x = 2$

b local maximum at $\left(\frac{2}{3}, \frac{32}{27}\right)$, local minimum at $(2, 0)$, non-horizontal inflection at $\left(\frac{4}{3}, \frac{16}{27}\right)$

c



5 **b** $A(x) = 200x - 2x^2 - \frac{1}{2}\pi x^2$ **c**

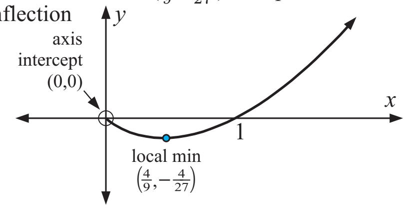


6 **a** $x \geqslant 0$

b y-intercept at $y = 0$, x-intercepts at $x = 0, 1$

c local minimum at $(\frac{4}{9}, \frac{-4}{27})$, no points of inflection

d

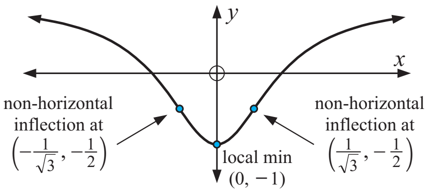


7 **a** y-int. at $y = -1$ x-int. at $x = 1$, $x = -1$

b $x^2 + 1 > 0$ for all real x (i.e., denominator is never 0)

c local minimum at $(0, -1)$

d

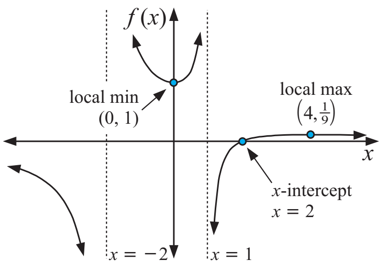


8 **a** $x = -2$ and $x = 1$

b local min. at $(0, 1)$, local max. at $(4, \frac{1}{9})$

c y-intercept at $y = 1$, x-intercept at $x = 2$

d



e $p < \frac{1}{9}$ or $p > 1$

9 **a** **Hint**: Use Pythagoras to find h as a function of x and then substitute into the equation for the volume of a cylinder.

b radius $= 4.08$ cm, height $= 5.77$ cm

10 **a** $\text{LQ} = \dfrac{8}{x}$ km **b** **Hint**: Show that $(\text{length of pipe})^2 = (LQ + 1)^2 + (8 + x)^2$ then simplify.

c 11.2 km (when $x = 2$ km)

11 **Hint**: Show that $V = x(k - 2x)^2$

12 **a** **Hint**: Draw in construction lines OA and OC to find the base length and height respectively.

b **Hint**: Show that $x = \dfrac{r}{2}$ and use ΔOAN and CNB to show that all sides have length $\sqrt{3}r$.

EXERCISE 23A

1 **a** $4e^{4x}$ **b** e^x **c** $-2e^{-2x}$ **d** $\frac{1}{2}e^{\frac{x}{2}}$ **e** $-e^{-\frac{x}{2}}$ **f** $2e^{-x}$

g $2e^{\frac{x}{2}} + 3e^{-x}$ **h** $\dfrac{e^x - e^{-x}}{2}$ **i** $-2xe^{-x^2}$ **j** $e^{\frac{1}{x}} \times \dfrac{-1}{x^2}$

k $20e^{2x}$ **l** $40e^{-2x}$ **m** $2e^{2x+1}$ **n** $\frac{1}{4}e^{\frac{x}{4}}$ **o** $-4xe^{1-2x^2}$

p $-0.02e^{-0.02x}$

2 **a** $e^x + xe^x$ **b** $3x^2e^{-x} - x^3e^{-x}$ **c** $\dfrac{xe^x - e^x}{x^2}$

d $\dfrac{1-x}{e^x}$ **e** $2xe^{3x} + 3x^2e^{3x}$ **f** $\dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $\frac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$ **h** $\dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** $4e^x(e^x + 2)^3$ **b** $\dfrac{-e^{-x}}{(1 - e^{-x})^2}$ **c** $\dfrac{e^{2x}}{\sqrt{e^{2x} + 10}}$

d $\dfrac{6e^{3x}}{(1 - e^{3x})^3}$ **e** $-\dfrac{e^{-x}}{2}\left(1 - e^{-x}\right)^{-\frac{3}{2}}$ **f** $\dfrac{1 - 2e^{-x} + xe^{-x}}{\sqrt{1 - 2e^{-x}}}$

4 **b** $\dfrac{d^n y}{dx^n} = k^n y$

5 **Hint:** Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute into the equation.

6 **a** local maximum at $(1, e^{-1})$

b local max. at $(-2, 4e^{-2})$, local min. at $(0, 0)$

c local minimum at $(1, e)$ **d** local maximum at $(-1, e)$

EXERCISE 23B

1 **a** $\ln N = \ln 50 + 2t$ **b** $\ln P = \ln 8.69 - 0.0541t$

c $\ln S = 2\ln a - kt$

2 **a** $D \doteqdot 8.17 \times e^{0.69t}$ **b** $G \doteqdot 1.815 \times 10^{-14} \times e^{0.0173t}$

c $P = ge^{-2t}$ **d** $F = x^2e^{-0.03t}$

3 **a** 2 **b** $\frac{1}{2}$ **c** -1 **d** $-\frac{1}{2}$ **e** 3 **f** 9 **g** $\frac{1}{5}$ **h** $\frac{1}{4}$

4 **a** $\ln 30$ **b** $\ln 16$ **c** $\ln 25$ **d** $\ln(2^3e^2)$

5 **a** $e^{\ln 2}$ **b** $e^{\ln 10}$ **c** $e^{\ln a}$ **d** $e^{x\ln a}$

6 **a** $x = \ln 2$ **b** no real solutions **c** no real solutions

d $x = \ln 2$ **e** $x = 0$ **f** $x = \ln 2$ or $\ln 3$ **g** $x = 0$

h $x = \ln 4$ **i** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

7 **a** $x = 2$ (note that $x > 0$) **b** x has no solutions

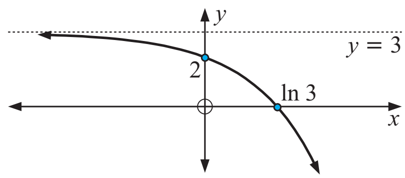
8 **a** $(\ln 3, 3)$ **b** $(\ln 2, 5)$ **c** $(0, 2)$ and $(\ln 5, -2)$

9 **a** $\text{A} = (\ln 3, 0)$ $\text{B} = (0, 2)$

b $f'(x) = -e^x$ which is < 0 for all x

c $f''(x) = -e^x$ which is < 0 for all x, so $f(x)$ is concave down

d



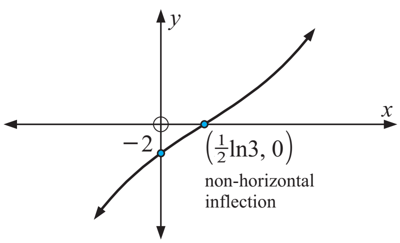
e as $x \to -\infty$, $e^x \to 0$, so $f(x) \to 3$

10 **a** $\text{P} = (\frac{1}{2}\ln 3, 0)$ $\text{Q} = (0, -2)$

b $f'(x) = e^x + 3e^{-x} > 0$ for all x

c $f(x)$ is concave down below the x-axis and concave up above the x-axis

d



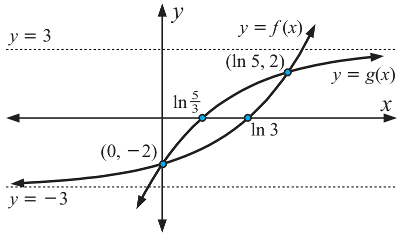
11 **a** $f(x)$: x-intercept is at $x = \ln 3$, y-intercept is at $y = -2$

$g(x)$: x-intercept is at $x = \ln\left(\frac{5}{3}\right)$ y-intercept is at $y = -2$

b $f(x)$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -3$ (above)
$g(x)$: as $x \to \infty$, $g(x) \to 3$ (below)
as $x \to -\infty$, $g(x) \to -\infty$

c intersect at $(0, -2)$ and $(\ln 5, 2)$

d



EXERCISE 23C

1 **a** $\dfrac{1}{x}$ **b** $\dfrac{2}{2x+1}$ **c** $\dfrac{1-2x}{x-x^2}$ **d** $-\dfrac{2}{x}$ **e** $2x\ln x + x$

f $\dfrac{1-\ln x}{2x^2}$ **g** $e^x\ln x + \dfrac{e^x}{x}$ **h** $\dfrac{2\ln x}{x}$ **i** $\dfrac{1}{2x\sqrt{\ln x}}$

j $\dfrac{e^{-x}}{x} - e^{-x}\ln x$ **k** $\dfrac{\ln(2x)}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$ **l** $\dfrac{\ln x - 2}{\sqrt{x}(\ln x)^2}$

m $\dfrac{4}{(1-x)}$ **n** $\dfrac{\ln(4x)}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$ **o** $\ln(x^2+1) + \dfrac{2x^2}{x^2+1}$

2 **a** $\ln 5$ **b** $\dfrac{3}{x}$ **c** $\dfrac{4x^3+1}{x^4+x}$ **d** $\dfrac{1}{x-2}$

e $\dfrac{6}{2x+1}[\ln(2x+1)]^2$ **f** $\dfrac{1-\ln(4x)}{x^2}$ **g** $-\dfrac{1}{x}$

h $\dfrac{1}{x\ln(\ln x)}$ **i** $\dfrac{-1}{x[\ln x]^2}$

3 **a** $\dfrac{-1}{1-2x}$ **b** $\dfrac{-2}{2x+3}$ **c** $1+\dfrac{1}{2x}$ **d** $\dfrac{1}{x} - \dfrac{1}{2(2-x)}$

e $\dfrac{1}{x+3} - \dfrac{1}{x-1}$ **f** $\dfrac{2}{x} + \dfrac{1}{3-x}$ **g** $\dfrac{9}{3x-4}$

h $\dfrac{1}{x} + \dfrac{2x}{x^2+1}$ **i** $\dfrac{2x+2}{x^2+2x} - \dfrac{1}{x-5}$ **j** $\dfrac{3}{x} + \dfrac{6}{2-3x}$

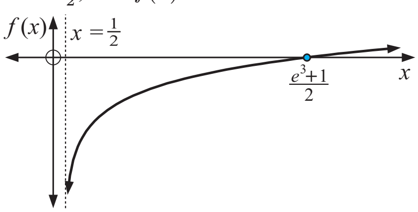
k $\dfrac{1}{2(x+1)} - \dfrac{1}{x} - \dfrac{1}{x+2}$ **l** $\dfrac{1}{x+1} + \dfrac{1}{2x-1} - \dfrac{2}{x}$

4 $\dfrac{dy}{dx} = 2^x\ln 2$

5 **a** $x = \dfrac{e^3+1}{2}$ $(\doteq 10.54)$ **b** no, $\therefore$ there is no y-int.

c slope $= 2$ **d** $x > \frac{1}{2}$ **e** $f''(x) = \dfrac{-4}{(2x-1)^2} < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down

f



6 **a** $x > 0$ **b** **Hint:** Find when $f'(x) = 0$ and show this point to be a local minimum. You must also consider $f(x)$ as $x \to 0$ and $x \to \infty$.

7 **Hint**: Show that as $x \to 0$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to 0$.

8 **Hint**: Show that $f(x) \geqslant 1$ for all $x > 0$.

EXERCISE 23D

1 $y = -\dfrac{1}{e}x + \dfrac{2}{e}$ **2** $3y = -x + 3\ln 3 - 1$

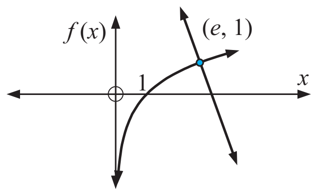
3 A is $\left(\frac{2}{3}, 0\right)$, B is $(0, -2e)$ **4** $y = -\dfrac{2}{e^2}x + \dfrac{2}{e^4} - 1$

5 $y = e^a x + e^a(1-a)$ so $y = ex$ is the tangent to $y = e^x$ from the origin

6 **a** $x > 0$

b $f'(x) > 0$ for all $x > 0$, so $f(x)$ is always increasing. Its slope is always positive.
$f''(x) < 0$ for all $x > 0$, so $f(x)$ is concave down for all $x > 0$.

c



normal has equation $f(x) = -ex + 1 + e^2$

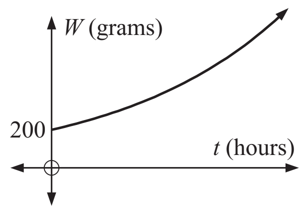
7 $\doteq 63.4^o$

8 **a** **i** 200 grams **ii** 256 grams **iii** 423 grams

b 3 hours 13 min

c **i** 100 g/hour **ii** 272 g/hour

d



9 **a** $k = \frac{1}{50}\ln 2$ $(\doteq 0.0139)$

b **i** 20 grams **ii** 14.3 grams **iii** 1.95 grams

c 9 days and 6 minutes (216 hours)

d **i** -0.0693 g/h **ii** -2.64×10^{-7} g/h

e **Hint**: You should find $\dfrac{dW}{dx} = -\frac{1}{50}\ln 2 \times 20e^{-\frac{1}{50}\ln 2t}$

10 **a** $k = \frac{-1}{15}\ln\left(\frac{95}{15}\right)$ $(\doteq 0.123)$ **b** 100^oC

d **i** decreasing by 11.7^oC per minute
ii decreasing by 3.42^oC per minute
iii decreasing by 0.998^oC per minute

11 **a** 43.9 cm **b** 10.4 years

c **i** growing by 5.45 cm per year
ii growing by 1.88 cm per year

12 **a** $A = 0$ **b** $k = \dfrac{\ln 2}{3}$ $(\doteq 0.231)$

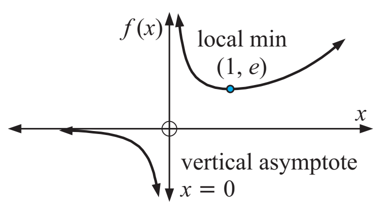
c 0.728 units of alcohol produced per hour

13 **a** $f(x)$ does not have any x or y-intercepts

b as $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to 0$ (negative)

c local minimum at $(1, e)$

d



e $ey = -2x - 3$

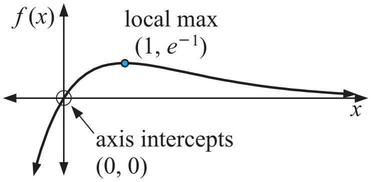
14 **a** cuts the axes at (0, 0)

b as $x \to \infty$, $f(x) \to 0$ (positive)

as $x \to -\infty$ $f(x) \to -\infty$

c yes - a local maximum at $(1, e^{-1})$

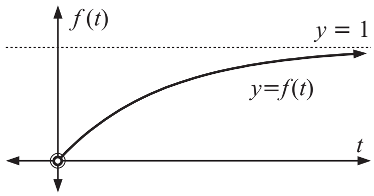
d



e **Hint**: Show that the normal has equation $y = e^2x + 2e^{-2} - 2e^2$

15 **a** No, because $f'(t) > 0$ for all t for any value a.

b



c $f'(t) > 0$ and $f''(t) < 0$ for all t, so the curve is always concave down and $f(t)$ is always increasing.

16 **a** $v(t) = 10e^{-\frac{t}{10}}$ cm/s , $a(t) = -e^{-\frac{t}{10}}$ cm/s^2

b $x(0) = 100$ cm, $v(0) = 10$ cm/s , $a(0) = -1$ cm/s^2

c $x(5) = 139$ cm $v(5) = 6.06$ cm/s

$a(5) = -0.606$ cm/s^2

d 6.93 seconds

e As $v(t)$ and $a(t)$ are opposite sign, speed is decreasing. Because $a(t) < 0$, velocity is decreasing also.

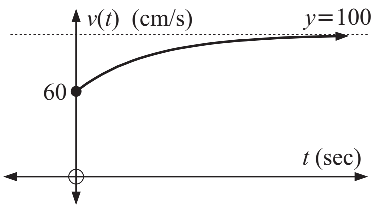
17 **a** $v(t) = 100 - 40e^{-\frac{t}{5}}$ cm/s, $a(t) = 8e^{-\frac{t}{5}}$ cm/s^2

b $s(0) = 200$ cm on positive side of origin

$v(0) = 60$ cm/s $a(0) = 8$ cm/s^2

c as $t \to \infty$, $v(t) \to 100$ ms^{-1} (below)

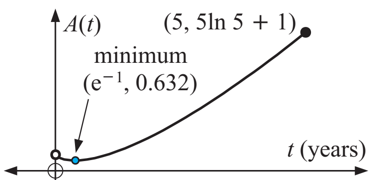
d



e after 3.47 seconds

18 **a** at 4.41 months old

b



19 **a** There is a local maximum at $\left(0, \dfrac{1}{\pi\sqrt{2}}\right)$.

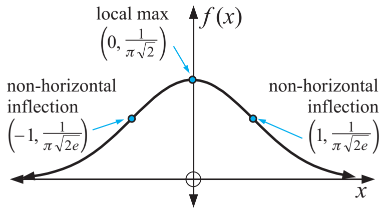
$f(x)$ is incr. for all $x \leqslant 0$ and decr. for all $x \geqslant 0$.

b Inflections at $\left(-1, \dfrac{1}{\pi\sqrt{2e}}\right)$ and $\left(1, \dfrac{1}{\pi\sqrt{2e}}\right)$

c as $x \to \infty$, $f(x) \to 0$ (positive)

as $x \to -\infty$, $f(x) \to 0$ (positive)

d



20 20 kettles **21** $C = \left(\frac{1}{\sqrt{2}}, e^{(-\frac{1}{2})}\right)$ **22** 267 torches

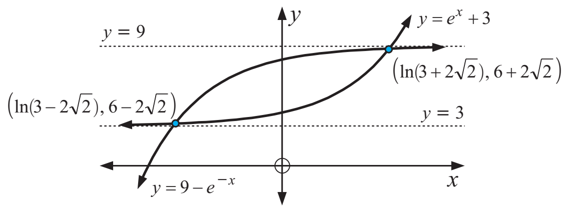
23 **a** **Hint**: They must have the same y-coordinate at $x = b$ and the same slope. **c** $a = \dfrac{1}{2e}$ **d** $y = e^{-\frac{1}{2}}x - \frac{1}{2}$

24 after 13.8 weeks **25** $a = \dfrac{\sqrt{e}}{2}$, $b = -\frac{1}{8}$

REVIEW SET 23A

1 **a** $3x^2e^{x^3+2}$ **b** $\dfrac{xe^x - 2e^x}{x^3}$ **2** $y = \dfrac{e}{2}x + \dfrac{1}{e} - \dfrac{e}{2}$

3



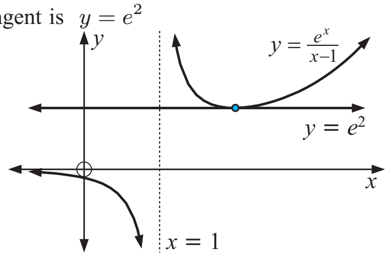
4 **a** y-intercept at $y = -1$, no x-intercept

b $f(x)$ is defined for all $x \neq 1$

c $f'(x) < 0$ for $x < 1$ and $1 < x \leqslant 2$ and $f'(x) > 0$ for $x \geqslant 2$, $f''(x) > 0$ for $x > 1$ and $f''(x) < 0$ for $x < 1$.

So, the slope of the curve is negative for all defined values of $x \leqslant 2$ and positive for all $x \geqslant 2$. The curve is concave down for $x \leqslant 1$ and concave up for $x \geqslant 1$.

d tangent is $y = e^2$



5 **a** 60 cm **b** **i** 4.244 years **ii** 201.2 years

c **i** 16 cm per year **ii** 1.951 cm per year

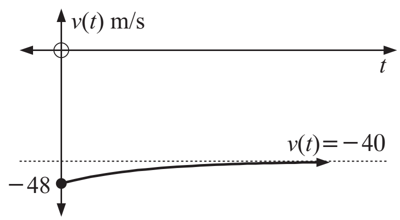
6 **a** $v(t) = -8e^{-\frac{t}{10}} - 40$ m/s $t \geqslant 0$

$a(t) = \frac{4}{5}e^{-\frac{t}{10}}$ m/s^2 $t \geqslant 0$

b $s(0) = 80$ m, $v(0) = -48$ m/s, $a(0) = 0.8$ m/s^2

c as $t \to \infty$, $v(t) \to -40$ m/s (below)

d



e $t = 6.93$ seconds

7 A$(1, e^{-1})$ **8** 100 or 101 shirts, \$938.63 profit

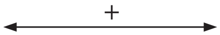
REVIEW SET 23B

1 **a** $\dfrac{3x^2-3}{x^3-3x}$ **b** $\dfrac{1}{x+3}-\dfrac{2}{x}$ **2** It does not.

3 **a** $x=\ln 3$ **b** $x=\ln 4$ or $\ln 3$

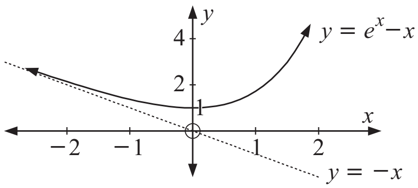
4 **a** local minimum at $(0,\ 1)$

b As $x\to\infty$, $f(x)\to\infty$,
as $x\to-\infty$, $f(x)\to -x$ (above)



c $f''(x)=e^x$
Thus $f(x)$ is concave up for all x.

d



5 **a** $\dfrac{e^x}{e^x+3}$ **b** $\dfrac{3x-(x+2)}{x(x+2)}$

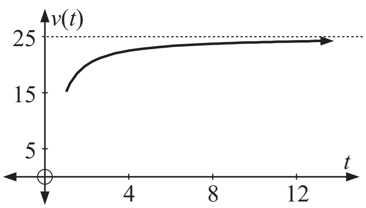
6 **a** $x=\ln\left(\frac{2}{3}\right)$ or 0 **b** $x=e^2$

7 **a** $v(t)=25-\dfrac{10}{t}$ cm/min , $a(t)=\dfrac{10}{t^2}$ cm/min^2

b $s(e)=25e-10$, $v(e)=25-\dfrac{10}{e}$, $a(e)=\dfrac{10}{e^2}$

c As $t\to\infty$, $v(t)\to 25$ cm/min from below

d



e $t=\frac{10}{13}$ min **8** 199

EXERCISE 24A

1 **a** $2\cos(2x)$ **b** $\cos x-\sin x$ **c** $-3\sin(3x)-\cos x$

d $\cos(x+1)$ **e** $2\sin(3-2x)$ **f** $\dfrac{5}{\cos^2(5x)}$

g $\frac{1}{2}\cos(\frac{x}{2})+3\sin x$ **h** $\dfrac{3\pi}{\cos^2(\pi x)}$ **i** $4\cos x+2\sin(2x)$

2 **a** $2x-\sin x$ **b** $\dfrac{1}{\cos^2 x}-3\cos x$ **c** $e^x\cos x-e^x\sin x$

d $-e^{-x}\sin x+e^{-x}\cos x$ **e** $\cot x$ **f** $2e^{2x}\tan x+\dfrac{e^{2x}}{\cos^2 x}$

g $3\cos(3x)$ **h** $-\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **i** $\dfrac{6}{\cos^2(2x)}$

j $\cos x-x\sin x$ **k** $\dfrac{x\cos x-\sin x}{x^2}$ **l** $\tan x+\dfrac{x}{\cos^2 x}$

3 **a** $\cos(x^2)\times 2x$ **b** $-\sin(\sqrt{x})\times\frac{1}{2}x^{-\frac{1}{2}}$

c $\frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$ **d** $2\sin x\cos x$

e $3\cos^2 x\times(-\sin x)$ **f** $-\sin x\times\sin(2x)+2\cos x\cos(2x)$

g $-\sin(\cos x)\times(-\sin x)$ **h** $3\cos^2(4x)\times(-4\sin(4x))$

i $-(\sin x)^{-2}\times\cos x$ **j** $-(\cos(2x))^{-2}\times(-2\sin(2x))$

k $-4(\sin(2x))^{-3}\times(2\cos(2x))$

l $-24\left(\tan(\frac{x}{2})\right)^{-4}\left(\dfrac{1}{2\cos^2(\frac{x}{2})}\right)$

4 **a** $\dfrac{dy}{dx}=\cos x$, $\dfrac{d^2y}{dx^2}=-\sin x$, $\dfrac{d^3y}{dx^3}=-\cos x$, $\dfrac{d^4y}{dx^4}=\sin x$

b The answers of **a** are cycled over and over.

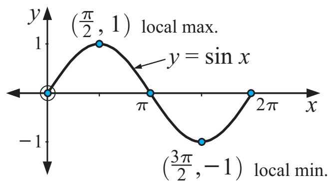
6 **a** $y=x$ **b** $y=x$ **c** $2x-y=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ **d** $x=\frac{\pi}{4}$

7 **a** falling **b** falling at 2.731 m per hour

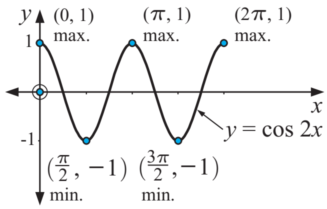
8 **a** $-34\,000\pi$ units per second **b** $V'(t)=0$

9 **b** **i** 0 **ii** 1 **iii** $\doteq 1.106$

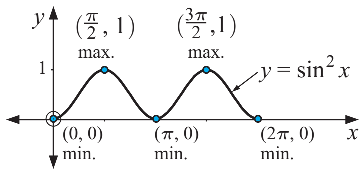
10 **a**



b



c

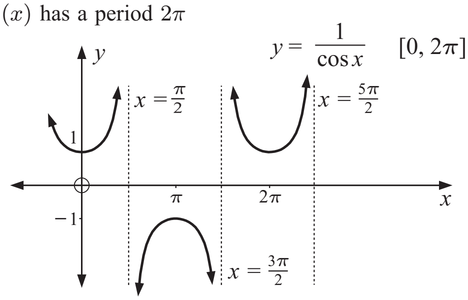


11 **a** $x=\frac{\pi}{2},\ \frac{3\pi}{2}$

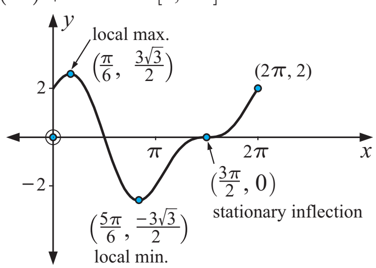
b $(0,\ 1)$ is a local minimum, $(\pi,\ -1)$ is a local maximum
$(2\pi,\ 1)$ is a local minimum

c $f(x)$ has a period 2π

d



12 $y=\sin(2x)+2\cos x$ $[0,\ 2\pi]$



13 **a** $x(0)=-1$ cm $v(0)=0$ cm/s $a(0)=2$ cm/s^2

b At $t=\frac{\pi}{4}$ seconds, the particle is
$(\sqrt{2}-1)$ cm, left of the origin moving right
at $\sqrt{2}$ cm/s, with increasing speed.

c At $t = 0$, $x(0) = -1$ cm, at $t = \pi$, $x(\pi) = 3$ cm, at $t = 2\pi$, $x(2\pi) = -1$ cm

d for $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

EXERCISE 24B

1 $\doteqdot 109.5^o$ **2** **c** $\theta = 30^o$

3 1 hour 34 min 53 sec when $\theta = 36.9^o$

4 **c** $4\sqrt{2}$ m **5** 9.866 m **6** 1.340 m from A

7 **e** AP + PB is a minimum when $\theta = \phi$

REVIEW SET 24

1 **a** $10-10\cos(10x)$ **b** $3\cos(3x)\cos(2x)-2\sin(3x)\sin(2x)$

c $-2e^{-2x}\sin x + e^{-2x}\cos x$ **d** $\tan x$

e $5\cos(5x)\ln(2x) + \dfrac{\sin(5x)}{x}$

3 **a** $f'(x) = 3\cos x + 8\sin(2x)$, $f''(x) = -3\sin x + 16\cos(2x)$

b $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)$,
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)$

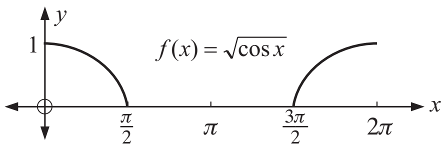
4 **a** $x(0) = 3$ cm, $x'(0) = 2$ cm/s, $x''(0) = 0$ cm/s^2

b $t = \frac{\pi}{4}$ sec and $\frac{3\pi}{4}$ sec **c** 4 cm

5 **a** for $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$, decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$

c



6 **a** $\ln(\sin x) + \dfrac{x\cos x}{\sin x}$ **b** $\frac{1}{2}(e^{\tan x})^{-\frac{1}{2}} \times \dfrac{e^{\tan x}}{\cos^2 x}$

c $\dfrac{-3\sqrt{x}\sin(3x) - \dfrac{\frac{1}{2}\cos(3x)}{\sqrt{x}}}{x}$

7 **a** $a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$

8 **a** **i** 5 km **ii** $2\sqrt{10}$ km **9** **b** $\frac{1}{\sqrt{2}}$ m above the floor

10 **a** 0 cm/s, $-\pi$ cm/s, 0 cm/s, π cm/s, 0 cm/s

b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, etc

EXERCISE 25A.1

1 **a** 0.34 **b** 0.70 **2** **a** 0.335 **b** 0.6938

EXERCISE 25A.2

1 **a** lower = 2.693, upper = 3

b lower = 2.69, upper = 3.09

c

n	*Lower*	*Upper*
10	2.79	2.99
50	2.87	2.91
100	2.88	2.90
500	2.883	2.887
5000	2.8852	2.8856

2 **a** lower = 2.55, upper = 2.90

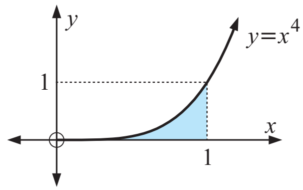
b

n	*Lower*	*Upper*
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

c Upper and lower sums converge.

EXERCISE 25B.1

1 **a**

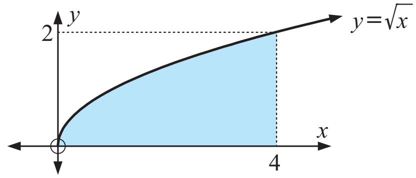


b

n	*Lower*	*Upper*
10	0.15	0.25
100	0.195	0.205
1000	0.1995	0.2005
10 000	0.199 95	0.20005

c 0.2

2 **a**

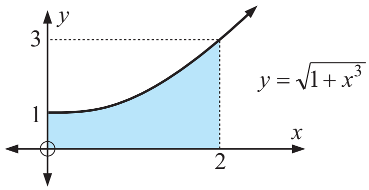


b

n	*Lower*	*Upper*
100	5.29	5.37
10 000	5.333	5.334

c $5\frac{1}{3}$

3 **a**

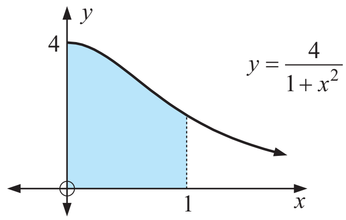


b

n	*Lower*	*Upper*
100	3.22	3.26
10 000	3.2411	3.2415

c 3.2413

4 **a**



b

n	*Lower*	*Upper*
100	3.13	3.15
1000	3.141	3.143
10 000	3.1415	3.1417

c 3.1416

5 10.203 22 **6** **a** 18 **b** 4.5 **c** 1 **d** 1 **e** 2π **f** $2+\pi$

EXERCISE 25B.2

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx = 4.667$ $\displaystyle\int_1^4 (-\sqrt{x})\,dx = -4.667$

b $\displaystyle\int_0^1 x^7\,dx = \frac{1}{8}$ $\displaystyle\int_0^1 (-x^7)dx = -\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1 **3** **a** -4 **b** 6.25 **c** 2.25

4 **a** $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$

b $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

5 **a** $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** 1

i.e., $\int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx = \int_0^1 (x^2+\sqrt{x})\,dx$

EXERCISE 25B.3

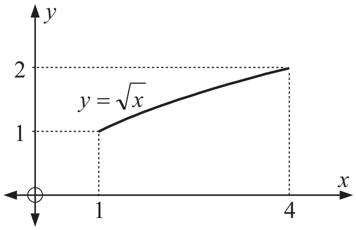
1 **a** 6.5 **b** -9 **c** 0 **d** -2.5

2 **a** 2π **b** -4 **c** $\frac{\pi}{2}$ **d** $\frac{5\pi}{2}-4$

3 **a** $\displaystyle\int_2^7 f(x)\,dx$ **b** $\displaystyle\int_1^9 g(x)\,dx$ **4** **a** -5 **b** 4

REVIEW SET 25

1 **a**

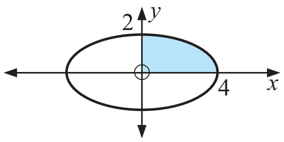


b 4.41

2 **c** $\displaystyle\int_0^4 y_A\,dx = 2\pi$ $\displaystyle\int_4^6 y_B\,dx = -\frac{\pi}{2}$ **d** $\frac{3}{2}\pi$

4 **a** $2+\pi$ **b** -2 **c** π

5 **a**



6 **a** 4 **b** 20

EXERCISE 26A

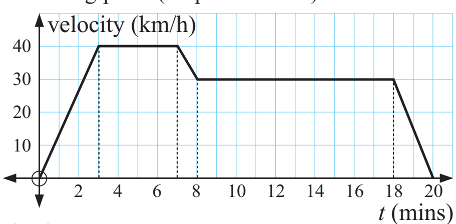
1 110 m

2 **a** **i** travelling forwards

ii travelling backwards (or in the opposite direction)

b 8 km from starting point (on positive side)

3 9.75 km



4 approx. 8.85 kWh

5 **a** 1.17 units^2 **b** 1.035 units^2 **c** 1.10 units^2

6 6.38 u^2 **7** 4.05 u^2 **8** **a** 6.39 u^2 **b** 4.05 u^2

EXERCISE 26C

1 **a** **i** $\dfrac{x^2}{2}$ **ii** $\dfrac{x^3}{3}$ **iii** $\dfrac{x^6}{6}$ **iv** $-\dfrac{1}{x}$ **v** $-\dfrac{1}{3x^3}$ **vi** $\frac{3}{4}x^{\frac{4}{3}}$ **vii** $2\sqrt{x}$

b the antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$.

2 **a** **i** $\frac{1}{2}e^{2x}$ **ii** $\frac{1}{5}e^{5x}$ **iii** $2e^{\frac{1}{2}x}$ **iv** $100e^{0.01x}$ **v** $\frac{1}{\pi}e^{\pi x}$

vi $3e^{\frac{x}{3}}$ **b** the antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$

3 **a** $\dfrac{d}{dx}(x^3+x^2) = 3x^2+2x$

$\therefore$ antiderivative of $6x^2+4x = 2x^3+2x^2$

b $\dfrac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$

$\therefore$ antiderivative of $e^{3x+1} = \frac{1}{3}e^{3x+1}$

c $\dfrac{d}{dx}(x\sqrt{x}) = \frac{3}{2}\sqrt{x}$

$\therefore$ antiderivative of $\sqrt{x} = \frac{2}{3}x\sqrt{x}$

d $\dfrac{d}{dx}(2x+1)^4 = 8(2x+1)^3$

$\therefore$ antiderivative of $(2x+1)^3 = \frac{1}{8}(2x+1)^4$

4 **a** $y = 6x+c$ **b** $y = \frac{4}{3}x^3+c$ **c** $y = \frac{5}{2}x^2 - \frac{1}{3}x^3 + c$

d $y = -\dfrac{1}{x}+c$ **e** $y = -\frac{1}{3}e^{-3x}+c$ **f** $y = x^4+x^3+c$

EXERCISE 26D

2 **a** $\frac{1}{4}$ units^2 **b** $3\frac{3}{4}$ units^2 **c** $24\frac{2}{3}$ units^2 **d** $\frac{4\sqrt{2}}{3}$ units^2

e 3.482 units^2 **f** 2 units^2 **g** 3.965 units^2

EXERCISE 26E.1

1 $\dfrac{dy}{dx} = 7x^6$; $\int x^6\,dx = \frac{1}{7}x^7 + c$

2 $\dfrac{dy}{dx} = 3x^2+2x$; $\int (3x^2+2x)\,dx = x^3+x^2+c$

3 $\dfrac{dy}{dx} = 2e^{2x+1}$; $\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1}+c$

4 $\dfrac{dy}{dx} = 8(2x+1)^3$; $\int (2x+1)^3\,dx = \frac{1}{8}(2x+1)^4+c$

5 $\dfrac{dy}{dx} = \frac{3}{2}\sqrt{x}$; $\int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x}+c$

6 $\dfrac{dy}{dx} = -\dfrac{1}{2x\sqrt{x}}$; $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx = -\frac{2}{\sqrt{x}}+c$

8 **a** $\dfrac{dy}{dx} = 12(2x-1)^5$; $\int (2x-1)^5\,dx = \frac{1}{12}(2x-1)^6+c$

b $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-4x}}$; $\displaystyle\int \frac{1}{\sqrt{1-4x}}\,dx = -\frac{1}{2}\sqrt{1-4x}+c$

c $\dfrac{dy}{dx} = -\dfrac{3}{2(3x+1)^{\frac{3}{2}}}$; $\displaystyle\int \frac{1}{(3x+1)^{\frac{3}{2}}}\,dx = \frac{-2}{3\sqrt{3x+1}}+c$

9 **a** $\dfrac{dy}{dx} = -3e^{1-3x}$; $\int e^{1-3x}\,dx = -\frac{1}{3}e^{1-3x}+c$

b $\dfrac{dy}{dx} = \dfrac{4}{4x+1}$;

$\displaystyle\int \frac{1}{4x+1}\,dx = \frac{1}{4}\ln(4x+1)+c \quad (4x+1>0)$

10 **a** $e^{x-x^2}+c$ **b** $2\ln(5-3x+x^2)+c \quad (5-3x+x^2>0)$

c $-\frac{1}{2}(x^2-5x+1)^{-2}+c$ **d** $xe^x - e^x + c$

e $\dfrac{1}{\ln 2}2^x + c$ **f** $x\ln x - x + c$

EXERCISE 26E.2

1 **a** $\frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c$ **b** $\frac{2}{3}x\sqrt{x} - 2\sqrt{x} + c$

c $2e^x + \dfrac{1}{x} + c$ **d** $\frac{2}{5}x^{\frac{5}{2}} - \ln|x| + c$

e $\frac{1}{6}(2x+1)^3 + c$ **f** $\frac{1}{2}x^2 + x - 3\ln|x| + c$

g $\frac{4}{3}x^{\frac{3}{2}} - 2\sqrt{x} + c$ **h** $-\dfrac{2}{\sqrt{x}} - 4\ln|x| + c$

i $\frac{1}{4}(x+1)^4 + c$

2 **a** $y = -\frac{1}{6}(1-2x)^3 + c$ **b** $y = \frac{2}{3}x^{\frac{3}{2}} - 4\sqrt{x} + c$

c $y = x + 2\ln|x| + \dfrac{5}{x} + c$

3 **a** $f(x) = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 3x + c$ **b** $f(x) = \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c$

c $f(x) = 3e^x - 4\ln|x| + c$

4 **a** $f(x) = x^2 - x + 3$ **b** $f(x) = x^3 + x^2 - 7$

c $f(x) = e^x + 2\sqrt{x} - 1 - e$ **d** $f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

5 **a** $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 26F

1 **a** $\frac{1}{8}(2x+5)^4 + c$ **b** $\dfrac{1}{2(3-2x)} + c$ **c** $\dfrac{-2}{3(2x-1)^3} + c$

d $\frac{1}{32}(4x-3)^8 + c$ **e** $\frac{2}{9}(3x-4)^{\frac{3}{2}} + c$ **f** $-4\sqrt{1-5x} + c$

g $-\frac{3}{5}(1-x)^5 + c$ **h** $-2\sqrt{3-4x} + c$ **i** $\frac{3}{8}(2x-1)^{\frac{4}{3}} + c$

2 **a** $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$ **b** $(-8, -19)$

3 **a** $\frac{1}{2}(2x-1)^3 + c$ **b** $\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $-\frac{1}{12}(1-3x)^4 + c$ **d** $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}} + c$ **f** $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

4 **a** $2e^x + \frac{5}{2}e^{2x} + c$ **b** $\frac{1}{3}x^3 + \frac{2}{3}e^{-3x} + c$

c $\frac{2}{3}x^{\frac{3}{2}} + 2e^{2x} + e^{-x} + c$ **d** $\frac{1}{2}\ln|2x-1| + c$

e $-\frac{5}{3}\ln|1-3x| + c$ **f** $-e^{-x} - 2\ln|2x+1| + c$

g $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$ **h** $-\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

i $\frac{1}{2}x^2 + 5\ln|1-x| + c$

5 **a** $y = x - 2e^x + \frac{1}{2}e^{2x} + c$ **b** $y = x - x^2 + 3\ln|x+2| + c$

c $y = -\frac{1}{2}e^{-2x} + 2\ln|2x-1| + c$

6 Both are correct. Recall that:

$$\frac{d}{dx}(\ln|Ax|) = \frac{d}{dx}(\ln|A| + \ln|x|) = \frac{1}{x}$$

7 **a** $f(x) = -e^{-2x} + 4$

b $f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

c $f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

8 $\displaystyle\int \frac{2x-8}{x^2-4}\,dx = 3\ln|x+2| - \ln|x-2| + c$

9 $\displaystyle\int \frac{2}{4x^2-1}\,dx = \frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|2x+1| + c$

EXERCISE 26G

1 **a** $\frac{1}{5}(x^3+1)^5 + c$ **b** $2\sqrt{x^2+3} + c$

c $\frac{2}{3}(x^3+x)^{\frac{3}{2}} + c$ **d** $\frac{1}{4}(2+x^4)^4 + c$

e $\frac{1}{5}(x^3+2x+1)^5 + c$ **f** $-\dfrac{1}{27(3x^3-1)^3} + c$

g $\dfrac{1}{8(1-x^2)^4} + c$ **h** $-\dfrac{1}{2(x^2+4x-3)} + c$

i $\frac{1}{5}(x^2+x)^5 + c$

2 **a** $e^{1-2x} + c$ **b** $e^{x^2} + c$ **c** $\frac{1}{3}e^{x^3+1} + c$

d $2e^{\sqrt{x}} + c$ **e** $-e^{x-x^2} + c$ **f** $e^{1-\frac{1}{x}} + c$

3 **a** $\ln\left|x^2+1\right| + c$ **b** $-\frac{1}{2}\ln\left|2-x^2\right| + c$

c $\ln\left|x^2-3x\right| + c$ **d** $2\ln\left|x^3-x\right| + c$

e $-2\ln\left|5x-x^2\right| + c$ **f** $-\frac{1}{3}\ln\left|x^3-3x\right| + c$

4 **a** $f(x) = -\frac{1}{9}(3-x^3)^3 + c$ **b** $f(x) = \frac{3}{2}\ln\left|x^2-2\right| + c$

c $f(x) = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$ **d** $f(x) = -\frac{1}{2}e^{1-x^2} + c$

e $f(x) = -\ln\left|x^3-x\right| + c$ **f** $f(x) = \frac{1}{4}(\ln x)^4 + c$

g $f(x) = \ln\left|x^3+2x^2-1\right| + c$ **h** $f(x) = 4\ln|\ln x| + c$

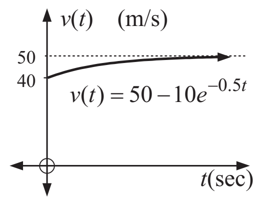
i $f(x) = \dfrac{-1}{\ln x} + c$

EXERCISE 26H

1 $\frac{1}{2}$ cm **2** $5\frac{1}{6}$ cm **3** **a** 41 units **b** 34 units

4 **a** 40 m/s **b** 47.77 m/s **c** 1.386 seconds

d as $t \to \infty$, $v(t) \to 50$ **f**



e $a(t) = 5e^{-0.5t}$ and as $e^x > 0$ for all x, $a(t) > 0$ for all t.

g 134.5 m

5 900 m

6 **a** Show that $v(t) = 100 - 80e^{-\frac{1}{20}t}$ m/s and as $t \to \infty$, $v(t) \to 100$ m/s **b** 370.4 m

EXERCISE 26I

1 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** $e - 1$ ($\doteqdot 1.718$) **d** $1\frac{1}{2}$ **e** $6\frac{2}{3}$

f $\ln 3$ ($\doteqdot 1.099$) **g** 1.524 **h** 2 **i** $e - 1$ ($\doteqdot 1.718$)

2 **a** $\frac{1}{12}$ **b** 1.557 **c** $20\frac{1}{3}$ **d** 0.0337

e $\frac{1}{2}\ln(\frac{2}{7})$ ($\doteqdot -0.6264$) **f** $\frac{1}{2}(\ln 2)^2$ ($\doteqdot 0.2402$)

g 0 **h** $2\ln 7$ ($\doteqdot 3.892$) **i** $\dfrac{3^{n+1}}{2n+2}$, $n \neq -1$

3 **Hint:** $\ln A - \ln B = \ln \dfrac{A}{B}$

EXERCISE 26J

1 **a** $\frac{1}{3}$ units^2 **b** $3\frac{3}{4}$ units^2 **c** $e - 1$ ($\doteqdot 1.718$) units^2

d $20\frac{5}{6}$ units^2 **e** 18 units^2 **f** $\ln 4$ ($\doteqdot 1.386$) units^2

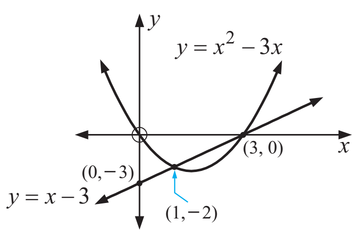
g $\ln 3$ ($\doteqdot 1.099$) u^2 **h** $4\frac{1}{2}$ u^2 **i** $2e - \dfrac{2}{e}$ ($\doteqdot 4.701$) u^2

2 **a** $4\frac{1}{2}$ u^2 **b** $1 + e^{-2}$ ($\doteqdot 1.135$) u^2 **c** $1\frac{5}{27}$ u^2 **d** $2\frac{1}{4}$ u^2

3 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2 **c** 8 units^2

4 **a** $10\frac{2}{3}$ units^2

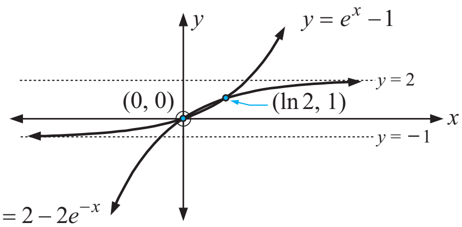
b **i, ii**



iii $1\frac{1}{3}$ units^2

c $\frac{1}{3}$ units^2

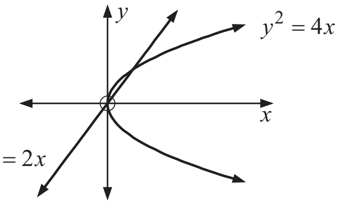
d



enclosed area $= 3\ln 2 - 2$ ($\doteqdot 0.0794$) units^2

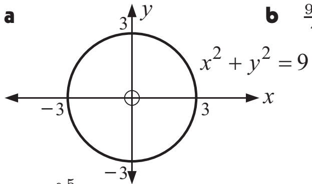
e $\frac{1}{2}$ units^2

5



enclosed area $= \frac{1}{3}$ units^2

6 **a**



b $\frac{9\pi}{4}$ units^2 ($\doteqdot 7.07$ units^2)

7 **a** $\int_3^5 f(x)\,dx = -$(area between $x=3$ and $x=5$)

b $\int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

8 $k \doteqdot 1.7377$ **9** $b \doteqdot 1.3104$ **10** $k \doteqdot 2.3489$ **11** $a=0$

EXERCISE 26K

1 \$4250 **2** \$1127.60 **3** \$4793.2 mill. dollars **4** 95.1 units

5 **a** $P(x) = 15x - 0.015x^2 - 650$ dollars

b maximum profit is \$3100, when 500 plates are made

c $46 \leqslant x \leqslant 954$ plates (you can't produce part of a plate)

6 14 400 calories **7** 76.3^o

8 **a** $y = -\frac{1}{120}(1-x)^4 - \dfrac{x}{30} + \frac{1}{120}$ **b** 2.5 cm (at $x=1$ m)

9 **a** $y = \left(\frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 - \frac{0.08}{3}x\right)$ metres **b** 3.33 cm

c 2.375 cm **d** 1.05^o

10 Extra hint: $\dfrac{dC}{dV} = \frac{1}{2}x^2 + 4$ and $\dfrac{dV}{dx} = \pi r^2$

11 **e** 0.974 km $\times$ 2.05 km

REVIEW SET 26A

1 **a** $8\sqrt{x} + c$ **b** $-\frac{3}{2}\ln|1-2x| + c$ **c** $-\frac{1}{2}e^{1-x^2} + c$

2 **a** $12\frac{4}{9}$ **b** $\frac{5}{54}$

3 $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-4}}$; $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4} + c$

4 29.6 cm **5** 4.5 units^2

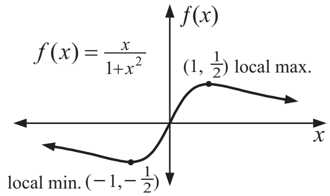
6 $I(t) = \dfrac{100}{t} + 100$ **a** 105 amps **b** as $t \to \infty$, $I \to 100$

7 π units^2 **8** no, $\int_1^3 f(x)\,dx = -$ (area from $x=1$ to $x=3$)

9 **a** local maximum at $(1, \frac{1}{2})$, local minimum at $(-1, -\frac{1}{2})$

b as $x \to \infty$, $f(x) \to 0$ (positive) as $x \to -\infty$, $f(x) \to 0$ (negative)

c



d $\frac{1}{2}\ln 5$ ($\doteqdot 0.805$) units^2

10 $\int_{-4}^{-1} [g(x) - f(x)]\,dx + \int_{-1}^{5} [f(x) - g(x)]\,dx$ **11** $k = 1\frac{1}{3}$

REVIEW SET 26B

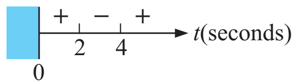
1 **a** $-2e^{-x} - \ln|x| + 3x + c$ **b** $\frac{1}{2}x^2 - 2x + \ln|x| + c$

2 **a** $2\frac{8}{15}$ **b** $4\frac{1}{2}$

3 $\dfrac{d}{dx}(3x^2+x)^3 = 3(3x^2+x)^2(6x+1)$

$\int (3x^2+x)^2(6x+1)\,dx = \frac{1}{3}(3x^2+x)^3 + c$

4 **a** $V(t)$:



b The particle moves in the positive direction initially, then at $t=2$, $6\frac{2}{3}$ from its starting point, it changes direction. It changes direction again at $t=4$, $5\frac{1}{3}$ from its starting point, and at $t=5$, it is $6\frac{2}{3}$ m from its starting point.

c $6\frac{2}{3}$ m **d** $9\frac{1}{3}$ m

5 $3 - \ln 4$ ($\doteqdot 1.614$) units^2 **6** $f(x) = 3x^3 + 5x^2 + 6x - 1$

7 $a = \ln 3$, $b = \ln 5$

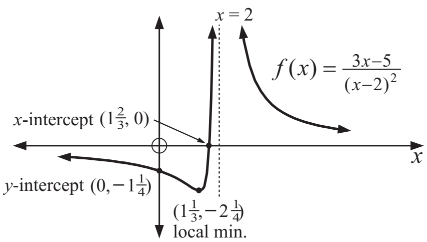
8 **a** $A = 2$, $B = -5$ **b** $\displaystyle\int_0^2 \frac{4x-3}{2x+1}\,dx = 4 - \frac{5}{2}\ln 5$ ($\doteqdot -0.0236$)

9 $a = -3$ A has x-coordinate $\sqrt[3]{4}$

10 **a** y-intercept is $-1\frac{1}{4}$, x-intercept is $1\frac{2}{3}$ **b** $x=2$

c local minimum at $(1\frac{1}{3}, -2\frac{1}{4})$

d



e $A=3$, $B=1$ **f** area $= 3.925$ units^2

11 $m=1$, $c=1$

REVIEW SET 26C

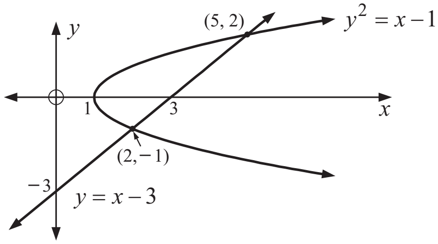
1 **a** $y = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c$ **b** $y = 400x + 40e^{-\frac{x}{2}} + c$

2 **a** $-2\ln 5$ ($\doteqdot -3.219$) **b** $\frac{10}{3}$

3 $\dfrac{dy}{dx} = \dfrac{3x}{\sqrt{3x^2+1}}$ $\therefore$ $\displaystyle\int \frac{x}{\sqrt{3x^2+1}}\,dx = \frac{1}{3}\sqrt{3x^2+1} + c$

4 269 cm

5



a $(2, -1)$ and $(5, 2)$ **b** 4.5 units^2

6 $k = \sqrt[3]{16}$ **7** $a = \frac{1}{3}$ slope > 0 for all x in the domain

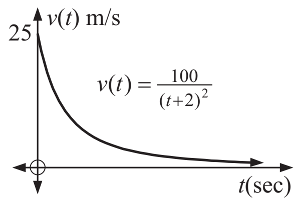
8 $\dfrac{-2x}{4 - x^2} = \dfrac{1}{x+2} - \dfrac{1}{2-x}$

9 **Hint**: Show that the areas represented by the integrals can be arranged to form a $1 \times e$ unit rectangle.

10 **a** $v(0) = 25$ m/s, $v(3) = 4$ m/s

b as $t \to \infty$, $v(t) \to 0$

c



d 3 seconds **e** $a(t) = \dfrac{-200}{(t+2)^3}$, $t \geqslant 0$ **f** $k = \frac{1}{5}$

11 $\dfrac{d}{dx}[\ln x]^2 = \dfrac{2 \ln x}{x}$ $\displaystyle\int \frac{\ln x}{x}\, dx = \tfrac{1}{2}(\ln x)^2 + c$

EXERCISE 27A

1 **a** $-3\cos x - 2x + c$ **b** $2x^2 - 2\sin x + c$

c $\frac{4}{3}x^{\frac{3}{2}} + 4\tan x + c$ **d** $\tan x - 2\cos x + c$

e $\dfrac{x^2}{4} - \tan x + c$ **f** $-\cos x - 2\sin x + e^x + c$

2 **a** $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}\sin x + c$ **b** $\dfrac{\theta^2}{2} + \cos\theta + c$ **c** $\frac{2}{5}t^{\frac{5}{2}} + 2\tan t + c$

d $2e^t + 4\cos t + c$ **e** $3\sin t - \ln|t| + c$

f $3\theta - 2\ln|\theta| + \tan\theta + c$

3 **a** $e^x \sin x + e^x \cos x$, $e^x \sin x + c$

b $-e^{-x}\sin x + e^{-x}\cos x$, $e^{-x}\sin x + c$

c $\cos x - x\sin x$, $\sin x - x\cos x + c$ **d** $\dfrac{1}{\cos x} + c$

4 **a** $f(x) = \dfrac{x^3}{3} - 4\sin x + 3$

b $f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$

c $f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2\tan x - \frac{2}{3}\pi^{\frac{3}{2}}$

EXERCISE 27B

1 **a** $-\frac{1}{3}\cos(3x) + c$ **b** $\frac{1}{2}\sin(4x) + c$ **c** $\frac{1}{2}\tan(2x) + c$

d $6\sin\left(\frac{x}{2}\right) + c$ **e** $-\frac{3}{2}\cos 2x + e^{-x} + c$

f $\frac{1}{2}e^{2x} - 4\tan\left(\frac{x}{2}\right) + c$ **g** $-\cos\left(2x + \frac{\pi}{6}\right) + c$

h $3\sin\left(\frac{\pi}{4} - x\right) + c$ **i** $-2\tan\left(\frac{\pi}{3} - 2x\right) + c$

j $\frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + c$ **k** $-\frac{2}{3}\cos(3x) + \frac{5}{4}\sin(4x) + c$

l $\frac{1}{16}\sin(8x) + 3\cos x + c$

2 **a** $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$ **b** $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

c $\frac{3}{2}x + \frac{1}{8}\sin(4x) + c$ **d** $\frac{5}{2}x + \frac{1}{12}\sin(6x) + c$

e $\frac{1}{4}x + \frac{1}{32}\sin(8x) + c$ **f** $\frac{3}{2}x + 2\sin x + \frac{1}{4}\sin(2x) + c$

3 $\frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$

4 **a** $\frac{1}{5}\sin^5 x + c$ **b** $-2(\cos x)^{\frac{1}{2}} + c$ **c** $-\ln|\cos x| + c$

d $\frac{2}{3}(\sin x)^{\frac{3}{2}} + c$ **e** $-(2 + \sin x)^{-1} + c$ **f** $\frac{1}{2}(\cos x)^{-2} + c$

g $\ln|1 - \cos x| + c$ **h** $\frac{1}{2}\ln|\sin(2x) - 3| + c$

i $\frac{1}{3}\ln|\sin(3x)| + c$

5 **a** $\sin x - \frac{1}{3}\sin^3 x + c$ **b** $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

c $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

6 **a** $-e^{\cos x} + c$ **b** $\frac{1}{8}\sin^4(2x) + c$ **c** $\ln|\sin x - \cos x| + c$

d $e^{\tan x} + c$

EXERCISE 27C

1 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\sqrt{3} - 1$ **d** $\frac{1}{3}$ **e** $\frac{\pi}{8} + \frac{1}{4}$ **f** $\frac{\pi}{4}$

2 **a** $2 - \sqrt{2}$ **b** $\frac{1}{24}$ **c** $\frac{1}{2}\ln 2$ **d** $\ln 2$ **e** $\ln 2$

EXERCISE 27D

1 **b** $\frac{\pi}{2}$ u^2 **2** $(\sqrt{2} - 1)$ u^2 **3** **a** $(\frac{\pi}{4}, 1)$ **b** $\ln\sqrt{2}$ u^2

4 **a** C_1 is $y = \sin 2x$, C_2 is $y = \sin x$

b $A(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ **c** $2\frac{1}{2}$ u^2

5 **a** C_1 is $y = \cos^2 x$, C_2 is $y = \cos(2x)$

b $A(0, 1)$ $B(\frac{\pi}{4}, 0)$ $C(\frac{\pi}{2}, 0)$ $D(\frac{3\pi}{4}, 0)$ $E(\pi, 1)$

6 $(\frac{\pi}{2} - 1)$ units^2

REVIEW SET 27A

1 **a** $\frac{1}{8}\sin^8 x + c$ **b** $-\frac{1}{2}\ln|\cos(2x)| + c$ **c** $e^{\sin x} + c$

2 $x\tan x + \ln|\cos x| + c$

3 $\left(1 - \frac{\pi}{4}\right)$ units^2

4 **a** $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$ **b** $\frac{1}{2}\ln 2$

5 $\tan x$, $\int \tan x\, dx = \ln(\sec x) + c$ **6** $\ln 2$ **7** $\left(\frac{\pi^2}{2} - 2\right)$ u^2

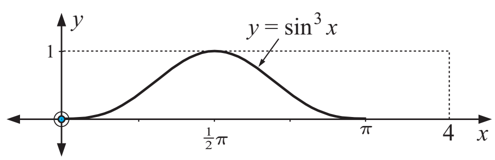
REVIEW SET 27B

1 **a** $2x - 2\sin x + c$ **b** $\dfrac{9x}{2} - 4\sin x + \frac{1}{4}\sin(2x) + c$

2 $\dfrac{1}{3\cos^3 x} + c$ **3** $\frac{1}{2}\sin(x^2) + c$ **4** $\ln 2$ units^2

5 $\doteqdot 2.35$ m **6** **a** $\frac{\pi}{12} - \frac{1}{4}$ **b** $\frac{1}{2}\ln 3$

7



EXERCISE 28A.1

1 **a** 36π units^3 **b** 8π units^3 **c** $\frac{127\pi}{7}$ units^3
d $\frac{255\pi}{4}$ units^3 **e** $\frac{992\pi}{5}$ units^3 **f** $\frac{250\pi}{3}$ units^3

2 **a** 186π units^3 **b** $\frac{146\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^8 - 1)$ units^3

3 **a** 63π units^3 **b** $\doteqdot 198$ cm^3

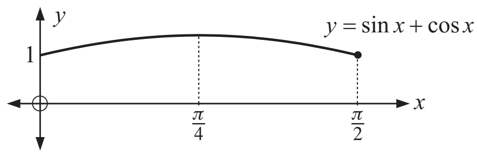
4 **a** a cone of base radius r and height h
b $y = -\left(\frac{r}{h}\right)x + r$ **c** $V = \frac{1}{3}\pi r^2 h$

5 **a** a sphere of radius r

EXERCISE 28A.2

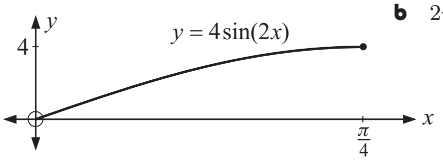
1 **a** $\frac{\pi^2}{4}$ units^3 **b** $\frac{\pi^2}{8}$ units^3 **c** 2π units^3 **d** $\pi\sqrt{3}$ units^3

2 **a**



b $\pi\left(\frac{\pi}{4} + \frac{1}{2}\right)$ units^3

3 **a**



b $2\pi^2$ units^3

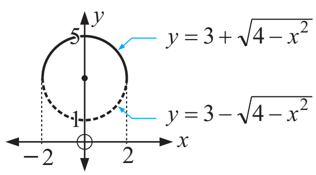
EXERCISE 28B

1 **a** A is at $(-1, 3)$, B$(1, 3)$ **b** $\frac{136\pi}{15}$ units^3

2 **a** A is at $(2, e)$ **b** $\pi(e^2 + 1)$ units^3

3 **a** A is at $(1, 1)$ **b** $\frac{11\pi}{6}$ u^3 **4** **a** A is at $(5, 1)$ **b** $\frac{9\pi}{2}$ u^3

5 **b**



c $\doteqdot 237$ units^3

REVIEW SET 28

1 **a** 312π units^3 **b** 402π units^3 **c** $\frac{\pi^2}{2}$ units^3 **d** 18π units^3

2 **a** $\pi\left(\frac{3\pi}{32} - \frac{1}{8\sqrt{2}}\right)$ units^3 **b** $\doteqdot 123.8$ units^3 **3** 2π units^3

4 $\frac{128\pi}{5}$ units^3 **5** $\frac{\pi}{2}$ units^3 **6** **a** $\frac{128\pi}{3}$ units^3 **b** $\frac{128\pi}{3}$ units^3

EXERCISE 29A

1 **a** continuous **b** discrete **c** continuous **d** continuous
e discrete **f** discrete **g** continuous **h** continuous

2 **a** **i** height of water in the rain gauge
ii $0 \leqslant x \leqslant 200$ mm **iii** continuous
b **i** stopping distance **ii** $0 \leqslant x \leqslant 50$ m **iii** continuous
c **i** time for the switch to fail
ii $0 \leqslant x \leqslant 10\,000$ hours **iii** continuous

3 **a** $0 \leqslant x \leqslant 4$
b

YYYY	YYYN YYNY YNYY NYYY	YYNN YNYN YNNY NNYY NYNY NYYN	NNNY NNYN NYNN YNNN	NNNN
$(x = 4)$	$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

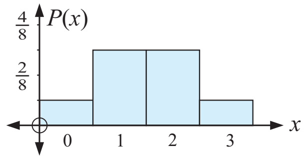
c **i** $x = 2$ **ii** $x = 2$, 3 or 4

4 **a** $x = 0, 1, 2, 3$
b

HHH	HHT HTH THH	TTH THT HTT	TTT
$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

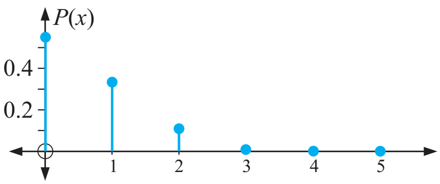
c $\Pr(x = 3) = \frac{1}{8}$, $\Pr(x = 2) = \frac{3}{8}$,
$\Pr(x = 1) = \frac{3}{8}$, $\Pr(x = 0) = \frac{1}{8}$

d



EXERCISE 29B

1 **a** $a = 0.5488$, the probability that Jason does not hit a home run in a game.
b $P(2) = 0.1088$
c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.
d



2 **a** $\sum P(x_i) > 1$ **b** $P(5) < 0$ which is not possible

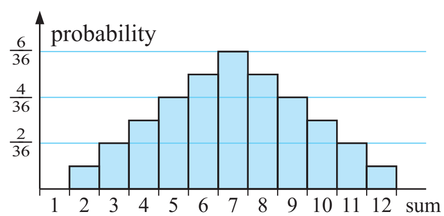
3 **a** The random variable represents the number of hits that Sally has in each game.
b $k = 0.23$ **c** **i** $P(x \geqslant 2) = 0.79$ **ii** $P(1 \leqslant x \leqslant 3) = 0.83$

4 **a**

roll 1							
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6
		roll 2					

b $\Pr(0) = 0$; $\Pr(1) = 0$; $\Pr(2) = \frac{1}{36}$; $\Pr(3) = \frac{2}{36}$;
$\Pr(4) = \frac{3}{36}$; $\Pr(5) = \frac{4}{36}$; $\Pr(6) = \frac{5}{36}$;
$\Pr(7) = \frac{6}{36}$; $\Pr(8) = \frac{5}{36}$; $\Pr(9) = \frac{4}{36}$;
$\Pr(10) = \frac{3}{36}$; $\Pr(11) = \frac{2}{36}$; $\Pr(12) = \frac{1}{36}$

c



5 **a** $k = \frac{1}{12}$ **b** $k = \frac{12}{25}$

6 **a** $P(0) = 0.1975k$; $P(1) = 0.0988k$; $P(2) = 0.0494k$; $P(3) = 0.0247k$; $P(4) = 0.0123k$
b $k = 2.6130$ $P(x \geqslant 2) = 0.2258$

7 **a** $P(0) = 0.6648$ **b** $P(x \geqslant 1) = 0.3352$

EXERCISE 29C

1 102 days **2** **a** $\frac{1}{8}$ **b** 25 **3** **a** $\doteq 0.385$ **b** 19 times

4 **a** **i** $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{2}$ **b** **i** \$1.33 **ii** \$0.50 **iii** \$3.50
c lose 50 cents **d** lose \$50

5 27 saves **6** 15 days **7** 30 doubles **8** $\doteq$ 86 times

9 **a** **i** 0.55 **ii** 0.29 **iii** 0.16
b **i** 4125 **ii** 2175 **iii** 1200

10 **a** \$3.50 **b** No **11** **a** \$2.75 **b** \$3.75 **12** \$1.50

13 **a** \$7.67 **b** lose \$233.33 **14** \$2

EXERCISE 29D

1 **a** $\mu = 0.74$ **b** $\sigma = 0.9962$

2 $P(1) = \frac{1}{10}$, $P(2) = \frac{3}{10}$, $P(3) = \frac{6}{10}$, $\mu = 2.5$, $\sigma = 0.6708$

3 **a** $P(0) = 0.216$, $P(1) = 0.432$, $P(2) = 0.288$, $P(3) = 0.064$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

b $\mu = 1.2$, $\sigma = 0.8485$

5 **a**

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

b $\mu = 3.0$, $\sigma = 1.0954$ **c** **i** $P(\mu - \sigma < x < \mu + \sigma) \doteq 0.8$ **ii** $P(\mu - 2\sigma < x < \mu + 2\sigma) \doteq 1$

6 \$390 **7** **a**

x_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b $\mu = 4.472$, $\sigma = 1.404$

8

x_i	1	2	3
$P(x_i)$	0.5	0.25	0.25

$\mu = 1.75$

EXERCISE 29E

1 **a** The binomial distribution applies, as tossing a coin has one of two possible outcomes (H or T) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.

2 **a** 0.0156 **b** 0.234 **c** 0.109 **d** 0.344

3 **a** 0.268 **b** 0.800 **c** 0.200

4 **a** 0.231 **b** 0.723

5 **a** 0.476 **b** 0.840 **c** 0.160 **d** 0.996

6 **a** 9.54×10^{-7} **b** 0.176 **c** 0.412 **d** 0.0207

7 **a** 0.0280 **b** 0.002 46 **c** 0.131 **d** 0.710

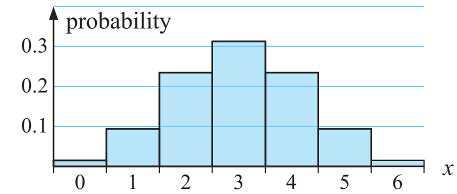
8 **a** 0.998 **b** 0.8065 **9** **a** 0.290 **b** 0.885

EXERCISE 29F

1 **a** **i** $\mu = 3$, $\sigma = 1.2247$

x_i	0 or 6	1 or 5	2 or 4	3
$P(x_i)$	0.0156	0.0938	0.2344	0.3125

ii

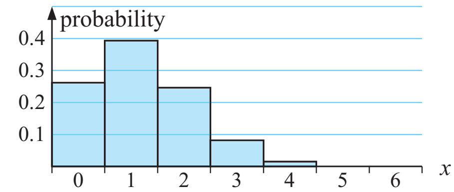


iii The distribution is bell-shaped.

b **i** $\mu = 1.2$, $\sigma = 0.980$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.262	0.393	0.246	0.082	0.015	0.002	0.000

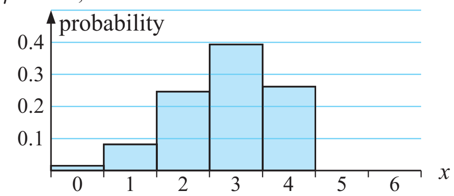
ii



iii The distribution is positively skewed.

c **i** $\mu = 4.8$, $\sigma = 0.980$

ii



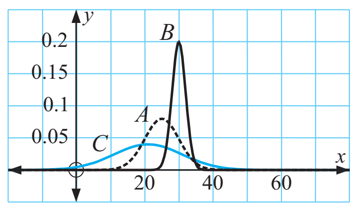
iii This distribution is negatively skewed and is the exact reflection of **b**.

2 $\mu = 5$, $\sigma = 1.58$ **4** $\mu = 1.2$, $\sigma = 1.07$

5 $\mu = 8.99$, $\sigma = 2.94$ **6** $\mu = 0.65$, $\sigma = 0.752$

EXERCISE 29G.1

1



2 **a** The mean volume (or life time, weight, diameter etc) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 **a** 84.1% **b** 2.3% **c** 2.14%, 95.5% **d** 97.7%, 2.3%

4 **a** 0.683 **b** 0.341 **c** 0.477 **d** 0.499

5 **a** 34.1% **b** 47.7% **c** 0.136 **d** 0.159
e 0.0228 **f** 0.841

6 **a** $\doteq 382$ **b** $\doteq 399$ **c** $\doteq 781$

7 **a** $\doteq$ 41 days **b** $\doteq$ 254 days **c** $\doteq$ 213 days

EXERCISE 29G.2

1 **a** 0.341 **b** 0.383 **c** 0.106

2 **a** 0.341 **b** 0.264 **c** 0.212 **d** 0.945 **e** 0.579
f 0.383

EXERCISE 29H.1

1 **a** 0.885 **b** 0.195 **c** 0.3015 **d** 0.947 **e** 0.431

2 **a** 0.201 **b** 0.525 **c** 0.809 **d** 0.249 **e** 0.249

3 **a** 0.383 **b** 0.950

4 **a** $a = 1.645$ **b** $a = -1.282$

EXERCISE 29H.2

1 **a** 0.159 **b** 0.3085 **c** 0.335
2 **a** 0.348 **b** 0.324 **c** 0.685
3 **a** 0.585 **b** 0.805 **c** 0.528

EXERCISE 29H.3

1 **a** $k \doteqdot 0.878$ **b** $k \doteqdot 0.202$ **c** $k \doteqdot -0.954$
2 **a** $k \doteqdot -0.295$ **b** $k \doteqdot 1.165$ **c** $k \doteqdot -1.087$
3 **a** $\Pr(-k \leqslant z \leqslant k)$
$= \Pr(z \leqslant k) - P(z \leqslant -k)$
$= \Pr(z \leqslant k) - [1 - P(z \leqslant k)]$ {as is symmetric about O}
$= 2\Pr(z \leqslant k) - 1$
b **i** $k \doteqdot 0.303$ **ii** $k \doteqdot 1.037$

EXERCISE 29I

1 0.378 **2** **a** 90.4% **b** 4.78% **3** 83
4 **a** 0.003 33 **b** 61.5% **c** 23
5 **a** 0.933 **b** 0.243
6 **a** $\mu = 52.4$, $\sigma = 21.6$ **b** 54.4%
7 **a** $\mu = 2.00$, $\sigma = 0.0305$ **b** 0.736
8 $\Pr(S \geqslant 70) = 0.1587$ $\Pr(G \geqslant 66) = 0.0913$ $\therefore$ only 9% achieved a higher grade in Geography while almost 16% achieved a higher grade in Science. The student did better in Geography where he/she achieved a grade higher than 91% of the class compared with a grade of higher than 84% of the class in Science.

REVIEW SET 29A

1 **a** $a = \frac{5}{9}$ **b** $\frac{4}{9}$
2 **a**

x_i	0	1	2	3	4
$P(x_i)$	0.0625	0.25	0.375	0.25	0.0625

b $\mu = 2$, $\sigma = 1$
3 $p = 0.18$, **a** 0.302 **b** 0.298 **c** 0.561
4 $p = 0.04$, $n = 120$ **a** $\mu = 4.8$ **b** $\sigma = 2.15$
5 **a** \$4 **b** \$75
6 **a** \$7 **b** No, he would lose \$1 per game in the long run
7 $\mu = 64$, $\sigma = 4$ **a** **i** 81.85% **ii** 84.1% **b** 81.85%
8 $\mu = 31.2$

REVIEW SET 29B

1 **a** $k = \frac{8}{5}$ **b** 0.975
2

x_i	0	10 000	30 000	45 000
$P(x_i)$	0.988 18	0.0088	0.0023	0.000 72

$\mu = 189.40$ $\therefore$ charges \$439.40
3 $p = 0.96$ **a** 0.849 **b** 2.56×10^{-6} **c** 0.991
d 0.000 25
5 **a** **i** 2.28% **ii** 84% **b** 0.840
6 **a** 0.260
b $k = 29.3$ $\therefore$ can expect that no more than 8% of batteries will fail in at most 37.13 weeks.

REVIEW SET 29C

1 **a** $k = 0.05$ **b** $\mu = 1.7$, $\sigma = 0.954$
2 **a** 0.259 **b** 0.337 **c** 0.922
3 $\mu = 6.43$, $\sigma = 2.52$
4 **a** $a = 6.3$ grams **b** $b = 32.3$ grams
5 $\Pr(x \leqslant 3) = 0.147$ $\therefore$ out of 2000, 294 will on average need to be replaced. Profit = \$28 530
6 **a** $\int_0^2 ax^2(2-x)\,dx = 1$ gives $a = \frac{3}{4}$
b mode occurs at the maximum of $f(x)$ and so mode is $\frac{4}{3}$
c median is 1.23
d $P(0.6 < x < 1.2) = \int_{0.6}^{1.2} \frac{3}{4}x^2(2-x)\,dx$
$= 0.3915$

INDEX

absolute value 120
addition law of probability 499
amplitude 259, 274
angle of depression 218
angle of elevation 218
angular velocity 626
antiderivative 656, 665
arc 239
area function 655
arithmetic sequence 39, 41
arithmetic series 51
asymptote 579
asymptotic 28, 75
average acceleration 566
average cost 597
average rate of change 512
average speed 508, 511
average velocity 564
axis intercepts 120, 134
axis of symmetry 157, 174
bell-shaped curve 427, 461, 728
bimodal 429
binomial 197
binomial coefficient 199
binomial expansion 196
binomial experiment 721, 724
binomial probability distribution 721
box-and-whisker plot 448
cardioid 259
cartesian equation 399, 411
cartesian plane 131
catenary curve 106
chain rule 539, 664
chord 214, 239, 522
class interval 421
coincident lines 135
collinear points 391
column matrix 308, 318
column vector 308
common difference 41
common ratio 39
complementary events 477, 494
completing the square 160, 166
component form 360, 370, 390
composite function 26, 538
compound interest 48, 105
concave downwards 583
concave upwards 583
continuous function 698
continuous numerical variable 421
continuous random variable 710, 727
coordinate planes 380
cosine curve 285
cosine function 285
cosine rule 241, 248
cost function 562, 597
cubic function 120
cumulative frequency 442
decreasing function 571
definite integral 643, 675, 696
demand function 598
dependent events 483
derivative 526, 530
determinant 329, 334
directed line segment 384
discrete random variable 710
discriminant 178
displacement 660
displacement function 564
displacement vector 352
distance between points 366, 381
distance formula 141
distinct real root 178
domain 16
dot product 371, 392
double angle formulae 292
empty set 495
equality of vectors 350, 366, 384
equidistant 145
Euler's equation 106
expected frequency 473

experimental probability	*468*
exponent	*62*
exponential function	*75, 120, 606*
exponential growth	*62*
first principles	*526*
five-number summary	*448*
frequency	*421, 470*
frequency histogram	*422*
function	*17*
Fundamental theorem of calculus	*658*
general form	*133*
general term	*38, 40, 45, 199*
geometric mean	*45*
geometric sequence	*39, 44*
geometric series	*54*
geometric test	*18*
gradient of line	*132*
histogram	*421*
horizontal asymptote	*75, 78, 579, 582*
horizontal inflection	*576, 584*
horizontal line test	*30*
horizontal step	*131*
horizontal translation	*125, 274*
hypotenuse	*209*
identity function	*31*
identity matrix	*326*
image	*23*
included angle	*236*
increasing function	*571*
indefinite integral	*696*
independent events	*481, 503*
independent trials	*721*
index laws	*65*
initial conditions	*566*
instantaneous acceleration	*566*
instantaneous rate of change	*526, 558*
instantaneous speed	*513*
instantaneous velocity	*564*
integral	*662*
integrand	*695*
integrating constant	*665*
interquartile range	*445*
intersecting lines	*414*
intersection	*494*
interval notation	*20*
inverse function	*28, 114, 611*
inverse matrix	*328*
laws of logarithms	*93, 612*
length of vector	*349, 384*
linear function	*120*
linear motion	*673*
linear speed	*626*
local maximum	*576, 588*
local minimum	*576, 588*
logarithm	*90*
logarithmic function	*120*
lower area sum	*639, 642*
lower limit	*695*
lower quartile	*445, 451*
magnitude of vector	*384*
marginal cost	*562, 597*
matrix equality	*311*
maximum point	*259*
maximum value	*186*
mean	*425*
median	*426, 451*
midpoint of data	*440*
midpoint of line	*132, 144, 382*
midpoint rule	*653*
minimum point	*259*
minimum value	*186*
modal class	*421*
mode	*428*
modulus	*386*
motion graph	*564*
mutually exclusive events	*495, 500*
mutually perpendicular	*380*
natural logarithm	*611*
negative base	*63*
negative definite	*181*
negative vector	*354*
non-horizontal inflection	*584*
non-stationary inflection	*584*
normal curve	*462, 728*
normal distribution	*461, 728*

normal to curve 545
Null Factor law 162
number sequence 36, 40
obtuse angle 228
optimisation 186
optimum solution 587
order of matrix 309
ordered data set 426
ordered pair 131
ordered triple 381
outcome 470
parabola 152, 174
parallel lines 135, 414, 416
parallel vectors 350, 368, 389
parameter 398, 728
parametric equation 232, 398, 411
parametric form 233
Pascal's triangle 196, 492
percentile 451
period 258, 274
period of investment 99
periodic function 258
perpendicular bisector 145
perpendicular lines 416
point of inflection 584
point of intersection 135
population mean 426, 717
population standard deviation 717
position vector 349, 381
positive base 63
positive definite 181
power rule 664
principal axis 259
probability density function 727, 734
probability distribution 711
product of matrix 319
product rule 541, 664
profit function 599
Pythagoras' Rule 205
quadratic formula 168
quadratic function 120, 174
quotient 543
quotient rule 543, 664
radian 262
random variable 710
range of data 445, 451
range of relation 17
rate of change 508, 560
rational function 579
reciprocal function 27, 120
relation 17
relative frequency 470
relative frequency histogram 422
repeated root 178
revenue function 598
row matrix 309, 318
row vector 309
sample mean 426
sample space 474, 494
scalar 348, 363
scalar multiplication 359, 363
scalar product 371, 392
secant 512
second derivative 550
second derivative test 588
sector 239
segment 239
sigma notation 57
sign diagram 567, 576
sine curve 270
sine function 270
sine rule 244, 248
sine waves 258
skew lines 414
skewed distribution 427
slope function 530, 571, 616
slope of chord 512, 522
slope of curve 526
slope of line 132
slope of tangent 514
slope-intercept form 133
solid of revolution 702
square matrix 309
standard deviation 445, 456, 461
standard normal distribution 734
stationary inflection 584
stationary point 576
statistic 728

sum rule *664*
symmetric distribution *427*
tangent *214, 522, 545*
tangent curve *297*
tangent function *295*
terms of sequence *36*
theoretical probability *475*
tree diagram *474, 485*
trial *470*
trigonometric ratios *209*
turning point *120, 152*
two dimensional grid *474, 479*
undefined slope *132*
union *494*
unique solution *335*
unit circle *231*
unit quarter circle *226*
unit vector *369, 390*
upper area sum *639, 642*
upper limit *695*
upper quartile *445*
variance *456*
vector equation *399, 411*
velocity vector *401*
Venn diagram *494*
vertex *152, 157*
vertical asymptote *579*
vertical line test *18*
vertical step *131*
vertical translation *125, 274*
volume of revolution *702, 706*
Voronoi diagram *130*
zero matrix *316*
zero vector *354*